On the regularization of linear descriptor systems

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Regularization of linear systems

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) + Bu(t)} \qquad\qquad [E, A, B]$$

$$E, A \in \mathbb{R}^{\ell \times n}, B \in \mathbb{R}^{\ell \times m}$$

Aim: $[E, A, B] \rightsquigarrow [\hat{E}, \hat{A}, \hat{B}]$
s.t. $s\hat{E} - \hat{A}$ is regular

1. approach: state feedback

2. approach: behavioral equivalence

3. approach: 1. + 2. + permutation

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Some basic notions

$sE - A$ is **regular**, if $\ell = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$

regular $sE - A$ has **index** $\nu \in \mathbb{N}_0$, if

$$S(sE - A)T = \begin{bmatrix} sI_r - J & 0 \\ 0 & sN - I_{n-r} \end{bmatrix},$$

then $\nu := \begin{cases} 0, & \text{if } r = n, \\ \min\{ k \in \mathbb{N}_0 \mid N^k = 0 \}, & \text{if } r < n \end{cases}$

augmented Wong sequences:

$$\mathcal{V}_0 := \mathbb{R}^n, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i + \operatorname{im} B), \ i \geq 0, \qquad \mathcal{V}^* := \bigcap_{i \geq 0} \mathcal{V}_i$$

$$\mathcal{W}_0 := \{0\}, \quad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i + \operatorname{im} B), \ i \geq 0, \quad \mathcal{W}^* := \bigcup_{i \geq 0} \mathcal{W}_i$$

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Approach 1: State feedback

Here: $\ell = n, \quad u(t) = Fx(t)$ for some $F \in \mathbb{R}^{m \times n}$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = (A + BF)x(t)$$

Thm. [Bunse-Gerstner et al. '92]:
$\exists\, F \in \mathbb{R}^{m \times n} : \ sE - (A + BF)$ is regular and has index ≤ 1
$\Longleftrightarrow [E, A, B]$ is impulse controllable $(\operatorname{im} E + A \ker E + \operatorname{im} B = \mathbb{R}^n)$

Thm.: $\exists\, F \in \mathbb{R}^{m \times n} : \ sE - (A + BF)$ is regular
$\quad \Longleftrightarrow \dim(E\mathcal{V}^* + \operatorname{im} B) = \dim \mathcal{V}^*$ [Özçaldiran/Lewis '90]
$\quad \Longleftrightarrow \dim(A\mathcal{W}^* + \operatorname{im} B) = \dim \mathcal{W}^*$ [Özçaldiran/Lewis '90]
$\quad \Longleftrightarrow \operatorname{rk}_{\mathbb{R}(s)}[sE - A, B] = n$ [Duan/Zhang '03]

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Def.: $[E, A]$ is *autonomous*, if

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) \ \wedge \ Ex(0) = 0 \implies x = 0$$

Lemma:

$sE - A$ is regular $\iff \ell = n \ \wedge \ [E, A]$ is autonomous

Thm. [Fletcher '86]:

$\exists \ F \in \mathbb{R}^{m \times n} : \ [E, A + BF]$ is aut. $\iff \mathrm{rk}_{\mathbb{R}(s)}[sE - A, B] \geq n$

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Approach 2: Behavioral equivalence

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) \overset{\text{beh.}}{\iff} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

$$s \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is regular}$$

Def.: The *behavior* of $[E, A, B]$ is

$$\mathfrak{B}_{[E,A,B]} = \{ (x, u) \in \mathcal{C}^{\infty} \times \mathcal{C}^{\infty} \mid E\dot{x}(t) = Ax(t) + Bu(t) \}.$$

Two systems $[E_i, A_i, B_i]$, $i = 1, 2$, are *behaviorally equivalent* $([E_1, A_1, B_1] \simeq_{\mathfrak{B}} [E_2, A_2, B_2])$, if

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Def.: $[E, A, B]$ is *minimal*, if

$$\forall\, k \in \{1, \dots, \ell\} \; \forall\, [\tilde{E}, \tilde{A}, \tilde{B}] \text{ with } k \text{ rows}:$$

$$\left([E, A, B] \simeq_{\mathfrak{B}} \left[\begin{bmatrix} \tilde{E} \\ 0_{\ell-k,n} \end{bmatrix}, \begin{bmatrix} \tilde{A} \\ 0_{\ell-k,n} \end{bmatrix}, \begin{bmatrix} \tilde{B} \\ 0_{\ell-k,m} \end{bmatrix} \right] \implies k = \ell \right).$$

Thm. : $\qquad [E, A, B]$ is minimal

$\iff \text{rk}_{\mathbb{R}(s)}[sE - A, B] = \ell$ [Polderman/Willems '98]

$\iff E\mathcal{V}^* + A\mathcal{W}^* + \text{im } B = \mathbb{R}^\ell$ [B./Reis '15]

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Approach 3: Combination of 1 and 2 + permutation

Feedback transformation of $[E, A, B]$:

$$S\,[sE - A, -B] \begin{bmatrix} T & 0 \\ F & V \end{bmatrix}$$

Lemma: $[E_1, A_1, B_1] \simeq_{\mathfrak{B}} [E_2, A_2, B_2] \iff \exists U(s) \in \mathbf{Gl}_\ell(\mathbb{R}[s]):$

$$[sE_1 - A_1, -B_1] = U(s)[sE_2 - A_2, -B_1]$$

Behavioral and feedback transformation of $[E, A, B]$:

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Example:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Approaches 1 and 2 do not work!

x_2, x_3 are free $\rightsquigarrow$ x_2, x_3 are inputs in the behavioral framework

$u = 0$ $\rightsquigarrow$ u is a state in the behavioral framework

Permutation:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{u}(t) \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1(t) \\ u(t) \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_2(t) \\ x_3(t) \end{pmatrix}$$

$$s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$ is regular and has index 1

$\rightarrow$ general (numerical) procedure derived by [Campbell et al. '12]

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$\exists U(s) \in \mathbf{Gl}_\ell(\mathbb{R}[s])$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$, $F \in \mathbb{R}^{m \times n}$ and a permutation matrix $P \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that

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where $s\hat{E} - \hat{A} \in \mathbb{R}[s]^{\hat{n} \times \hat{n}}$ is regular and has index at most 1

- $[\hat{E}, \hat{A}, \hat{B}]$ is minimal
- explicit construction of transformation matrices using feedback canonical form [Loiseau et al. '91]
- the subsystem $[\hat{E}, \hat{A}, \hat{B}]$ is fully determined by the augmented Wong sequences

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Proof: Feedback canonical form [Loiseau et al. '91]

$$N_k := \begin{bmatrix} 0 \\ 1 \\ & \ddots \\ & & 1\, 0 \end{bmatrix} \in \mathbb{R}^{k \times k}, \quad K_k := \begin{bmatrix} 0\,1 \\ & \ddots \\ & & 0\,1 \end{bmatrix}, \quad L_k := \begin{bmatrix} 1\,0 \\ & \ddots \\ & & 1\,0 \end{bmatrix} \in \mathbb{R}^{(k-1) \times k}$$

$\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ multi-index, $|\alpha| = \sum_{i=1}^{k} \alpha_i$, $\ell(\alpha) = k$

$$N_\alpha := \mathrm{diag}(N_{\alpha_1}, \ldots, N_{\alpha_k}) \in \mathbb{R}^{|\alpha| \times |\alpha|},$$

$$K_\alpha := \mathrm{diag}(K_{\alpha_1}, \ldots, K_{\alpha_k}) \in \mathbb{R}^{(|\alpha|-k) \times |\alpha|},$$

$$L_\alpha := \mathrm{diag}(L_{\alpha_1}, \ldots, L_{\alpha_k}) \in \mathbb{R}^{(|\alpha|-k) \times |\alpha|},$$

$$E_\alpha := \mathrm{diag}(e_{\alpha_1}^{[\alpha_1]}, \ldots, e_{\alpha_k}^{[\alpha_k]}) \in \mathbb{R}^{|\alpha| \times k}$$

$$[SET, SAT + SBF, SBV] =$$

$$\left[\begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0 & 0 \\ 0 & L_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & K_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & L_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & N_\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_{\overline{c}}} \end{bmatrix}, \begin{bmatrix} N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & K_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & L_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & K_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{\overline{c}} \end{bmatrix}, \begin{bmatrix} E_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]$$

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$$N_\alpha := \mathrm{diag}(N_{\alpha_1}, \ldots, N_{\alpha_k}) \in \mathbb{R}^{|\alpha| \times |\alpha|},$$
$$K_\alpha := \mathrm{diag}(K_{\alpha_1}, \ldots, K_{\alpha_k}) \in \mathbb{R}^{(|\alpha|-k) \times |\alpha|},$$
$$L_\alpha := \mathrm{diag}(L_{\alpha_1}, \ldots, L_{\alpha_k}) \in \mathbb{R}^{(|\alpha|-k) \times |\alpha|},$$
$$E_\alpha := \mathrm{diag}(e_{\alpha_1}^{[\alpha_1]}, \ldots, e_{\alpha_k}^{[\alpha_k]}) \in \mathbb{R}^{|\alpha| \times k}$$

$[SET, SAT + SBF, SBV] =$

$$\left[\begin{array}{cccccc} I_{|\alpha|} & 0 & 0 & 0 & 0 & 0 \\ 0 & L_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & K_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & L_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & N_\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_{\overline{c}}} \end{array} \right], \left[\begin{array}{cccccc} N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & K_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & L_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & K_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{\overline{c}} \end{array} \right], \left[\begin{array}{ccc} E_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

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$$\left(\sum_{k=0}^{\nu_\kappa - 1} s^k N_\kappa^k \right) (s N_\kappa - I_{|\kappa|}) = -I_{|\kappa|}, \quad \nu_\kappa = \max_i \kappa_i$$

$$\left(\sum_{k=0}^{\nu_\gamma - 1} s^k N_\gamma^k \right) [s K_\gamma^\top - L_\gamma^\top, -E_\gamma] = [-L_\gamma^\top, -E_\gamma], \quad \nu_\gamma = \max_i \gamma_i$$

Shift: $N_k L_k^\top = \begin{bmatrix} 0 \\ 1 \quad \diagdown \\ \quad 1\,0 \end{bmatrix} \begin{bmatrix} 1\,0 \\ \quad \diagdown \\ \quad\quad 1\,0 \end{bmatrix}^\top = \begin{bmatrix} 0\,1 \\ \quad \diagdown \\ \quad\quad 0\,1 \end{bmatrix}^\top = K_k^\top$

$$\left(\sum_{k=0}^{\nu_\gamma - 1} s^k N_\gamma^k \right) (s K_\gamma^\top - L_\gamma^\top) = s^{\nu_\gamma} N_\gamma^{\nu_\gamma - 1} K_\gamma^\top - L_\gamma^\top$$

and: $0 = N_k^{\nu_\gamma} = N_k^{\nu_\gamma - 1}[K_k^\top, 0]$

$$\left(\sum_{k=0}^{\nu_\kappa-1} s^k N_\kappa^k\right)(sN_\kappa - I_{|\kappa|}) = -I_{|\kappa|}, \quad \nu_\kappa = \max_i \kappa_i$$

$$\left(\sum_{k=0}^{\nu_\gamma-1} s^k N_\gamma^k\right)[sK_\gamma^\top - L_\gamma^\top, -E_\gamma] = [-L_\gamma^\top, -E_\gamma], \quad \nu_\gamma = \max_i \gamma_i$$

Shift: $N_k L_k^\top = \begin{bmatrix} 0 \\ 1 \\ & \diagdown \\ & & 1\,0 \end{bmatrix}\begin{bmatrix} 1\,0 \\ & \diagdown \\ & & 1\,0 \end{bmatrix}^\top = \begin{bmatrix} 0\,1 \\ & \diagdown \\ & & 0\,1 \end{bmatrix}^\top = K_k^\top$

$$\left(\sum_{k=0}^{\nu_\gamma-1} s^k N_\gamma^k\right)(sK_\gamma^\top - L_\gamma^\top) = s^{\nu_\gamma} N_\gamma^{\nu_\gamma-1} K_\gamma^\top - L_\gamma^\top$$

and: $0 = N_k^{\nu_\gamma} = N_k^{\nu_\gamma-1}[K_k^\top, 0]$

$$\left(\sum_{k=0}^{\nu_\kappa-1} s^k N_\kappa^k\right)(sN_\kappa - I_{|\kappa|}) = -I_{|\kappa|}, \quad \nu_\kappa = \max_i \kappa_i$$

$$\left(\sum_{k=0}^{\nu_\gamma-1} s^k N_\gamma^k\right)[sK_\gamma^\top - L_\gamma^\top, -E_\gamma] = [-L_\gamma^\top, -E_\gamma], \quad \nu_\gamma = \max_i \gamma_i$$

Shift: $N_k L_k^\top = \begin{bmatrix} 0 \\ 1 & \searrow \\ & & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ & \searrow \\ & & 1 & 0 \end{bmatrix}^\top = \begin{bmatrix} 0 & 1 \\ & \searrow \\ & & 0 & 1 \end{bmatrix}^\top = K_k^\top$

$$\left(\sum_{k=0}^{\nu_\gamma-1} s^k N_\gamma^k\right)(sK_\gamma^\top - L_\gamma^\top) = s^{\nu_\gamma} N_\gamma^{\nu_\gamma-1} K_\gamma^\top - L_\gamma^\top$$

$$\text{and: } 0 = N_k^{\nu_\gamma} = N_k^{\nu_\gamma-1}[K_k^\top, 0]$$

$$\left(\sum_{k=0}^{\nu_\kappa-1} s^k N_\kappa^k\right)(sN_\kappa - I_{|\kappa|}) = -I_{|\kappa|}, \quad \nu_\kappa = \max_i \kappa_i$$

$$\left(\sum_{k=0}^{\nu_\gamma-1} s^k N_\gamma^k\right)[sK_\gamma^\top - L_\gamma^\top, -E_\gamma] = [-L_\gamma^\top, -E_\gamma], \quad \nu_\gamma = \max_i \gamma_i$$

Shift: $N_k L_k^\top = \begin{bmatrix} 0 \\ 1 \diagdown \\ \quad 1\,0 \end{bmatrix}\begin{bmatrix} 1\,0 \\ \diagdown \\ \quad 1\,0 \end{bmatrix}^\top = \begin{bmatrix} 0\,1 \\ \diagdown \\ \quad 0\,1 \end{bmatrix}^\top = K_k^\top$

$$\left(\sum_{k=0}^{\nu_\gamma-1} s^k N_\gamma^k\right)(sK_\gamma^\top - L_\gamma^\top) = s^{\nu_\gamma} N_\gamma^{\nu_\gamma-1} K_\gamma^\top - L_\gamma^\top$$

and: $0 = N_k^{\nu_\gamma} = N_k^{\nu_\gamma-1}[K_k^\top, 0]$

$$U(s)[sE - A, -B] \begin{bmatrix} T & 0 \\ F & V \end{bmatrix} =$$

$$\left[\begin{bmatrix} sI_{|\alpha|} - N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & sL_\beta - K_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -L_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & sL_\delta^\top - K_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}} - A_{\overline{c}} \end{bmatrix}, \begin{bmatrix} -E_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -E_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]$$

$$U(s)[sE - A, -B]\begin{bmatrix} T & 0 \\ F & V \end{bmatrix} =$$

$$\left[\begin{bmatrix} sI_{|\alpha|}-N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & sL_\beta-K_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -L_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & sL_\delta^\top-K_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}}-A_{\overline{c}} \end{bmatrix}, \begin{bmatrix} -E_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -E_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$$

$$P_\delta(s)\left(sL_\delta^\top - K_\delta^\top\right) = \begin{bmatrix} -I_{|\delta|-\ell(\delta)} \\ 0_{\ell(\delta),|\delta|-\ell(\delta)} \end{bmatrix}$$

$$P_\delta(s) = \begin{bmatrix} L_\delta \\ E_\delta^\top \end{bmatrix} \operatorname{diag}\left(\begin{bmatrix} I_{\delta_i-1} & 0 \\ s(e_1^{[\delta_i-1]})^\top & 1 \end{bmatrix} \begin{bmatrix} 0 & \sum_{k=0}^{\delta_i-2} s^k\left(N_{\delta_i-1}^\top\right)^k \\ 1 & 0 \end{bmatrix}\right)_{i=1,\ldots,\ell(\delta)}$$

$$U(s)[sE - A, -B] \begin{bmatrix} T & 0 \\ F & V \end{bmatrix} =$$

$$\left[\begin{bmatrix} sI_{|\alpha|} - N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & sL_\beta - K_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -L_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & {\color{red}-I_{|\delta|-\ell(\delta)}} & 0 & 0 \\ 0 & 0 & 0 & {\color{red}0_{\ell(\delta),|\delta|-\ell(\delta)}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}} - A_{\overline{c}} \end{bmatrix}, \begin{bmatrix} -E_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -E_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$$

$$P_\delta(s)\left(sL_\delta^\top - K_\delta^\top\right) = \begin{bmatrix} -I_{|\delta|-\ell(\delta)} \\ 0_{\ell(\delta),|\delta|-\ell(\delta)} \end{bmatrix}$$

$$P_\delta(s) = \begin{bmatrix} L_\delta \\ E_\delta^\top \end{bmatrix} \mathrm{diag}\left(\begin{bmatrix} I_{\delta_i-1} & 0 \\ s(e_1^{[\delta_i-1]})^\top & 1 \end{bmatrix} \begin{bmatrix} 0 & \sum_{k=0}^{\delta_i-2} s^k \left(N_{\delta_i-1}^\top\right)^k \\ 1 & 0 \end{bmatrix}\right)_{i=1,\dots,\ell(\delta)}$$

$$U(s)[sE - A, -B] \begin{bmatrix} T & 0 \\ F & V \end{bmatrix} =$$

$$\left[\begin{bmatrix} sI_{|\alpha|} - N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & sL_\beta - K_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -L_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{|\delta|-\ell(\delta)} & 0 & 0 \\ 0 & 0 & 0 & 0_{\ell(\delta),|\delta|-\ell(\delta)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}} - A_{\overline{c}} \end{bmatrix}, \begin{bmatrix} -E_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -E_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]$$

$$\begin{bmatrix} sL_\beta - K_\beta & 0 & \Big| & 0 \\ 0 & -L_\gamma^\top & \Big| & -E_\gamma \end{bmatrix} \begin{bmatrix} L_\beta^\top & 0 & E_\beta \\ 0 & L_\gamma & 0 \\ 0 & E_\gamma^\top & 0 \end{bmatrix}$$

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$$\begin{bmatrix} sL_\beta - K_\beta & 0 & \bigg| & 0 \\ 0 & -L_\gamma^\top & \bigg| & -E_\gamma \end{bmatrix} \begin{bmatrix} L_\beta^\top & 0 & E_\beta \\ 0 & L_\gamma & 0 \\ 0 & E_\gamma^\top & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \, 0 \\ \diagdown \\ 1 \, 0 \end{bmatrix} \begin{bmatrix} 1 \, 0 \\ \diagdown \\ 1 \, 0 \end{bmatrix}^\top = I_{k-1}, \qquad \begin{bmatrix} 0 \, 1 \\ \diagdown \\ 0 \, 1 \end{bmatrix} \begin{bmatrix} 1 \, 0 \\ \diagdown \\ 1 \, 0 \end{bmatrix}^\top = N_{k-1}^\top$$

$$\implies \quad (sL_\beta - K_\beta)L_\beta^\top = sI_{|\beta|-\ell(\beta)} - N_{\beta-1}^\top$$

$$\begin{bmatrix} sL_\beta - K_\beta & 0 & 0 \\ 0 & -L_\gamma^\top & -E_\gamma \end{bmatrix} \begin{bmatrix} L_\beta^\top & 0 & E_\beta \\ 0 & L_\gamma & 0 \\ 0 & E_\gamma^\top & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1\,0 & \\ & 1\,0 \end{bmatrix} \begin{bmatrix} 1\,0 & \\ & 1\,0 \end{bmatrix}^\top = I_{k-1}, \quad \begin{bmatrix} 0\,1 & \\ & 0\,1 \end{bmatrix} \begin{bmatrix} 1\,0 & \\ & 1\,0 \end{bmatrix}^\top = N_{k-1}^\top$$

$$\implies \quad (sL_\beta - K_\beta)L_\beta^\top = sI_{|\beta| - \ell(\beta)} - N_{\beta-1}^\top$$

$$\begin{bmatrix} 1\,0 & \\ & 1\,0 \end{bmatrix}^\top \begin{bmatrix} 1\,0 & \\ & 1\,0 \end{bmatrix} = \begin{bmatrix} I_{k-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad e_k e_k^\top = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\implies \quad L_\gamma^\top L_\gamma + E_\gamma E_\gamma^\top = I_{|\gamma|}$$

$$\begin{bmatrix} sL_\beta - K_\beta & 0 & \Big| & 0 \\ 0 & -L_\gamma^\top & \Big| & -E_\gamma \end{bmatrix} \begin{bmatrix} L_\beta^\top & 0 & E_\beta \\ 0 & L_\gamma & 0 \\ 0 & E_\gamma^\top & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \, 0 \\ & \diagdown \\ & 1 \, 0 \end{bmatrix} \begin{bmatrix} 1 \, 0 \\ & \diagdown \\ & 1 \, 0 \end{bmatrix}^\top = I_{k-1}, \quad \begin{bmatrix} 0 \, 1 \\ & \diagdown \\ & 0 \, 1 \end{bmatrix} \begin{bmatrix} 1 \, 0 \\ & \diagdown \\ & 1 \, 0 \end{bmatrix}^\top = N_{k-1}^\top$$

$$\implies \quad (sL_\beta - K_\beta)L_\beta^\top = sI_{|\beta| - \ell(\beta)} - N_{\beta-1}^\top$$

$$\begin{bmatrix} 1 \, 0 \\ & \diagdown \\ & 1 \, 0 \end{bmatrix}^\top \begin{bmatrix} 1 \, 0 \\ & \diagdown \\ & 1 \, 0 \end{bmatrix} = \begin{bmatrix} I_{k-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad e_k e_k^\top = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\implies \quad L_\gamma^\top L_\gamma + E_\gamma E_\gamma^\top = I_{|\gamma|}$$

$$\begin{bmatrix} 1 \, 0 \\ & \diagdown \\ & 1 \, 0 \end{bmatrix} e_k = 0, \quad \begin{bmatrix} 0 \, 1 \\ & \diagdown \\ & 0 \, 1 \end{bmatrix} e_k = e_{k-1} \implies (sL_\beta - K_\beta)E_\beta = -E_{\beta-1}$$

$$\begin{bmatrix} sL_\beta - K_\beta & 0 & \Big| & 0 \\ 0 & -L_\gamma^\top & \Big| & -E_\gamma \end{bmatrix}\begin{bmatrix} L_\beta^\top & 0 & E_\beta \\ 0 & L_\gamma & 0 \\ 0 & E_\gamma^\top & 0 \end{bmatrix} = \begin{bmatrix} sI_{|\beta|-\ell(\beta)} - N_{\beta-1}^\top & 0 & \Big| & -E_{\beta-1} \\ 0 & -I_{|\gamma|} & \Big| & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \\ & \ddots & \ddots \\ & & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & \\ & \ddots & \ddots \\ & & 1 & 0 \end{bmatrix}^\top = I_{k-1}, \quad \begin{bmatrix} 0 & 1 & \\ & \ddots & \ddots \\ & & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & \\ & \ddots & \ddots \\ & & 1 & 0 \end{bmatrix}^\top = N_{k-1}^\top$$

$$\implies \quad (sL_\beta - K_\beta)L_\beta^\top = sI_{|\beta|-\ell(\beta)} - N_{\beta-1}^\top$$

$$\begin{bmatrix} 1 & 0 & \\ & \ddots & \ddots \\ & & 1 & 0 \end{bmatrix}^\top\begin{bmatrix} 1 & 0 & \\ & \ddots & \ddots \\ & & 1 & 0 \end{bmatrix} = \begin{bmatrix} I_{k-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad e_k e_k^\top = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\implies \quad L_\gamma^\top L_\gamma + E_\gamma E_\gamma^\top = I_{|\gamma|}$$

$$\begin{bmatrix} 1 & 0 & \\ & \ddots & \ddots \\ & & 1 & 0 \end{bmatrix} e_k = 0, \quad \begin{bmatrix} 0 & 1 & \\ & \ddots & \ddots \\ & & 0 & 1 \end{bmatrix} e_k = e_{k-1} \implies (sL_\beta - K_\beta)E_\beta = -E_{\beta-1}$$

$$U(s)[sE - A, -B]\begin{bmatrix} T & 0 \\ F & V \end{bmatrix} P = \begin{bmatrix} s\hat{E} - \hat{A} & -\hat{B} \\ 0_{\ell(\delta), n-\mu} & 0_{\ell(\delta), m+\mu} \end{bmatrix},$$

where $\mu = \ell(\beta) - \ell(\gamma) \in \mathbb{Z}$

$$s\hat{E} - \hat{A} = \begin{bmatrix} sI_{|\alpha|} - N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & sI_{|\beta|-\ell(\beta)} - N_{\beta-1}^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_{|\gamma|} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{|\delta|-\ell(\delta)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}} - A_{\overline{c}} \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} E_\alpha & 0 & 0 \\ 0 & E_{\beta-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\square$

- the number of redundant equations is $\ell(\delta)$

- the number of free states is $\ell(\beta)$

- the number of constrained inputs is $\ell(\gamma)$

$$U(s)[sE - A, -B] \begin{bmatrix} T & 0 \\ F & V \end{bmatrix} P = \begin{bmatrix} s\hat{E} - \hat{A} & -\hat{B} \\ 0_{\ell(\delta), n-\mu} & 0_{\ell(\delta), m+\mu} \end{bmatrix},$$

where $\mu = \ell(\beta) - \ell(\gamma) \in \mathbb{Z}$

$$s\hat{E} - \hat{A} = \begin{bmatrix} sI_{|\alpha|} - N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & sI_{|\beta|-\ell(\beta)} - N_{\beta-1}^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_{|\gamma|} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{|\delta|-\ell(\delta)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}} - A_{\overline{c}} \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} E_\alpha & 0 & 0 \\ 0 & E_{\beta-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\square$

- the number of redundant equations is $\ell(\delta)$
- the number of free states is $\ell(\beta)$
- the number of constrained inputs is $\ell(\gamma)$