Funnel control for linear DAEs

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y(t) & = & Cx(t) \\
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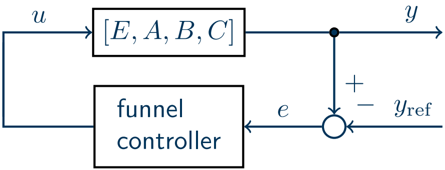

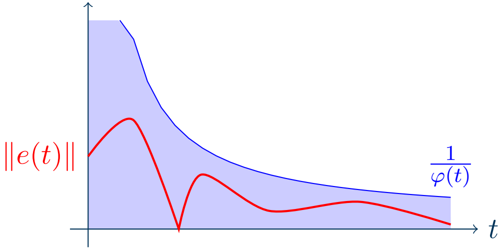

Zero dynamics:

$$\mathcal{ZD} := \left\{ (x,u,y) \;\middle|\; \begin{array}{rcl} E\dot{x} & = & Ax + Bu \\ 0 = \quad y & = & Cx \end{array} \right\}$$

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$\forall\, w_1, w_2 \in \mathcal{Z}D \;\forall\, I \subseteq \mathbb{R}$ open interval :

$$w_1|_I = w_2|_I \implies w_1 = w_2$$

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Prop.: $\mathcal{ZD}$ autonomous $\quad\Longleftrightarrow\quad \operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} = n + m$

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Lem.: $\mathcal{ZD}$ stable $\implies$ $\mathcal{ZD}$ autonomous

Theorem ("normal form")

$[E, A, B, C] \in \Sigma_{l,n,m}$ with $\mathcal{ZD}$ autonomous

$\Longrightarrow \quad \exists\, S \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}) :$

$$SET = \begin{bmatrix} I_k & E_2 \\ 0 & E_4 \\ 0 & E_6 \end{bmatrix}, \; SAT = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \\ 0 & A_6 \end{bmatrix}, \; SB = \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix},$$

$$CT = [0, C_2]$$

Theorem ("normal form" DAE)

$[E, A, B, C] \in \Sigma_{l,n,m}$ with

- $\operatorname{rk} C = m$
- $\mathcal{ZD}$ autonomous
- $\Gamma = -\lim_{s\to\infty} s^{-1}[0, I_m]L(s)[0, I_m]^\top \in \mathbb{R}^{m\times m}$ exists, where $L(s)$ is left inverse of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ over $\mathbb{R}(s)$

$\Longrightarrow$ $[E, A, B, C]$ can be put into the form

$$\begin{aligned} \dot{x}_1 &= Q\,x_1 + A_{12}\,y - E_{13}\,\dot{x}_3 \\ \Gamma\,\dot{y} &= \tilde{A}_{22}\,y + \Psi(x_1(0), y) + u \\ x_3 &= \textstyle\sum_{k=0}^{\nu-1} N^k E_{32}\,y^{(k+1)} \\ 0 &= A_{42}\,y - E_{42}\,\dot{y} - E_{43}\,\dot{x}_3 \end{aligned}$$

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Lem.: $\mathcal{ZD}$ stable $\iff$ $\sigma(Q) \subseteq \mathbb{C}_-$

$[E, A, B, C]$ right-invertible $:\Longleftrightarrow$

$$\forall\, y \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^m)\ \exists\,(x,u) \in \mathcal{C}(\mathbb{R};\mathbb{R}^n) \times \mathcal{C}(\mathbb{R};\mathbb{R}^m) :$$
$$(x,u,y) \text{ solves } [E,A,B,C]$$

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Proposition

- $\mathcal{ZD}$ autonomous
- $\Gamma = -\lim_{s\to\infty} s^{-1}[0, I_m]L(s)[0, I_m]^\top$ exists

$[E,A,B,C]$ right-invertible $\iff$ $\operatorname{rk} C = m\ \wedge$

$$\begin{array}{rcl}
\dot{x}_1 &=& Q\,x_1 + A_{12}\,y - E_{13}\,\dot{x}_3 \\
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0 &=& \underbrace{A_{42}}_{=0}\, y - \underbrace{E_{42}}_{=0}\,\dot{y} - \sum_{k=0}^{\nu-1} \underbrace{E_{43} N^k E_{32}}_{=0}\, y^{(k+2)}
\end{array}$$

Theorem (funnel control)

$[E, A, B, C] \in \Sigma_{l,n,m}$ with

- $\mathcal{ZD}$ stable
- $[E, A, B, C]$ right-invertible
- $\Gamma = -\lim_{s\to\infty} s^{-1}[0, I_m]L(s)[0, I_m]^\top$ exists and $\Gamma = \Gamma^\top \geq 0$

Then the *funnel controller*

$$u(t) = -k(t)\, e(t), \qquad \text{where} \qquad e(t) = y(t) - y_{\text{ref}}(t)$$

$$k(t) = \frac{\hat{k}}{1 - \varphi(t)^2 \|e(t)\|^2},$$

applied to $[E, A, B, C]$, achieves that

$$x \in L^\infty,\ k \in L^\infty \quad \wedge \quad \exists\, \varepsilon > 0\ \forall\, t > 0 :\ \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon$$

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

$$E, A \in \mathbb{R}^{l\times n}, \quad B \in \mathbb{R}^{l\times m}, \quad C \in \mathbb{R}^{m\times n}$$