Institute for Mathematics, Paderborn University

FUNNEL CONTROL AND APPLICATIONS

Thomas Berger

June 29, 2022

Research group “Systems theory”, Institute for Mathematics

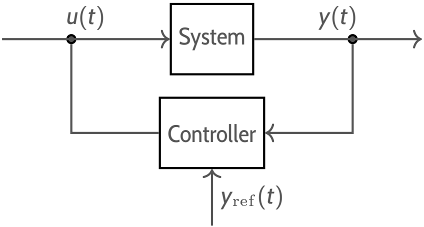
PhD students

- Lukas Lanza: Funnel control of multibody systems
- Dario Dennstädt: Funnel MPC and applications to magnetic levitation systems

Research areas

- adaptive control under input and output constraints
- nonlinear systems, multi-agent systems, differential-algebraic systems
- applications: multibody dynamics, electrical circuits, autonomous driving, defibrillation processes of the human heart, spread of epidemics

Control objective



$$
\begin{aligned}
\dot{x}(t) &= f\big(t, x(t), u(t)\big), \quad x(t) \in X \\
y(t) &= h\big(x(t)\big)
\end{aligned}
$$

- **Goal:** simple controller, so that “$y(t)$ tracks $y_{\mathrm{ref}}(t)$”
- only uses $y(t)$, no knowledge of $x(t) \in X$ or system parameters

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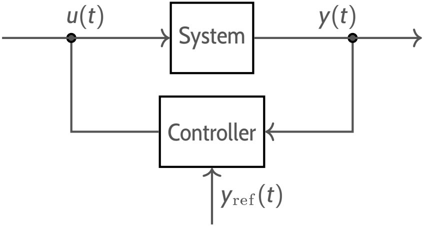

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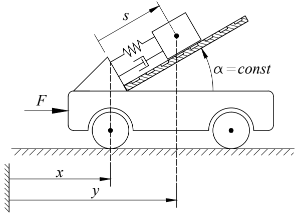
- **Goal:** simple controller, so that "$y(t)$ tracks $y_{\mathrm{ref}}(t)$"
- only uses $y(t)$, no knowledge of $x(t) \in X$ or system parameters
  $\rightarrow$ ODEs and PDEs in the same class!

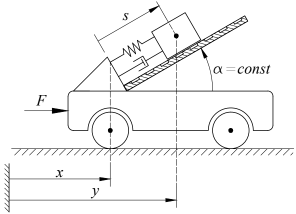


[Seifried, Blajer '13]

angle: $0^\circ \leq \alpha \leq 90^\circ$

spring, damper with nonlinear characteristics: $K(s),\ D(\dot{s})$

$$u(t) = F$$
$$y(t) = x(t) + s(t)\cos\alpha$$



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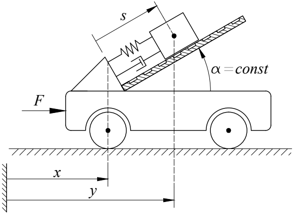
$$\begin{bmatrix} m_1 + m_2 & m_2 \cos\alpha \\ m_2 \cos\alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} u \\ -K(s) - D(\dot{s}) + m_2 g \sin\alpha \end{pmatrix}$$



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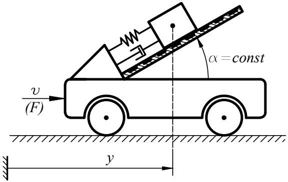
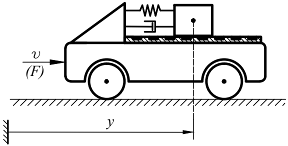
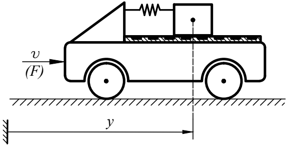
$$\dot y = \dot x + \dot s \cos\alpha$$
$$\ddot y = -c_1\big(K(s) + D(\dot s) - m_2 g \sin\alpha\big) + \frac{\sin^2\alpha}{m_1 + m_2\sin^2\alpha} u$$



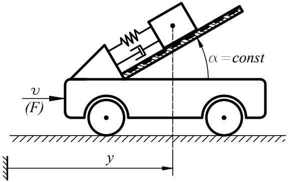


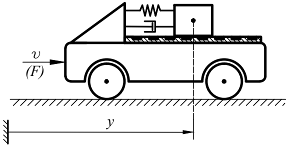


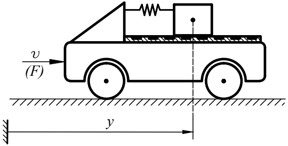
$0° < \alpha \leq 90°$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

relative degree $= 2$







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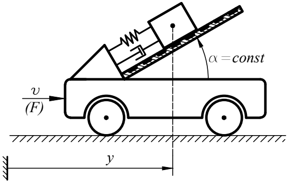
$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$

$$y^{(3)} = f_2(s, \dot{s}) + \frac{D'(\dot{s})}{m_1 m_2} u$$

relative degree $= 3$



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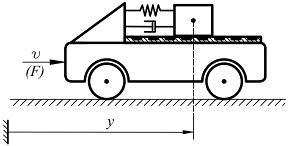

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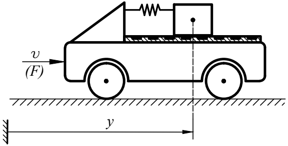
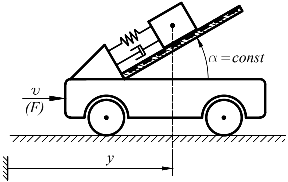


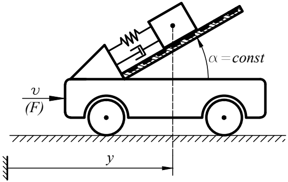
$$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$$

$$y^{(4)} = f_3(s, \dot{s}) + \frac{K'(s)}{m_1 m_2} u$$

relative degree $= 4$

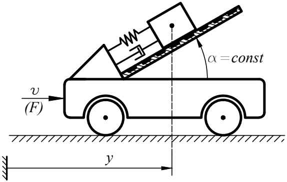


Internal dynamics: remaining dynamics when output is fixed



Internal dynamics: remaining dynamics when output is fixed

$$\ddot{\eta} = -c_3 K \left(\frac{\eta - y \cos\alpha}{\sin^2\alpha} \right) - c_3 D \left(\frac{\dot{\eta} - \dot{y}\cos\alpha}{\sin^2\alpha} \right) + c_4 g \sin\alpha$$



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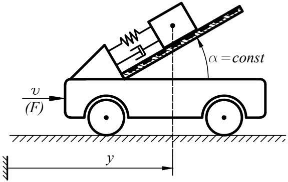
$$\alpha = 90^\circ, m_2 = 1: \qquad \ddot{s} = -K(s) - D(\dot{s}) + g$$



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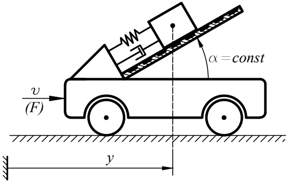
$$\alpha = 90^\circ, m_2 = 1: \qquad \ddot{s} = -K(s) - D(\dot{s}) + g$$

- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s})\,\dot{s} \geq 0$



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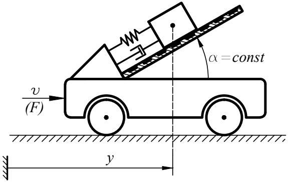
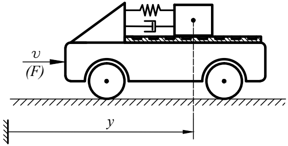
- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s})\,\dot{s} \geq 0$

$\Longrightarrow \; s, \dot{s} \in L^\infty$ **(stable internal dynamics)**



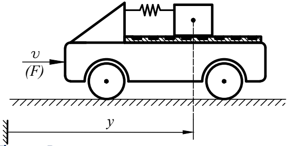
$0^\circ < \alpha \leq 90^\circ$

stable internal dynamics



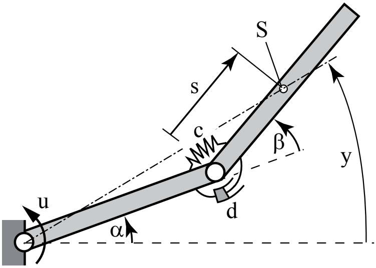
$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$

stable internal dynamics



$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$

no internal dynamics



[Seifried, Blajer '13]

Rotational Manipulator Arm

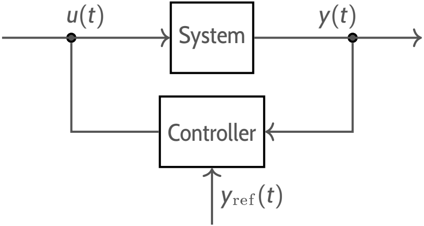
Input: angular velocity of first link

Output: position of S described by angle y

relative degree $= 1$

unstable internal dynamics

Reminder



$$\begin{aligned} \dot{x}(t) &= f\big(t, x(t), u(t)\big), \quad x(t) \in X \\ y(t) &= h\big(x(t)\big) \end{aligned}$$

- no knowledge of system parameters, only: **known relative degree** and assumption of **stable internal dynamics**
- **Goal:** design simple controller such that "$y(t)$ tracks $y_{\mathrm{ref}}(t)$"

High-gain based adaptive control

assumption: relative degree $=1$, stable internal dynamics

classical (non-adaptive) high-gain controller

$$u(t) = -ky(t), \qquad k > 0 \text{ suff. large} \implies y(t) \to 0$$

drawbacks: k unnecessary large; restricted to linear systems

High-gain based adaptive control

assumption: relative degree $= 1$, stable internal dynamics

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$$u(t) = -ky(t), \qquad k > 0 \text{ suff. large} \implies y(t) \to 0$$

drawbacks: k unnecessary large; restricted to linear systems

adaptive high-gain controller (since 1983)

$$u(t) = -k(t)y(t), \qquad \dot{k}(t) = \|y(t)\|^2$$

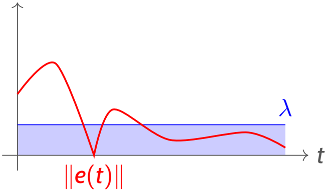
[Byrnes, Ilchmann, Logemann, Mareels, Mårtensson, Morse, Nussbaum, Owens, Prätzel-Wolters, Willems, ...]

drawbacks: $k(t)$ mon. increasing; restricted to linear systems

adaptive λ-tracker (since 1994)

$$u(t) = -k(t)\underbrace{\big(y(t) - y_{\mathrm{ref}}(t)\big)}_{=:e(t)},$$

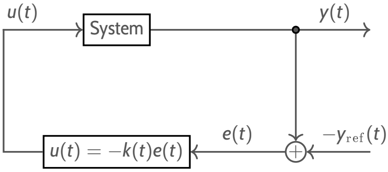
$$\dot k(t) = \begin{cases} \frac{\max\{\|e(t)\|-\lambda,0\}}{\|e(t)\|}, & e(t) \neq 0, \\ 0, & e(t) = 0 \end{cases}$$

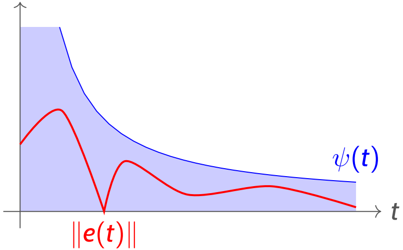


[Allgöwer, Ashman, Bullinger, Ilchmann, Logemann, Ryan, Sangwin, ...]

drawbacks: $k(t)$ mon. increasing; no transient behavior

Funnel control





$$k(t) = \frac{1}{\psi(t) - \|e(t)\|}$$

[Ilchmann, Ryan, Sangwin '02]:
Works, if

- relative degree $= 1$
- stable internal dynamics

Problem: higher relative degree

relative degree 1:

$$\dot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -ky(t) \implies \text{as. stable for } k \gg 0$$

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relative degree 2:

$$\ddot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -ky(t) \implies \text{not as. stable}$$

Problem: higher relative degree

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relative degree 2:

$$\ddot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -ky(t) \implies \text{not as. stable}$$

$$\ddot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -k_1 y(t) - k_2 \dot{y}(t) \implies \text{as. stable for } k_1, k_2 \gg 0$$

Funnel control for systems with higher relative degree

Funnel control via backstepping: [Ilchmann, Ryan, Townsend ’06 & ’07]

drawbacks: escalating controller complexity for relative degree ≥ 2, hence a typically bad controller performance

relative degree $= 2$: [Hackl, Hopfe, Ilchmann, Müller, Trenn ’13]

drawbacks: no generalization to arbitrary relative degree

Bang-bang funnel controller: [Liberzon & Trenn ’13]

drawbacks: restricted to SISO systems, strong compatibility assumptions

“Prescribed-Performance Control”: [Bechlioulis & Rovithakis ’14]

drawbacks: restricted to systems with trivial internal dynamics

Funnel control for systems with arbitrary relative degree

[B., Lê, Reis '18] & [B., Ilchmann, Ryan '21]

$$\begin{aligned} e_1(t) &= e(t), \qquad e(t) = y(t) - y_{\mathrm{ref}}(t), \\ e_2(t) &= \dot e(t) + k_1(t)e_1(t), \\ e_3(t) &= \ddot e(t) + k_2(t)e_2(t), \\ &\vdots \\ e_r(t) &= e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t), \end{aligned}$$

$$u(t) = -k_r(t)e_r(t)$$

$$k_i(t) = 1/(\psi_i(t) - \|e_i(t)\|), \quad i = 1, \ldots, r$$

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System class

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t), u(t)\big)$$

$d \in L^\infty$, $T : C \to L^\infty_{\mathrm{loc}}$ causal, locally Lipschitz, BIBO, $f \in C$ with HG property

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Linear prototype:

$\dot x(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ with

(A1) $\mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} = n + m$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$;

(A2) $CB = CAB = \ldots = CA^{r-2}B = 0$ and $CA^{r-1}B > 0$

System class

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t), u(t)\big)$$

$d \in L^\infty$, $T : C \to L^\infty_{\mathrm{loc}}$ causal, locally Lipschitz, BIBO, $f \in C$ with HG property

Linear prototype:

Is equivalent to $\dot{z}(t) = \hat{A}z(t) + \hat{B}u(t), y(t) = \hat{C}z(t)$ with

$$\hat{A} = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_r & S \\ P & 0 & \cdots & 0 & Q \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ CA^{r-1}B \\ 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} I_m & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \sigma(Q) \subseteq \mathbb{C}_-$$

$$\Longrightarrow \quad y^{(r)}(t) = R_1 y(t) + \ldots R_r y^{(r-1)}(t) + Se^{Qt}\eta(0) + \int_0^t Se^{Q(t-\tau)}Py(\tau)d\tau + CA^{r-1}Bu(t)$$

System class

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t), u(t)\big)$$

$d \in L^\infty$, $T : C \to L^\infty_{\mathrm{loc}}$ causal, locally Lipschitz, BIBO, $f \in C$ with HG property

Linear prototype:

$$\implies \quad y^{(r)}(t) = T(y, \dot{y}, \ldots, y^{(r-1)})(t) + \Gamma u(t)$$

with $\Gamma = CA^{r-1}B$ and

$$T(y, \ldots, y^{(r-1)})(t)$$
$$= R_1 y(t) + \ldots R_r y^{(r-1)}(t) + Se^{Qt}\eta(0) + \int_0^t Se^{Q(t-\tau)}Py(\tau)d\tau$$

Funnel control for systems with arbitrary relative degree $r \in \mathbb{N}$

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t), u(t)\big)$$

$$\begin{aligned}
e_1(t) &= e(t), \qquad e(t) = y(t) - y_{\mathrm{ref}}(t),\\
e_2(t) &= \dot{e}(t) + k_1(t)e_1(t),\\
&\vdots\\
e_r(t) &= e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t),
\end{aligned}$$

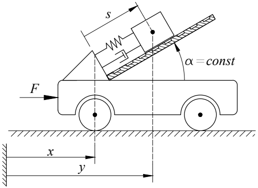
$$u(t) = -k_r(t)e_r(t)$$

$$k_i(t) = 1/(\psi_i(t) - \|e_i(t)\|), \quad i = 1, \ldots, r$$

Theorem [B., Lê, Reis '18] & [B., Ilchmann, Ryan '21]

$$y_{\mathrm{ref}} \in W^{r,\infty} \implies u, k_i, y^{(i)} \in L^\infty \text{ and } \|e_i(t)\| \le \psi_i(t) - \varepsilon_i$$

Control of multibody systems – jointly with R. Seifried (TU Hamburg, Germany)



[B., Lê, Reis '18]: $u(t) = u_{\mathrm{FC}}(t)$

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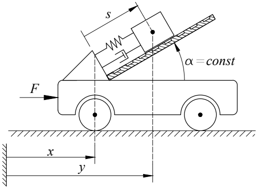

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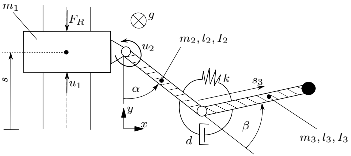

[B., Otto, Reis, Seifried '19]:
$u(t) = u_{\mathrm{FC}}(t) + u_{\mathrm{FF}}(t)$

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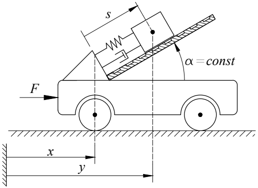
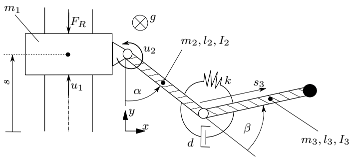
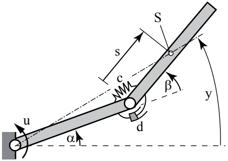


[B., Lê, Reis '18]: $u(t) = u_{\mathrm{FC}}(t)$

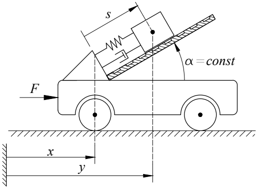


[B., Otto, Reis, Seifried '19]:
$u(t) = u_{\mathrm{FC}}(t) + u_{\mathrm{FF}}(t)$



[B., Lanza '20]: $u(t) = u_{\mathrm{FC}}(t)$
unstable internal dynamics

Control of multibody systems – jointly with R. Seifried (TU Hamburg, Germany)



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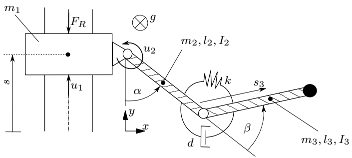

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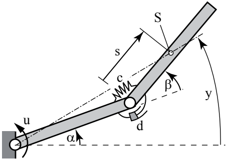
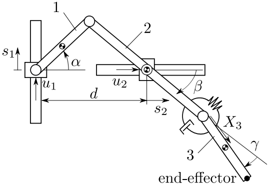
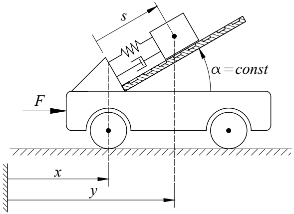


[B., Lanza '20]: $u(t) = u_{\rm FC}(t)$
unstable internal dynamics

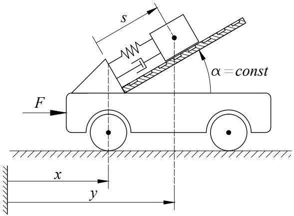


[B., Drücker, Lanza, Reis, Seifried '21]
$u(t) = u_{\rm FC}(t) + u_{\rm FF}(t)$
unstable internal dynamics, DAE formulation

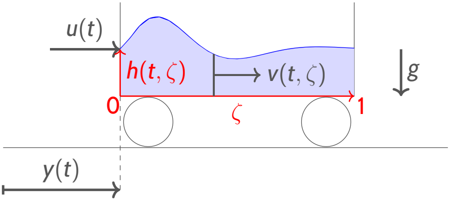


$\rightarrow$

$$\ddot{y}(t) = T(y, \dot{y})(t) + \gamma\, u(t)$$



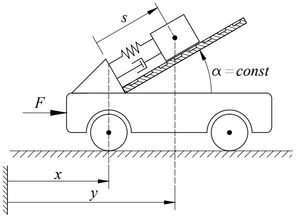
→



$$\ddot{y}(t) = T(y,\dot{y})(t) + \gamma\, u(t)$$

$$\ddot{y}(t) = \hat{T}(y,\dot{y})(t) + \hat{\gamma}\, u(t)$$

[B., Puche, Schwenninger '22]



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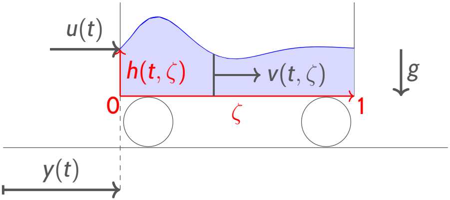

$$\ddot{y}(t) = T(y,\dot{y})(t) + \gamma\, u(t) \qquad \ddot{y}(t) = \hat{T}(y,\dot{y})(t) + \hat{\gamma}\, u(t)$$

[B., Puche, Schwenninger '22]

Finite and infinite dimensional systems in the same class!

Funnel control for ∞-dimensional systems

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t), u(t)\big)$$

allows for a "simple" class of ∞-dimensional systems
$\rightarrow$ internal dynamics described by PDE

Funnel control for ∞-dimensional systems

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allows for a "simple" class of ∞-dimensional systems
$\rightarrow$ internal dynamics described by PDE

$r = 1$: [Ilchmann, Ryan, Sangwin '02, etc.]

$$(Ty)(t) = A_1 y(t) + A_2 \int_0^t \mathcal{T}(t-s) A_3 y(s) \, \mathrm{d}s$$

- $(\mathcal{T}(t))_{t \geq 0}$ exp. stable C^0-semigroup on real Hilbert space X with generator $A_4 : \mathcal{D}(A_4) \subseteq X \to X$ *(finite dimensional: $\mathcal{T}(t) = e^{A_4 t}$)*
- (A_4, A_3, A_2) "regular well-posed", $A_1 \in \mathbb{R}^{m \times m}$

Funnel control for ∞-dimensional systems

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t), u(t)\big)$$

$r \in \mathbb{N}$: [Ilchmann, Selig, Trunk '16]
Byrnes-Isidori form for linear ∞-dimensional systems

$$\begin{aligned} \dot{\eta}(t) &= A_4\eta(t) + A_3 y(t), \\ y^{(r)}(t) &= R_1 y(t) + \ldots R_r y^{(r-1)}(t) + A_2\eta(t) + \gamma u(t) \end{aligned}$$

Funnel control for ∞-dimensional systems

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t), u(t)\big)$$

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Byrnes-Isidori form for linear ∞-dimensional systems

$$\begin{aligned}
\dot{\eta}(t) &= A_4 \eta(t) + A_3 y(t), \\
y^{(r)}(t) &= R_1 y(t) + \ldots R_r y^{(r-1)}(t) + A_2 \eta(t) + \gamma u(t) \\
&= T(y, \ldots, y^{(r-1)})(t) + \gamma u(t)
\end{aligned}$$

with $T(y, \ldots, y^{(r-1)})(t)$

$$= R_1 y(t) + \ldots R_r y^{(r-1)}(t) + A_2 \int_0^t \mathcal{T}(t-s) A_3 y(s)\, \mathrm{d}s$$

Funnel control for ∞-dimensional systems

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t), u(t)\big)$$

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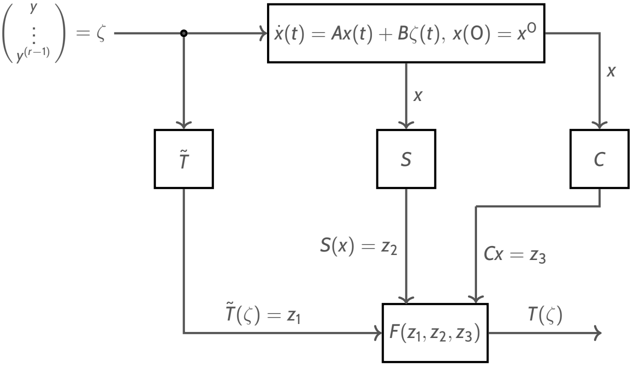
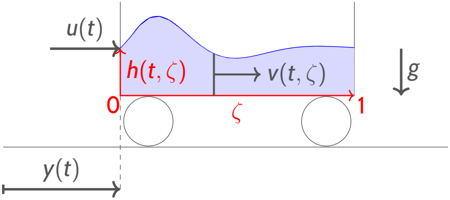


- (A, B, C) regular well-posed
- $f \mapsto \mathcal{L}^{-1}(G) * f$ bounded
- S nonlinear, $\mathcal{D}(S) = X$, (A, B, S) BIBO stable

Rolling water tank – [B., Puche, Schwenninger '22]

$$
\begin{aligned}
\partial_t h + \partial_\zeta (hv) &= 0,\\
\partial_t v + \partial_\zeta \left(\frac{v^2}{2} + gh \right) &\\
+ hS\left(\frac{v}{h}\right) &= -\ddot{y}
\end{aligned}
$$



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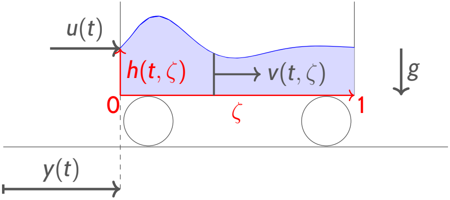

Linearized model:

$$\partial_t z_1 = -h_0 \partial_\zeta z_2, \qquad \partial_t z_2 = -g\partial_\zeta z_1 - \mu z_2 - \ddot{y}, \qquad z_2(t,0) = z_2(t,1) = 0$$

$$\ddot{y}(t) = \frac{g}{2m_T}\left(z_1(t,1) - z_1(t,0)\right)\left(2h_0 + z_1(t,1) + z_1(t,0)\right) + \frac{\mu h_0}{m_T}\int_0^1 z_2(t,\zeta)\,\mathrm{d}\zeta + \frac{\mu}{m_T}\int_0^1 z_1(t,\zeta) z_2(t,\zeta)\,\mathrm{d}\zeta + \frac{u(t)}{m_T}$$

Rolling water tank – [B., Puche, Schwenninger '22]

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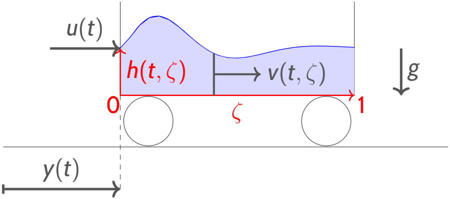


Linearized model:

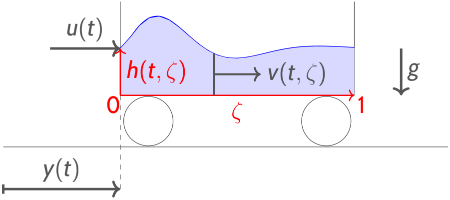
$$\dot{z}(t) = Az(t) + Ab\dot{y}(t), \quad Az = -\begin{pmatrix} h_0 \partial_\zeta z_2 \\ g\partial_\zeta z_1 + \mu z_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$z_1(t,1) - z_1(t,0) = Cz(t) = C\mathcal{T}(t)z(0) + C\int_0^t \mathcal{T}(t-s)Ab\dot{y}(s)\,\mathrm{d}s$$
$$= c(t) + \left((\mathfrak{h}_{L^1} + \mathfrak{h}_\delta) * \dot{y}\right)(t)$$

Rolling water tank – [B., Puche, Schwenninger '22]

$$\partial_t h + \partial_\zeta (hv) = 0,$$

$$\partial_t v + \partial_\zeta \left(\frac{v^2}{2} + gh \right) + hS\left(\frac{v}{h}\right) = -\ddot{y}$$



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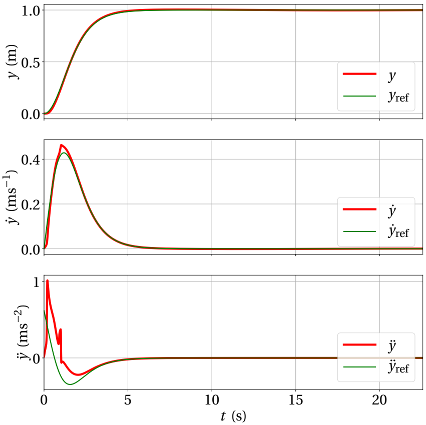
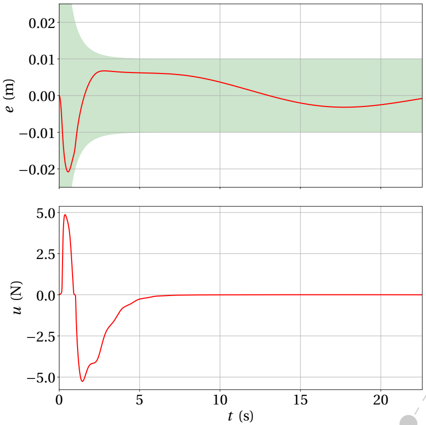
$$\ddot{y}(t) = T(\dot{y})(t) + \frac{u(t)}{m_T}$$

Simulation


1.0
0.5
0.0
y (m)
y
y_{ref}
0.4
0.2
0.0
$\dot{y}$ (ms^{-1})
$\dot{y}$
$\dot{y}_{\mathrm{ref}}$
1
0
$\ddot{y}$ (ms^{-2})
$\ddot{y}$
$\ddot{y}_{\mathrm{ref}}$
0
5
10
15
20
t (s)


0.02
0.01
0.00
−0.01
−0.02
e (m)
5.0
2.5
0.0
−2.5
−5.0
u (N)
0
5
10
15
20
t (s)

Funnel control for ∞-dimensional systems

"Hard" ∞-dimensional systems – there is NO concept of relative degree

- boundary controlled heat equation [Reis, Selig '15]

$$\partial_t x(t) = \Delta x(t), \qquad u(t) = \left(\nu^\top \cdot \nabla x(t)\right)|_{\partial\Omega},$$
$$y(t) = \int_{\partial\Omega} x(t)(\zeta)\,\mathrm{d}\zeta$$

- general class of boundary control problems based on m-dissipative operators [Puche, Reis, Schwenninger '21, Puche '19]

$$\dot{x}(t) = \mathfrak{A}x(t), \qquad x(0) = x_0 \in \mathcal{D}(\mathfrak{A}) \subseteq X,$$
$$u(t) = \mathfrak{B}x(t), \qquad y(t) = \mathfrak{C}x(t)$$

  e.g. lossy transmission line, wave equation, diffusion equation
- Fokker-Planck equation [B. '21] $\rightarrow$ *video clip*

Monodomain equations [B., Breiten, Puche, Reis ’21] – (simple) model for the electric activity of the human heart to describe defibrillation processes

$$\begin{aligned} \partial_t v(t) &= \nabla \cdot \big(D \nabla v(t)\big) + p_3(v)(t) - w(t) + I_{s,i}(t) + B I_{s,e}(t), \\ \partial_t w(t) &= c v(t) - d w(t), \quad y(t) = B' v(t) \end{aligned}$$

Monodomain equations [B., Breiten, Puche, Reis '21] – (simple) model for the electric activity of the human heart to describe defibrillation processes

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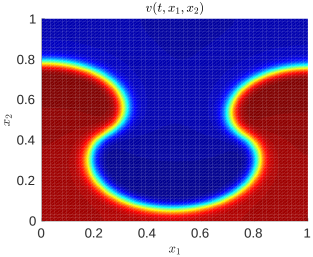
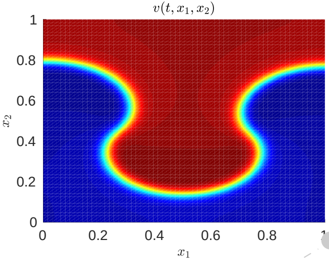
$$\partial_t w(t) = c v(t) - d w(t), \quad y(t) = B' v(t)$$

Control objective: "reentry waves", which can be interpreted as fibrillation processes, should be terminated





Recent research: Funnel MPC [B., Kästner, Worthmann '20]

OCP:

$$\underset{u\in L^\infty([\hat{t},\hat{t}+T],\mathbb{R}^m)}{\text{minimize}} \int_{\hat{t}}^{\hat{t}+T} \ell(t,x(t),u(t))\,\mathrm{d}t$$

subject to

$$\dot{x}(t) = f\big(t,x(t),u(t)\big),$$

$$x(\hat{t}) = \hat{x}, \quad \|u\|_\infty \le M$$

Recent research: Funnel MPC [B., Kästner, Worthmann '20]

OCP:

$$\underset{u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)}{\text{minimize}} \int_{\hat{t}}^{\hat{t}+T} \ell(t, x(t), u(t))\, \mathrm{d}t$$

$$\text{subject to} \quad \dot{x}(t) = f\big(t, x(t), u(t)\big),$$

$$x(\hat{t}) = \hat{x}, \quad \|u\|_\infty \leq M$$

Idea:

$$\ell(t, x, u) = \|h(x) - y_{\mathrm{ref}}(t)\|^2 + \lambda \|u\|^2$$

$$\longrightarrow \quad \ell(t, x, u) = \frac{1}{\psi(t) - \|h(x) - y_{\mathrm{ref}}(t)\|} + \lambda \|u\|^2$$

Recent research: Funnel MPC

Funnel MPC Algorithm

Choose time shift $\delta > 0$, prediction horizon $T \geq \delta$, initialize $\hat{t} := t_0$

1. Obtain a measurement of the state at $\hat{t}$, set $\hat{x} = x(\hat{t})$
2. Compute a solution $u^* \in L^\infty([\hat{t}, \hat{t} + T], \mathbb{R}^m)$ of the OCP
3. Apply the feedback law $\mu(t, \hat{x}) = u^*(t), t \in [\hat{t}, \hat{t} + \delta)$
4. Set $\hat{t} := \hat{t} + \delta$ and go to 1.

Recent research: Funnel MPC

Theorem [B., Dennstädt, Ilchmann, Worthmann ’22]

relative degree $= 1$, stable internal dynamics $\implies \exists\, M > 0$: Funnel MPC is initially and recursively feasible such that $\|u(t)\| \leq M$ and $\|e(t)\| \leq \psi(t)$

Recent research: Funnel MPC

Theorem [B., Dennstädt, Ilchmann, Worthmann '22]

relative degree $= 1$, stable internal dynamics $\implies \exists\, M > 0$: Funnel MPC is initially and recursively feasible such that $\|u(t)\| \leq M$ and $\|e(t)\| \leq \psi(t)$

1. no terminal costs or conditions required
2. no additional state constraints in the OCP
3. independent of the length of the prediction horizon $T > 0$

Recent research: Funnel MPC

Future goals:

- extension to arbitrary relative degree (first results in [B., Dennstädt ’22], however requiring feasibility constraints in the OCP)
- extend control scheme so that robustness is inherited from funnel control
- combine with machine learning for continuous model adaptation and improvement of controller performance