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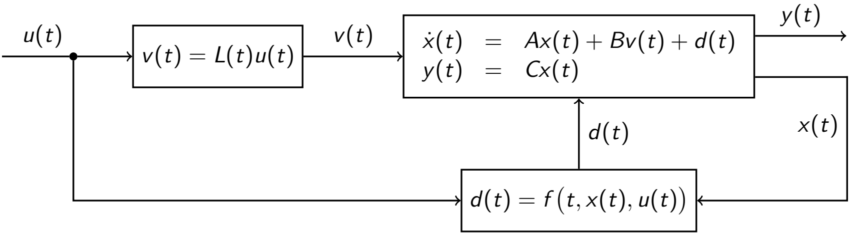
# Fault tolerant funnel control

# Linear systems with uncertainties and actuator faults

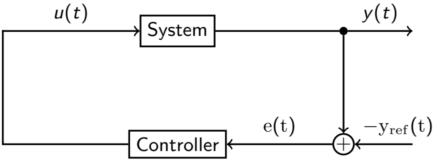


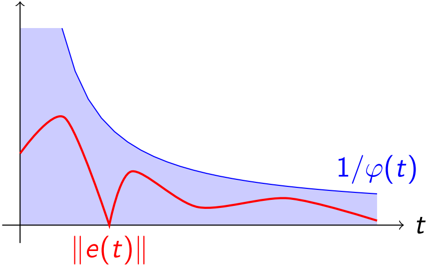

- $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $m \geq p$
- $f$ continuous and bounded *(modelling errors, uncertainties, disturbances, actuator saturation, etc.)*
- $L \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m\times m})$ s.t. $L, \dot{L}, \ldots, L^{(n)}$ bounded *(reliability of actuators)*

$$\dot{x}(t) = Ax(t) + BL(t)u(t) + f\big(t, x(t), u(t)\big), \quad y(t) = Cx(t)$$

- $\operatorname{rk} BL(t) = q \geq p$ *(e.g. q groups of actuators which perform the same control task; in each at least one (partially) functional)*
  **typical:** $q = p$
- $CA^k BL(\cdot) = 0,\ CA^k f(\cdot) = 0$ for all $k = 0, \ldots, r-2$
- $\Gamma := CA^{r-1}B \in \mathbb{R}^{p \times m}$ satisfies $\operatorname{rk} \Gamma L(t) = p$ for all $t \in \mathbb{R}$

# Control objective




$$\Phi_r = \left\{ \varphi \in \mathcal{C}^r(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot\varphi, \ldots, \varphi^{(r)} \text{ bounded}, \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau\to\infty} \varphi(\tau) > 0 \end{array} \right\}$$

## Theorem [B. '17] - ext. of [Ilchmann & Müller '07]

$\exists$ Lyapunov transf. $U$ (under mild conditions)

s.t. $$(\hat{A}, \hat{B}, \hat{C}) := \Big((UA + \dot{U})U^{-1}, UBL, CU^{-1}\Big),$$
$$\hat{f}(t, z, u) := U(t) f\big(t, U(t)^{-1} z, u\big)$$

have the form

$$\hat{A}(t) = \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_p & & & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & I_p & 0 \\ R_1(t) & R_2(t) & \cdots & R_{r-1}(t) & R_r(t) & S(t) \\ P_1(t) & P_2(t) & \cdots & P_{r-1}(t) & P_r(t) & Q(t) \end{bmatrix}, \quad \hat{B}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma L(t) \\ N(t) \end{bmatrix}, \quad \hat{f} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f_r \\ f_\eta \end{pmatrix},$$

$$\hat{C} = \begin{bmatrix} I_p & 0 & \cdots & 0 \end{bmatrix}$$

$$\hat{A}(t) = \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_p & & & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & I_p & 0 \\ R_1(t) & R_2(t) & \cdots & R_{r-1}(t) & R_r(t) & S(t) \\ P_1(t) & P_2(t) & \cdots & P_{r-1}(t) & P_r(t) & Q(t) \end{bmatrix}, \quad \hat{B}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma L(t) \\ N(t) \end{bmatrix}, \quad \hat{f} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f_r \\ f_\eta \end{pmatrix},$$

$$\hat{C} = \left[ I_p \ 0 \ \cdots \ 0 \right]$$

$$\left( \operatorname{rk} BL(t) = \right) \ q = p \quad \Longrightarrow \quad P_2 = \ldots = P_r = 0 \ \wedge \ N = 0$$

Transformation: $\big(y(t), \dot{y}(t), \ldots, y^{(r-1)}(t), \eta(t)\big) := U(t)x(t)$

$$\implies \quad y^{(r)}(t) = \sum_{i=1}^{r} R_i(t) y^{(i-1)}(t) + S(t)\eta(t) + \Gamma L(t)u(t) + f_r\big(t, y(t), \ldots, y^{(r-1)}(t), \eta(t), u(t)\big),$$

$$\dot{\eta}(t) = \sum_{i=1}^{r} P_i(t) y^{(i-1)}(t) + Q(t)\eta(t) + N(t)u(t) + f_\eta\big(t, y(t), \ldots, y^{(r-1)}(t), \eta(t), u(t)\big)$$

Transformation: $\big(y(t), \dot{y}(t), \ldots, y^{(r-1)}(t), \eta(t)\big) := U(t)x(t)$

$$\Longrightarrow \quad y^{(r)}(t) = \sum_{i=1}^{r} R_i(t) y^{(i-1)}(t) + S(t)\eta(t) + \underbrace{\Gamma L(t)}_{>0} u(t) + \cancel{f_r\big(t, y(t), \ldots, y^{(r-1)}(t), \eta(t), u(t)\big)},$$

$$\dot{\eta}(t) = \sum_{i=1}^{r} P_i(t) y^{(i-1)}(t) + \underbrace{Q(t)}_{\text{exp. st.}} \eta(t) + \cancel{N(t)u(t)} + \cancel{f_\eta\big(t, y(t), \ldots, y^{(r-1)}(t), \eta(t), u(t)\big)}$$

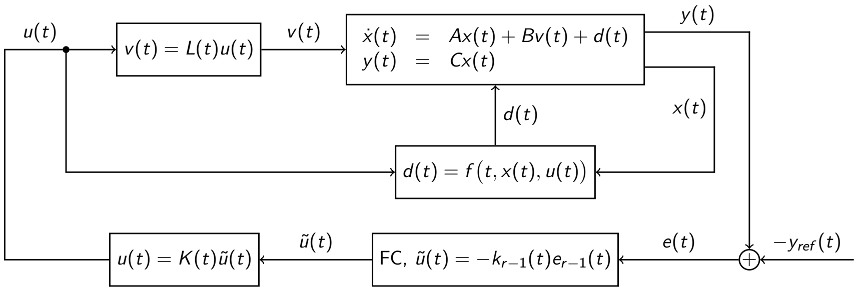
# Funnel controller for systems with known relative degree $r \in \mathbb{N}$

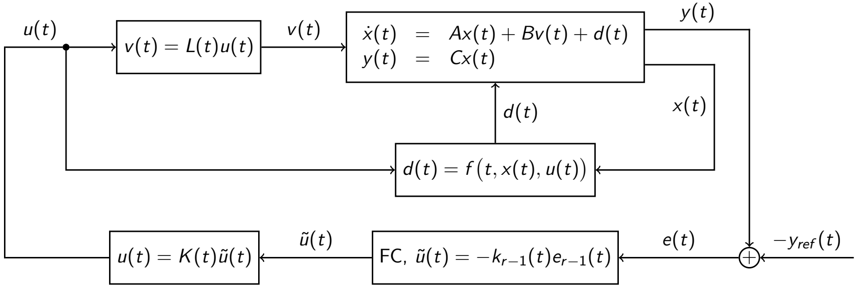
$$
\begin{aligned}
e_0(t) &= e(t) = y(t) - y_{ref}(t),\\
e_1(t) &= \dot{e}_0(t) + k_0(t)\, e_0(t),\\
e_2(t) &= \dot{e}_1(t) + k_1(t)\, e_1(t),\\
&\vdots\\
e_{r-1}(t) &= \dot{e}_{r-2}(t) + k_{r-2}(t)\, e_{r-2}(t),\\
k_i(t) &= 1/(1 - \varphi_i(t)^2 \|e_i(t)\|^2), \quad i = 0, \ldots, r-1\\
u(t) &= -k_{r-1}(t) e_{r-1}(t)
\end{aligned}
$$

**Theorem** [B., Hoang, Reis '18]

$y_{ref} \in \mathcal{W}^{r,\infty}, \ \varphi_i \in \Phi_{r-i}, \ \varphi_i(0)\|e_i(0)\| < 1 \implies u, k_i, y^{(i)}, \eta \in L^\infty$
and

$$\|e_i(t)\| \le \varphi_i(t)^{-1} - \varepsilon_i$$

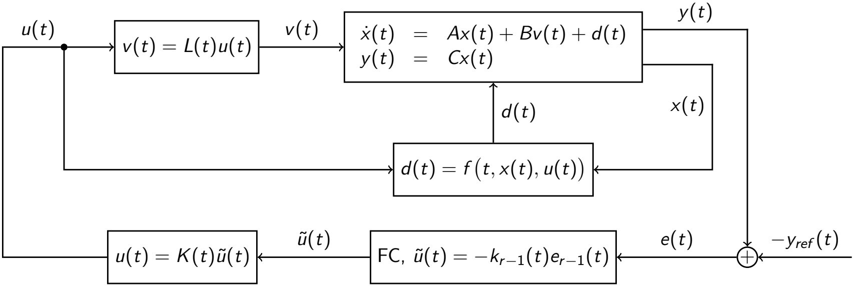

u(t)
v(t) = L(t)u(t)
v(t)
ẋ(t) = Ax(t) + Bv(t) + d(t)
y(t) = Cx(t)
y(t)
d(t)
x(t)
d(t) = f(t, x(t), u(t))
u(t) = K(t)ũ(t)
ũ(t)
FC, ũ(t) = −k_{r−1}(t)e_{r−1}(t)
e(t)
−y_ref(t)




$$\boxed{u(t) = -k_{r-1}(t)K(t)e_{r-1}(t)}$$

$$y^{(r)}(t) = \sum_{i=1}^{r} R_i(t)y^{(i-1)}(t) + S(t)\eta(t) + \Gamma L(t)K(t)\tilde{u}(t) + f_r$$

$$\dot{\eta}(t) = \sum_{i=1}^{r} P_i(t)y^{(i-1)}(t) + Q(t)\eta(t) + N(t)K(t)\tilde{u}(t) + f_\eta$$




$$u(t) = -k_{r-1}(t)K(t)e_{r-1}(t)$$

for controller weight matrix $K \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m\times p})$ with

- $\exists\, \alpha > 0:\ \Gamma L(t)K(t) + \big(\Gamma L(t)K(t)\big)^\top \geq \alpha I_p$,
- $N(t)K(t) = 0$

$$\dot{x}(t) = Ax(t) + BL(t)u(t) + f\big(t, x(t), u(t)\big), \quad y(t) = Cx(t)$$

**Theorem** [B. '17]

- $U$ is a Lyapunov transformation
- $\dot{\eta}(t) = Q(t)\eta(t)$ is uniformly exponentially stable (min. phase)
- $\exists\, \alpha > 0 :\ \Gamma L(t)K(t) + \big(\Gamma L(t)K(t)\big)^\top \geq \alpha I_p$
- $N(t)K(t) = 0$
- $y_{ref} \in \mathcal{W}^{r,\infty},\ \varphi_i \in \Phi_{r-i},\ \varphi_i(0)\|e_i(0)\| < 1$

$\Longrightarrow\ u, k_i, x \in L^\infty$ and

$$\|e_i(t)\| \leq \varphi_i(t)^{-1} - \varepsilon_i$$

## Choice of $K(t)$

$q > p$: Only possible when system parameters and $L$ are known!

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$q = p$: Recall: $N(t) = 0$;
choose $K(t) = \Gamma^\top$ and assume that

$$\Gamma\big(L(t) + L(t)^\top\big)\Gamma^\top \geq \alpha I_p$$

$\rightarrow$ there are at least $p$ lin. ind. actuators, the reliability of which does not converge to 0;

**Advantage:** only $\Gamma$ must be known

## Example: Lateral motion of a Boeing 737 aircraft (linearized); taken from [Tao et al. '04]

$$A = \begin{bmatrix} -0.13858 & 14.326 & -219.04 & 32.167 & 0 \\ -0.02073 & -2.1692 & 0.91315 & 0.000256 & 0 \\ 0.00289 & -0.16444 & -0.15768 & -0.00489 & 0 \\ 0 & 1 & 0.00618 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.15935 & 0.15935 & 0.00211 & 0.00211 \\ 0.01264 & 0.01264 & 0.21326 & 0.21326 \\ -0.12879 & -0.12879 & 0.00171 & 0.00171 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = (v_b, p_b, r_b, \phi, \psi)^\top, \quad u = (d_{r1}, d_{r2}, d_{a1}, d_{a2})^\top$$

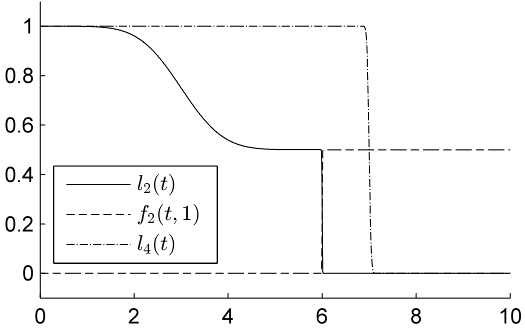
- $v_b$ - lateral velocity
- $p_b$ - roll rate
- $r_b$ - yaw rate
- $\phi$ - roll angle
- $\psi$ - yaw angle

- $d_{r1} + d_{r2}$ - rudder position
- $d_{a1} + d_{a2}$ - aileron position

$$L(t) = \operatorname{diag}\big(1, l_2(t), 1, l_4(t)\big), \quad f(t,u) = B\begin{pmatrix} 0 \\ f_2(t,u_2) \\ 0 \\ 0 \end{pmatrix}$$

$d_{r2}$: slowly decreasing efficiency to 50% on $[0,6]$; at $t = 6$ additional fault which leads to actuator saturation by 1

$d_{a2}$: sudden total fault at $t = 7$



$$y_{ref,1}(t) = 2\sin t, \quad y_{ref,2}(t) = \cos t, \quad x(0) = 0,$$

$$\varphi_0(t) = (5e^{-t} + 0.1)^{-1}, \qquad \varphi_1(t) = \left(\tfrac{5}{2}e^{-\frac{1}{2}t} + 0.1\right)^{-1},$$

$$K(t) = \Gamma^\top = \begin{bmatrix} 0.01184 & -0.12879 \\ 0.01184 & -0.12879 \\ 0.21327 & 0.00171 \\ 0.21327 & 0.00171 \end{bmatrix}$$