

Controlled invariance for DAEs

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Linear systems

$$\boxed{\frac{d}{dt}Ex(t) = Ax(t) + Bu(t)} \quad (E, A, B)$$

$$E, A \in \mathbb{R}^{\ell \times n}, B \in \mathbb{R}^{\ell \times m}$$

$$\mathfrak{B} = \{ (x, u) \in C^1 \times C \mid E\dot{x}(t) = Ax(t) + Bu(t) \}$$

Def.: $\mathcal{V} \subseteq \mathbb{R}^n$ is controlled invariant $:\Leftrightarrow$

$$\forall x^0 \in \mathcal{V} \exists (x, u) \in \mathfrak{B} \forall t \geq 0 : x(0) = x^0 \wedge x(t) \in \mathcal{V}$$

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Theorem (controlled invariance)

The following is equivalent for (E, A, B) and $\mathcal{V} \subseteq \mathbb{R}^n$:

- (1) \mathcal{V} is controlled invariant
- (2) $A\mathcal{V} \subseteq E\mathcal{V} + \text{im } B$
- (3) $\exists F \in \mathbb{R}^{m \times n} : (A + BF)\mathcal{V} \subseteq E\mathcal{V}$

Proof: (1) \Rightarrow (2):

$$x^0 \in \mathcal{V} \Rightarrow Ax^0 = Ax(0) = E\dot{x}(0) - Bu(0) \in E\mathcal{V} + \text{im } B$$

(2) \Rightarrow (3):

$$\text{im } V = \mathcal{V}, AV = EVW + BU \quad \Rightarrow \quad F := -U(V^\top V)^{-1}V^\top$$

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Existence lemma: $E, A \in \mathbb{R}^{\ell \times n}$ such that $\text{im } A \subseteq \text{im } E$, then

$$\forall x^0 \in \mathbb{R}^n \exists x \in C^\infty \forall t \in \mathbb{R} : x(0) = x^0 \wedge E\dot{x}(t) = Ax(t)$$

(3) \Rightarrow (1):

$$x^0 = Vw^0 \in \mathcal{V}, \quad \text{im}(A + BF)V \subseteq \text{im } EV$$

$$\stackrel{\text{lemma}}{\implies} \exists w \in C^\infty : w(0) = w^0 \wedge EV\dot{w}(t) = (A + BF)Vw(t),$$

$$\implies x := Vw, u := FVw \text{ satisfy}$$

$$(x, u) \in \mathfrak{B}, x(0) = x^0 \text{ and } x(t) \in \mathcal{V}, t \geq 0 \quad \square$$

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Nonlinear systems

$$\boxed{\frac{d}{dt}E(x(t)) = f(x(t)) + g(x(t))u(t)} \quad (E, f, g)$$

$X \subseteq \mathbb{R}^n$ open, $0 \in X$, $E, f : X \rightarrow \mathbb{R}^\ell$, $g : X \rightarrow \mathbb{R}^{\ell \times m}$ diff., $f(0) = 0$

$\mathfrak{B} = \{ (x, u) \in C^1 \times C \mid (x, u) \text{ is a maximal solution of } (E, f, g) \}$

M – connected submanifold of X with $0 \in M$

Def.: M is locally controlled invariant : \iff

\exists open neighborhood U of $0 \in X$ such that

$\forall x^0 \in M \cap U \exists (x, u) \in \mathfrak{B} \exists t_0 \in \text{dom } x, x(t_0) = x^0 :$

$(\forall t \in \text{dom } x, t \geq t_0 : x(t) \in M \cap U)$

$\vee (\exists \hat{t} \in \text{dom } x, \hat{t} > t_0 \forall t \in [t_0, \hat{t}) : x(t) \in M \cap U \wedge x(\hat{t}) \in \partial(M \cap U))$

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Theorem (local controlled invariance)

$E \in \mathcal{C}^2$, $f, g \in \mathcal{C}^1$, M connected submanifold of X with $0 \in M$ such that, in a neighborhood of $0 \in M$,

$$\dim E'(x)T_x M = \text{const} \quad \wedge \quad \dim (E'(x)T_x M + \text{im } g(x)) = \text{const}$$

Then the following is equivalent:

- (1) M is locally controlled invariant.
- (2) $f(x) \in E'(x)T_x M + \text{im } g(x)$ in $M \cap U$.
- (3) $\exists u \in \mathcal{C}^1(M \cap U \rightarrow \mathbb{R}^m) : f(x) + g(x)u(x) \in E'(x)T_x M$ in $M \cap U$.

Proof: (1) \Rightarrow (2): $x^0 \in M \cap U$

$$\Rightarrow f(x^0) = E'(x(0))\dot{x}(0) - g(x(0))u(0) \in E'(x^0)T_{x^0} M + \text{im } g(x^0)$$

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Existence lemma: $U \subseteq \mathbb{R}^n$ open, $E, f : U \rightarrow \mathbb{R}^\ell$ diff. and

$$\forall x \in U : \text{rk } E'(x) = r \quad \wedge \quad f(x) \in E'(x)T_x U,$$

$$\implies \forall x^0 \in U \exists x \in C^1(I \rightarrow \mathbb{R}^n) \forall t \in I : x(0) = x^0 \wedge \frac{d}{dt} E(x(t)) = f(x(t))$$

(3) \implies (1): $x^0 = \psi(w^0) \in M \cap U$, $\psi : G \rightarrow M \cap U$ parametrization of M

def. $\tilde{E} := E \circ \psi$, $\tilde{f} := f \circ \psi + (g \circ \psi)(u \circ \psi)$

$$\implies \text{rk } \tilde{E}'(x) = \text{rk } E'(\psi(x))\psi'(x) = \dim E'(\psi(x))T_{\psi(x)}M = \text{const},$$

$$\tilde{f}(x) \in E'(\psi(x))T_{\psi(x)}M = E'(\psi(x))\psi'(x)T_x G = \tilde{E}'(x)T_x G$$

lemma $\implies \exists w \in C^1(I \rightarrow G) : w(0) = w^0 \wedge \frac{d}{dt} \tilde{E}(w(t)) = \tilde{f}(w(t)),$

$$\implies (x := \psi \circ w, u \circ w) \in \mathfrak{B}, x(0) = x^0$$

$$\text{and } x(t) \in M \cap U, \forall t \in I, t \geq 0$$

Remains: show that $(x, u \circ x)$ can be extended to a maximal sln.

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Zero dynamics

$$\boxed{\frac{d}{dt}E(x(t)) = f(x(t)) + g(x(t))u(t), \quad y(t) = h(x(t))} \quad (E, f, g, h)$$

$E, f : X \rightarrow \mathbb{R}^\ell, g : X \rightarrow \mathbb{R}^{\ell \times m}, h : X \rightarrow \mathbb{R}^p$ diff., $f(0) = 0, h(0) = 0$

Zero dynamics:

$$\mathcal{ZD} = \{ (x, u) \in \mathfrak{B} \mid y(t) = h(x(t)) = 0 \}$$

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Example

$$\begin{aligned}
 \dot{x}_1 &= 0 \\
 0 &= x_1 + u \\
 y &= x_1 - x_2^2
 \end{aligned}
 \quad M := \{ x \in \mathbb{R}^2 \mid x_1 = x_2^2 \} \subseteq h^{-1}(0)$$

$x^0 = (x_1^0, x_2^0)^\top \in M \Rightarrow x(t) := x^0, u(t) := -x_1^0, t \in \mathbb{R}$, satisfy

$(x, u) \in \mathfrak{B}$ and $x(0) = x^0, x(t) \in M$ for $t \in \mathbb{R}$

$\Rightarrow M$ is output zeroing submanifold

Note: u must satisfy the algebraic constraint $u = -x_1$

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Theorem (zero dynamics algorithm)

$E, f, g, h \in C^\infty$; def. $M_0 := h^{-1}(0)$ and

$$M_k := \{ x \in M_{k-1} \mid f(x) \in E'(x)T_x M_{k-1} + \text{im } g(x) \};$$

suppose that M_k is a connected submanifold with $0 \in M_k$

- (1) $\exists k^* \in \mathbb{N}_0 \forall j \in \mathbb{N} : M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_{k^*} =: Z^* = M_{k^*+j}$
- (2) $\dim E'(x)T_x Z^* = \text{const}$
 $\wedge \dim (E'(x)T_x Z^* + \text{im } g(x)) = \text{const}$ in $Z^* \cap U$,
 $\implies Z^*$ is a **locally maximal** output zeroing submanifold
- (3) \exists open neighborhood U of $0 \in X \forall$ open $O \subseteq U \forall (x, u) \in \mathfrak{B}$
 with $x(t) \in O, t \in \text{dom } x$:

$$(x, u) \in \mathcal{ZD} \iff x(t) \in Z^* \cap O \quad \forall t \in \text{dom } x.$$

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\implies M is locally maximal output zeroing submanifold

BUT!

$$\dim E'(x)T_x Z^* = \dim \operatorname{im} \begin{bmatrix} 2x_2 \\ 0 \end{bmatrix}$$

$$\text{and } \dim (E'(x)T_x Z^* + \operatorname{im} g(x)) = \dim \operatorname{im} \begin{bmatrix} 2x_2 & 0 \\ 0 & 1 \end{bmatrix}$$

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