UNIVERSITÄT PADERBORN

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FUNNEL CONTROL FOR THE FOKKER-PLANCK EQUATION

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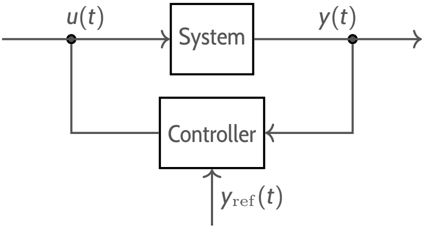
Oselot, July 2, 2020

Control objective



$$
\begin{aligned}
\dot{x}(t) &= f\big(t, x(t), u(t)\big), \quad x(t) \in X \\
y(t) &= h\big(x(t)\big)
\end{aligned}
$$

- **Goal:** simple controller, so that "$y(t)$ tracks $y_{\mathrm{ref}}(t)$"
- only uses $y(t)$, no knowledge of $x(t) \in X$ or system parameters

High-gain adaptive control

Consider $\dot{y}(t) = F\big(T(y)(t)\big) + G\,u(t)$, $G > 0$, $T : C \to L^\infty_{\rm loc}$ BIBO stable

classical (non-adaptive) high-gain controller

$$u(t) = -ky(t), \qquad k > 0 \text{ suff. large} \implies y(t) \to 0$$

Drawbacks: k may be unnecessary large; restricted to linear systems

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adaptive high-gain control (since approx. 1983)

$$u(t) = -k(t)y(t), \qquad \dot{k}(t) = \|y(t)\|^2$$

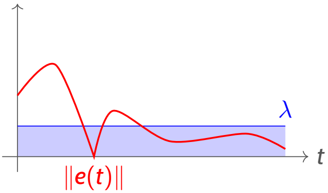
[Byrnes, Ilchmann, Logemann, Mareels, Mårtensson, Morse, Nussbaum, Owens, Prätzel-Wolters, Willems, ...]

Drawbacks: $k(t)$ mon. increasing; restricted to linear systems

adaptive λ-tracker (since approx. 1994)

$$u(t) = -k(t)\big(\underbrace{y(t) - y_{\mathrm{ref}}(t)}_{=:e(t)}\big),$$

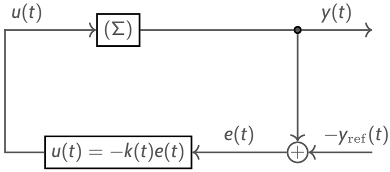
$$\dot{k}(t) = \begin{cases} \frac{\max\{\|e(t)\|-\lambda,0\}}{\|e(t)\|}, & e(t) \neq 0, \\ 0, & e(t) = 0 \end{cases}$$

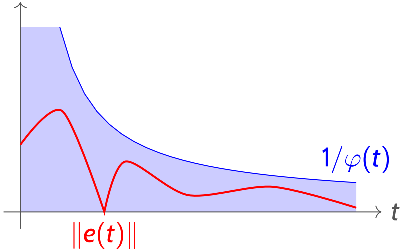


[Allgöwer, Ashman, Bullinger, Ilchmann, Logemann, Ryan, Sangwin, ...]

Drawbacks: $k(t)$ mon. increasing; transient behavior not controlled

Funnel control





$$(\Sigma):\ \dot{y}(t) = F\big(T(y)(t)\big) + G\,u(t)$$

$$k(t) = \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2}$$

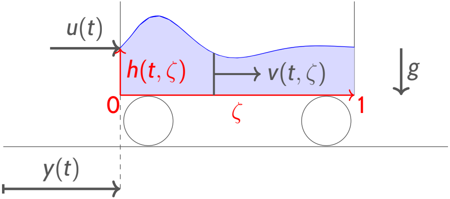
[Ilchmann, Ryan, Sangwin '02]:
Feasible, if $T : C \to L^\infty_{\text{loc}}$ is

- causal and loc. Lipschitz
- BIBO stable

Funnel control for ∞-dimensional systems

- systems which have a relative degree: [Ilchmann, Selig, Trunk '16], [B., Puche, Schwenninger '20]
- moving water tank [B., Puche, Schwenninger '19]

$$\begin{aligned} \partial_t z_1 &= -h_0 \partial_\zeta z_2, \\ \partial_t z_2 &= -g \partial_\zeta z_1 - \mu z_2 - \ddot{y}, \\ z_2(t,0) &= z_2(t,1) = 0 \end{aligned}$$



$$\ddot{y}(t) = \frac{g}{2m_T}\big(z_1(t,1) - z_1(t,0)\big)\big(2h_0 + z_1(t,1) + z_1(t,0)\big) + \frac{\mu h_0}{m_T}\int_0^1 z_2(t,\zeta)\,\mathrm{d}\zeta + \frac{\mu}{m_T}\int_0^1 z_1(t,\zeta) z_2(t,\zeta)\,\mathrm{d}\zeta + \frac{u(t)}{m_T}$$

Funnel control for ∞-dimensional systems

- boundary controlled heat equation [Reis, Selig '15]

$$\partial_t x(t) = \Delta x(t), \qquad u(t) = \big(\nu^\top \cdot \nabla x(t)\big)|_{\partial\Omega},$$
$$y(t) = \int_{\partial\Omega} \big(x(t)\big)(\zeta)\,\mathrm{d}\zeta$$

- general class of boundary control systems based on m-dissipative operators [Puche, Reis, Schwenninger '18, Puche '19]

$$\dot{x}(t) = \mathfrak{A}x(t), \qquad x(0) = x_0 \in \mathcal{D}(\mathfrak{A}) \subseteq X,$$
$$u(t) = \mathfrak{B}x(t), \qquad y(t) = \mathfrak{C}x(t)$$

e.g. lossy transmission line, wave equations, diffusion equation

Funnel control for ∞-dimensional systems

- monodomain equations [B., Breiten, Puche, Reis '19] – model for the electric activity of the human heart to describe defibrillation processes

$$\begin{aligned}\partial_t v(t) &= \nabla \cdot \big(D \nabla v(t)\big) + p_3(v)(t) - w(t) + I_{s,i}(t) + B I_{s,e}(t),\\ \partial_t w(t) &= c_5 v(t) - c_4 w(t),\\ y(t) &= B' v(t)\end{aligned}$$

where $p_3(v) = -c_1 v + c_2 v^2 - c_3 v^3$,

allows for distributed control/observation with $B \in \mathcal{L}(\mathbb{R}, L^2(\Omega))$ and boundary control/observation with $B \in \mathcal{L}(\mathbb{R}, W^{1,2}(\Omega)')$

Itô stochastic differential equation

$$\mathrm{d}X_t = b(t, X_t, u(t))\mathrm{d}t + \sigma(t, X_t, u(t))\mathrm{d}W_t, \quad X(t=0) = X_0,$$

- $X_t : \Omega \to \mathbb{R}^n$ – random vectors
- Ω – sample space of a probability space $(\Omega, \mathcal{F}, P)$
- $(W_t)_{t\geq 0}$ – d-dimensional Wiener process with zero mean value and unit variance
- $b : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ – drift function
- $\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times d}$ – covariance matrix
- $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ – the control input

Fokker-Planck equation

$\rightarrow$ models evolution of probability density p associated with process $(X_t)_{t\geq 0}$

$$\frac{\partial p}{\partial t}(t,x) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(b_i(t,x,u(t))p(t,x)\big) + \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\big(C_{ij}(t,x,u(t))p(t,x)\big), \qquad \text{in } (0,\infty)\times\mathbb{R}^n,$$

$$p(0,x) = p_0(x), \qquad \text{in } \mathbb{R}^n,$$

where $\quad C(t,x,u) = \frac{1}{2}\sigma(t,x,u)\sigma(t,x,u)^\top,$

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where $C(t,x,u) = \frac{1}{2}\sigma(t,x,u)\sigma(t,x,u)^\top$,

Solution properties: $p(t,x) \geq 0 \quad \wedge \quad \int_{\mathbb{R}^n} p(t,x)\mathrm{d}x = 1$

The Ornstein-Uhlenbeck process

$\rightarrow$ describes the motion of a massive Brownian particle under the influence of friction; applications in neurobiology and finance

$$b(t,x,u) = u - \gamma x, \quad \sigma(t,x,u) = \sigma > 0, \quad \gamma > 0$$

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$$\begin{aligned} \frac{\partial p}{\partial t}(t,x) &= c\frac{\partial^2 p}{\partial x^2}(t,x) + \gamma \frac{\partial}{\partial x}\big(x\,p(t,x)\big) - u(t)\frac{\partial p}{\partial x}(t,x), && \text{in } (0,\infty)\times\mathbb{R}, \\ p(0,x) &= p_0(x), && \text{in } \mathbb{R} \end{aligned}$$

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$$y(t) = E[X_t] = \int_{-\infty}^{\infty} x\, p(t,x) \mathrm{d}x$$

- $y(t)$ is assumed to be available for control purposes for $t \geq 0$
- approximate by data-driven methods, e.g. Monte Carlo integration

Comparison with the literature

recent results available by [Breiten, Kunisch, Pfeiffer '18], [Hosfeld, Jacob, Schwenninger '20] consider

$$\begin{aligned} \frac{\partial p}{\partial t}(t,x) &= c\Delta p(t,x) + \nabla \cdot \big(p(t,x)\nabla V(t,x)\big), && \text{in } (0,\infty)\times\Omega, \\ 0 &= \nu^\top \cdot \big(c\nabla p(t,x) + p(t,x)\nabla V(t,x)\big), && \text{in } (0,\infty)\times\partial\Omega, \\ p(0,x) &= p_0(x), && \text{in } \Omega \end{aligned}$$

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$$\begin{aligned} \Omega \subset \mathbb{R}^n \text{ bounded domain} \quad &\longleftrightarrow \quad \Omega = \mathbb{R}, \\ V(t,x) = W(x) + M(u(t))(x), \quad &\longleftrightarrow \quad W(x) = \frac{\gamma x^2}{2}, \\ W \in W^{2,\infty}(\Omega) \quad & \end{aligned}$$

The Fokker-Planck operator

$$\phi(x) := \frac{\gamma x^2}{2c}, \quad H := L^2(\mathbb{R}; e^{-\phi}) \quad \text{and} \quad V := W^{1,2}(\mathbb{R}; e^{-\phi})$$

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$\to$ use form methods, cf. [Arendt et al. '15, Lecture Notes 18th ISEM]

$$a : V \times V \to \mathbb{R}, \ (v_1, v_2) \mapsto \langle v_1', v_2' \rangle_H$$

Proposition

$\exists!\, A : \mathcal{D}(A) \subset V \to H$ with

$$\mathcal{D}(A) = \{\, v \in V \mid \exists\, u \in H \, \forall\, z \in V : \ a(v,z) = \langle u, z \rangle_H \,\},$$

$$\forall\, v \in \mathcal{D}(A) \, \forall\, z \in V : \ a(v,z) = \langle Av, z \rangle_H.$$

A is self-adjoint, positive, has compact resolvent

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Crucial ingredient: injection $j : V \to H$ is compact [JOHNSON '00]

The Fokker-Planck operator

Recall the Hermite polynomials defined by

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2} \right), \quad x \in \mathbb{R},\ n \in \mathbb{N}_0$$

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- $\sigma(A) = \{ \lambda_j \mid j \in \mathbb{N}_0 \}, \quad Av_j = \lambda_j v_j,$
  $\lambda_j = 2j\theta^2, \quad v_j(x) = \alpha_j H_j(\theta x), \quad \alpha_j := \sqrt{\frac{\theta}{\sqrt{\pi} 2^j j!}}, \quad \theta := \sqrt{\frac{\gamma}{2c}}$
- $(v_j)_{j \in \mathbb{N}_0}$ is complete orthonormal system in H, orthogonal system in V
- $v_j'(x) = \sqrt{\lambda_j} v_{j-1}(x),$
- $\lim_{x \to \pm\infty} e^{-\phi(x)} v_j(x) v(x) = 0$ for all $v \in V$
  (proof via Barbălat's Lemma)

The Fokker-Planck operator

$$\mathfrak{H} := \left\{ e^{-\phi} f \;\middle|\; f \in H \right\} = L^2(\mathbb{R}; e^{\phi}), \quad \mathfrak{V} := \left\{ e^{-\phi} f \;\middle|\; f \in V \right\},$$

$$T : H \to \mathfrak{H},\ f \mapsto e^{-\phi} f,$$

$$\langle z_1, z_2 \rangle_{\mathfrak{H}} := \langle T^{-1}(z_1), T^{-1}(z_2) \rangle_H = \langle e^{\phi} z_1, e^{\phi} z_2 \rangle_H,$$

$$\langle z_1, z_2 \rangle_{\mathfrak{V}} := \langle T^{-1}(z_1), T^{-1}(z_2) \rangle_V = \langle e^{\phi} z_1, e^{\phi} z_2 \rangle_H + \langle (e^{\phi} z_1)', (e^{\phi} z_2)' \rangle_H,$$

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$$\mathfrak{a} : \mathfrak{V} \times \mathfrak{V} \to \mathbb{R},\ (z_1, z_2) \mapsto a\left(T^{-1}(z_1), T^{-1}(z_2) \right) = \langle (e^{\phi} z_1)', (e^{\phi} z_2)' \rangle_H,$$

$$\mathfrak{A} := T \circ A \circ T^{-1} : \mathcal{D}(\mathfrak{A}) := T(\mathcal{D}(A)) \subset \mathfrak{V} \to \mathfrak{H}$$

The Fokker-Planck operator

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$$y = \mathfrak{A} v \quad \Longleftrightarrow \quad \forall z \in V :\ a\big(T^{-1}(v), z\big) = \langle T^{-1}(y), z \rangle_H$$

$$\overset{w = T(z)}{\Longleftrightarrow} \quad \forall w \in \mathfrak{V} :\ \mathfrak{a}\big(v, w\big) = \langle y, w \rangle_{\mathfrak{H}}$$

The Fokker-Planck operator

The operator $\mathfrak{A}$ satisfies

- $\sigma(\mathfrak{A}) = \sigma(A)$,
- z is an eigenfunction of $\mathfrak{A}$ if, and only if, $e^{\phi} z$ is an eigenfunction of A,
- $z_j := e^{-\phi} v_j$ defines a complete orthonormal system of eigenfunctions in $\mathfrak{H}$, orthogonal system in $\mathfrak{V}$,
- $z_j'(x) = -\sqrt{\lambda_{j+1}} z_{j+1}(x)$,
- $\lim_{x \to \pm\infty} e^{\phi(x)} z_j(x) z(x) = 0$ for all $z \in \mathfrak{V}$

The Fokker-Planck operator

Recall:

- $z \in \mathfrak{V} \iff e^{\phi} z, \left(e^{\phi} z\right)' \in H$
- $z \in W^{1,2}(\mathbb{R}; e^{\phi}) \iff e^{\phi} z, e^{\phi} z' \in H$
- $\|v\|_{\mathfrak{V}} = \sqrt{\|e^{\phi} v\|_H + \|(e^{\phi} v)'\|_H} \neq \|v\|_{W^{1,2}(\mathbb{R}; e^{\phi})}$

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Proposition

- $\mathfrak{V} = W^{1,2}(\mathbb{R}; e^{\phi})$
- $\|v\|_{W^{1,2}(\mathbb{R};e^{\phi})} \leq C\|v\|_{\mathfrak{V}}$

The Fokker-Planck operator

Now: $-c\mathfrak{A} :=$ *Fokker-Planck operator*

Motivation: for $v \in \mathfrak{V}$ such that $\left(e^{\phi}v\right)' \in V$:

$$-c\mathfrak{A}v = c\left(e^{-\phi}\left(e^{\phi}v\right)'\right)' = cv'' + c(\phi' v)', \quad c\phi'(x) = \gamma x$$

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$$\begin{aligned} \dot p(t,x) &= -c\mathfrak{A}p(t,x) - \mathfrak{B}\left(p(t,\cdot), u(t)\right)(x), && \text{in } (0,\infty)\times\mathbb{R}, \\ p(0,x) &= p_0(x), && \text{in } \mathbb{R}, \end{aligned}$$

$$\mathfrak{B} : \mathfrak{V}\times\mathbb{R} \to \mathfrak{H},\ (v,u) \mapsto u\cdot v'$$

Weak solutions

$$\begin{aligned} \dot p(t,x) &= -c\mathfrak{A}p(t,x) - \mathfrak{B}\big(p(t,\cdot),u(t)\big)(x), && \text{in } (0,\infty)\times\mathbb{R},\\ p(0,x) &= p_0(x), && \text{in } \mathbb{R} \end{aligned}$$

for $u \in C([0,T])$, p is called *(weak) solution* on $[0,T]$, if

- $p \in L^2(0,T;\mathfrak{V}) \cap C([0,T];\mathfrak{H})$, $\dot p \in L^2(0,T;\mathfrak{V}')$, $p(0) = p_0$,
- for all $v \in \mathfrak{V}$ and almost all $t \in [0,T]$ we have

$$\langle \dot p(t), v\rangle_{\mathfrak{H}} = -c\mathfrak{a}(p(t),v) - \langle \mathfrak{B}\big(p(t),u(t)\big), v\rangle_{\mathfrak{H}}$$

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$$\langle \dot{p}(t), v\rangle_{\mathfrak{H}} = -c\mathfrak{a}(p(t),v) - \langle \mathfrak{B}\big(p(t),u(t)\big), v\rangle_{\mathfrak{H}}$$

Proposition

- $\int_{-\infty}^{\infty} p(t,x)\mathrm{d}x = \int_{-\infty}^{\infty} p_0(x)\mathrm{d}x$
- $p_0(\cdot) \geq 0 \quad \Longrightarrow \quad p(t,\cdot) \geq 0$
- $\int_{-\infty}^{\infty} p_0(x)\mathrm{d}x = 1$

  $\Longrightarrow \quad \int_{-\infty}^{\infty} \big(x - y(t)\big)^2 p(t,x)\mathrm{d}x = \frac{c}{\gamma}\left(1 + Ke^{-2\gamma t}\right)$

Weak solutions

Sketch of proof: $\mu_i(t) := \langle p(t), z_i\rangle_{\mathfrak{H}}$, $\mu_{-1}(t) := 0$ then

$$\dot{\mu}_i(t) = -c\mathfrak{a}(p(t), z_i) - \langle \mathfrak{B}\left(p(t), u(t)\right), z_i\rangle_{\mathfrak{H}}$$
$$= -c\lambda_i\mu_i(t) + \sqrt{\lambda_i}u(t)\mu_{i-1}(t)$$

Weak solutions

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$$\begin{aligned}\dot{\mu}_i(t) &= -c\mathfrak{a}(p(t), z_i) - \langle \mathfrak{B}\left(p(t), u(t)\right), z_i\rangle_{\mathfrak{H}} \\ &= -c\lambda_i\mu_i(t) + \sqrt{\lambda_i}u(t)\mu_{i-1}(t)\end{aligned}$$

$$\sigma(t)^2 := \int_{-\infty}^{\infty} \big(x - y(t)\big)^2 p(t,x)\mathrm{d}x = \int_{-\infty}^{\infty} x^2 p(t,x)\mathrm{d}x - y(t)^2,$$

$$x^2 = \frac{1}{4\theta^2}\left(\frac{v_2(x)}{\alpha_2} + \frac{2v_0(x)}{\alpha_0}\right), \quad x \in \mathbb{R}$$

Weak solutions

Sketch of proof: $\mu_i(t) := \langle p(t), z_i\rangle_{\mathfrak{H}}$, $\mu_{-1}(t) := 0$ then

$$\begin{aligned}\dot{\mu}_i(t) &= -c\mathfrak{a}(p(t), z_i) - \langle \mathfrak{B}\left(p(t), u(t)\right), z_i\rangle_{\mathfrak{H}} \\ &= -c\lambda_i \mu_i(t) + \sqrt{\lambda_i} u(t) \mu_{i-1}(t)\end{aligned}$$

$$\sigma(t)^2 := \int_{-\infty}^{\infty} \left(x - y(t)\right)^2 p(t,x)\mathrm{d}x = \int_{-\infty}^{\infty} x^2 p(t,x)\mathrm{d}x - y(t)^2,$$

$$x^2 = \frac{1}{4\theta^2}\left(\frac{v_2(x)}{\alpha_2} + \frac{2v_0(x)}{\alpha_0}\right), \quad x \in \mathbb{R}$$

$$\implies \quad \sigma(t)^2 = \frac{c}{\gamma}\left(1 + \frac{\mu_2(t)}{2\alpha_2} - \frac{\mu_1(t)^2}{2\alpha_1^2}\right) =: \frac{c}{\gamma}(1 + g(t))$$

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$$\dot{g}(t) = -2\gamma g(t) \quad \implies \quad \sigma(t)^2 = \tfrac{c}{\gamma}\left(1 + Ke^{-2\gamma t}\right) \qquad \square$$

Feedforward control

$$\dot{p}(t,x) = -c\mathfrak{A}p(t,x) - \mathfrak{B}\big(p(t,\cdot),u(t)\big)(x), \qquad \text{in } (0,\infty)\times\mathbb{R},$$
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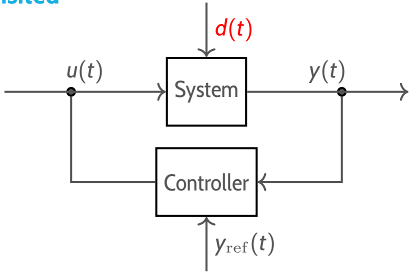
Proposition

$\exists!$ solution p on $\mathbb{R}_{\geq 0}$ such that

- $p \in L^\infty(0,\infty;\mathfrak{H})$
- $y(t) = y_{\mathrm{ref}}(t) + \big(y(0) - y_{\mathrm{ref}}(0)\big)e^{-\gamma t}$

Control objective revisited



$$\begin{aligned}\dot{x}(t) &= f\big(d(t), x(t), u(t)\big), \quad x(t) \in X\\ y(t) &= h\big(x(t)\big)\end{aligned}$$

- **Goal:** simple controller, so that "$y(t)$ tracks $y_{\mathrm{ref}}(t)$"
- only uses $y(t)$, no knowledge of $x(t) \in X$ or system parameters or disturbance $d(t)$

Funnel control for the Fokker-Planck equation

$$\begin{aligned} \dot p(t,x) &= -c\mathfrak{A}p(t,x) - \mathfrak{B}\big(p(t,\cdot),u(t)\big)(x) + d(t,x), && \text{in } (0,\infty)\times\mathbb{R},\\ p(0,x) &= p_0(x), && \text{in } \mathbb{R}, \end{aligned}$$

$$u(t) = -\frac{e(t)}{1-\varphi(t)^2 e(t)^2}, \quad e(t) = y(t) - y_{\mathrm{ref}}(t),$$

$$y(t) = \int_{-\infty}^{\infty} x\, p(t,x)\mathrm{d}x$$

where $d \in L^\infty(0,\infty;\mathfrak{H})$ such that $\int_{-\infty}^{\infty} d(t,x)\mathrm{d}x = 0$ for a.a. $t \geq 0$

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Theorem [B. '20]

$\exists!$ solution p on $\mathbb{R}_{\geq 0}$ such that

- $p \in L^\infty(0,\infty;\mathfrak{H}), u, y \in W^{1,\infty}(\mathbb{R}_{\geq 0})$
- $\exists \varepsilon > 0\ \forall t > 0 : \ |e(t)| \leq \varphi(t)^{-1} - \varepsilon$

Proof – Galerkin approximation

$\mu_i(t) := \langle p(t), z_i\rangle_{\mathfrak{H}}, d_i(t) := \langle d(t), z_i\rangle_{\mathfrak{H}}, i \in \mathbb{N}_0, \mu_{-1}(t) := 0$, then

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Observations:

- $z_0 = \alpha_0 e^{-\phi} \Rightarrow d_0(t) = \alpha_0 \int_{-\infty}^{\infty} d(t,x) \mathrm{d}x = 0$
- $\lambda_0 = 0, d_0 = 0 \Rightarrow \dot{\mu}_0(t) = 0$
- $y(t) = c_1 \mu_1(t)$ for some $c_1 > 0$

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$$\dot{y}(t) = -c\lambda_1 y(t) - \sqrt{\lambda_1} \frac{\mu_0}{c_1} \frac{e(t)}{1 - \varphi(t)^2 e(t)^2} + d_1(t), \quad e(t) = y(t) - y_{\mathrm{ref}}(t)$$

Proof – Energy estimate

$p_n(t) := \sum_{i=0}^{n} \mu_i(t) z_i \in \mathfrak{V}, \quad t \geq 0, n \in \mathbb{N},$

$$\implies \quad \sup_{t \geq 0} \|p_n(t)\|_{\mathfrak{H}} + \|p_n\|_{L^2(0,T;\mathfrak{V})} + \|\dot{p}_n\|_{L^2(0,T;\mathfrak{V}')} \leq C$$

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One important ingredient:

$\langle \mathfrak{B}\left(p_n(t), u(t)\right), p_n(t)\rangle_{\mathfrak{H}} = u(t)\langle p_n'(t), p_n(t)\rangle_{\mathfrak{H}} \leq \|u\|_\infty \|p_n'(t)\|_{\mathfrak{H}} \|p_n(t)\|_{\mathfrak{H}}$

$\leq \|p_n(t)\|_{W^{1,2}(\mathbb{R};e^{\phi})} \|p_n(t)\|_{\mathfrak{H}} \|u\|_\infty \leq C\|p_n(t)\|_{\mathfrak{V}} \|p_n(t)\|_{\mathfrak{H}} \|u\|_\infty$

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$$\leq \|p_n(t)\|_{W^{1,2}(\mathbb{R};e^{\phi})} \|p_n(t)\|_{\mathfrak{H}} \|u\|_\infty \leq C\|p_n(t)\|_{\mathfrak{V}} \|p_n(t)\|_{\mathfrak{H}} \|u\|_\infty$$

$$\begin{aligned} p_n &\to p \quad \text{weakly in } L^2(0,T;\mathfrak{V}),\\ \dot{p}_n &\to \dot{p} \quad \text{weakly in } L^2(0,T;\mathfrak{V}'),\\ p_n &\to p \quad \text{weak}^* \text{ in } L^\infty(0,\infty;\mathfrak{H}) \qquad \square \end{aligned}$$

Simulation

$$\begin{aligned} \dot{p}(t,x) &= -c\mathfrak{A}p(t,x) - \mathfrak{B}\big(p(t,\cdot),u(t)\big)(x) + d(t,x), && \text{in } (0,\infty)\times\mathbb{R}, \\ p(0,x) &= p_0(x), && \text{in } \mathbb{R} \end{aligned}$$

$$u(t) = -\frac{e(t)}{1-\varphi(t)^2 e(t)^2}, \quad e(t) = y(t) - y_{\mathrm{ref}}(t),$$

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with $c = 0.1, \gamma = 1, y_{\mathrm{ref}}(t) = \sin t$ and

$$p_0 : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} 1, & -1 \le x \le -\frac{1}{2} \vee \frac{1}{4} \le x \le \frac{3}{4}, \\ 0, & \text{otherwise} \end{cases} \quad \in \mathfrak{H} \setminus \mathfrak{V},$$

$$d : \mathbb{R}_{\ge 0} \times \mathbb{R} \to \mathbb{R},\ (t,x) \mapsto 3\cos(4t)\, x e^{-3x^2},$$