UNIVERSITÄT PADERBORN

Institute for Mathematics, Paderborn University

ASYMPTOTIC TRACKING BY FUNNEL CONTROL WITH INTERNAL MODELS

Thomas Berger, Christoph M. Hackl, Stephan Trenn

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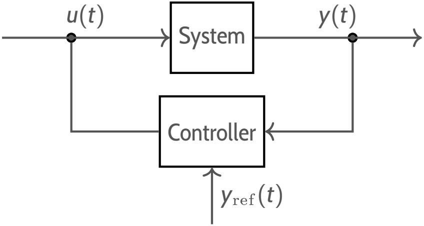
Control objective



$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x^0 \in \mathbb{R}^n$$
$$y(t) = Cx(t)$$

Goal: $\|y(t) - y_{\text{ref}}(t)\| < \psi(t)$ and $y(t) \to y_{\text{ref}}(t)$

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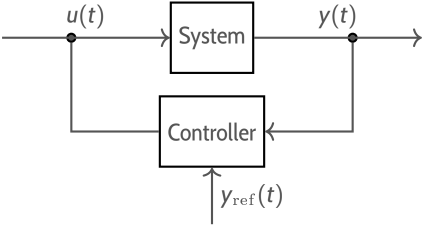

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Goal: $\|y(t) - y_{\mathrm{ref}}(t)\| < \psi(t)$ and $y(t) \to y_{\mathrm{ref}}(t)$

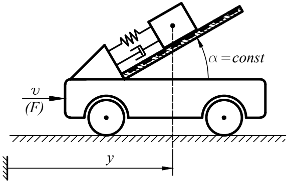
No knowledge of system parameters!

Class of linear systems with relative degree $r \in \mathbb{N}$

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = x^0$$

- $A \in \mathbb{R}^{n\times n}, B, C^\top \in \mathbb{R}^{n\times m}, x^0 \in \mathbb{R}^n$
- $CB = CAB = \ldots = CA^{r-2}B = 0, \quad CA^{r-1}B \in \mathbf{Gl}_n(\mathbb{R})$
- minimum phase:

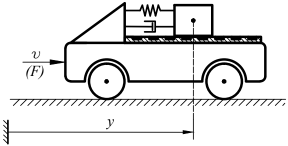
$$\forall\, \lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda \geq 0 : \quad \mathrm{rk} \begin{bmatrix} A - \lambda I_n & B \\ C & 0 \end{bmatrix} = n + m$$

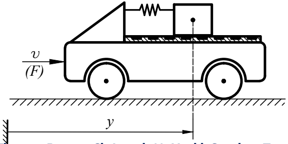


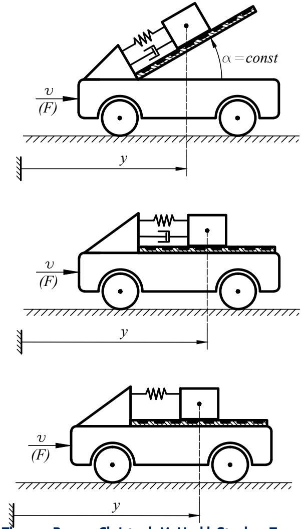
$$0^\circ < \alpha \leq 90^\circ$$

$$\ddot{y} = f_1(y, \dot{y}, \eta) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

relative degree $= 2$







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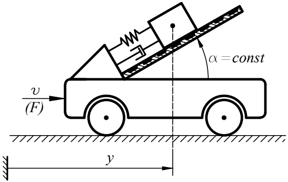
relative degree $= 2$

$$\alpha = 0^\circ, \quad d \neq 0$$

$$y^{(3)} = f_2(y, \dot{y}, \ddot{y}, \eta) + \frac{d}{m_1 m_2} u$$

relative degree $= 3$



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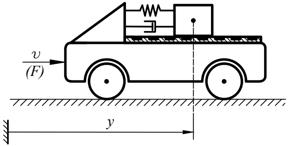

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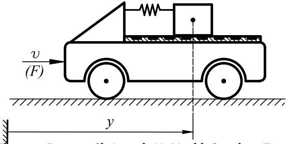
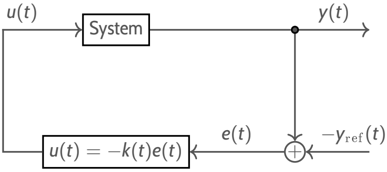


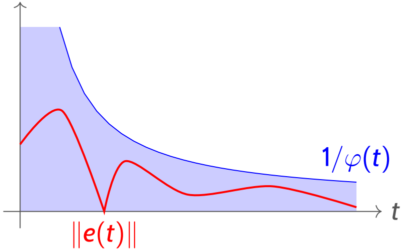
$\alpha = 0°, \quad d = 0, \quad k \neq 0$

$$y^{(4)} = f_3(y, \ldots, y^{(3)}) + \frac{k}{m_1 m_2} u$$

relative degree $= 4$

Funnel control





$$k(t) = \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2}$$

[Ilchmann, Ryan, Sangwin '02]:
Works, if

- relative degree $= 1$
- minimum phase

Funnel control for systems with arbitrary relative degree $r \in \mathbb{N}$

$$\begin{aligned} e_1(t) &= e(t), \qquad e(t) = y(t) - y_{\mathrm{ref}}(t), \\ e_2(t) &= \dot e_1(t) + k_1(t) e_1(t), \\ &\vdots \\ e_r(t) &= \dot e_{r-1}(t) + k_{r-1}(t) e_{r-1}(t), \end{aligned}$$

$$u(t) = -k_r(t) e_r(t)$$

$$k_i(t) = 1/(1 - \varphi_i(t)^2 \|e_i(t)\|^2), \quad i = 1, \dots, r$$

Theorem [B., Lê, Reis '18]

$$y_{\mathrm{ref}} \in W^{r,\infty} \implies u, k_i, y^{(i)} \in L^\infty \text{ and } \varphi_i(t)\|e_i(t)\| \le 1 - \varepsilon_i$$

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&\;\vdots \\
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[B., Ilchmann, Ryan '21]: $1/\varphi_1(t) \to 0$ possible, but very high noise sensitivity!

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$$u(t) = -k_r(t) e_r(t)$$

$$k_r(t) = \hat{k} / (1 - \varphi_r(t)^2 \|e_r(t)\|^2)$$

$$k_1, \dots, k_{r-1}, \hat{k} > 0$$

Funnel control with constant gains

Given: $\varphi_1 \quad \to \quad$ choose $\varphi_2, \dots, \varphi_r$ and $k_1, \dots, k_{r-1} > 0$ such that

(K1) $k_i > \left\| \frac{\dot{\varphi}_i}{\varphi_i} \right\|_\infty + \left\| \frac{\varphi_i}{\varphi_{i+1}} \right\|_\infty, \quad i = 1, \dots, r-1$

(K2) $\varphi_i(0)\|e_i(0)\| < 1, \quad i = 1, \dots, r$

Lemma [B., Hackl, Trenn '24]

$\varphi_r(t)\|e_r(t)\| < 1$ for all $t \in I$

$\implies \quad \forall\, t \in I : \; \varphi_i(t)\|e_i(t)\| \le \varepsilon_i < 1, \quad i = 1, \dots, r-1$

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Idea for proof: If $\varphi_{i+1}(t)\|e_{i+1}(t)\| < 1$, then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_i(t)^2\|e_i(t)\|^2$$
$$\le \left(\left\| \frac{\dot\varphi_i}{\varphi_i} \right\|_\infty + \left\| \frac{\varphi_i}{\varphi_{i+1}} \right\|_\infty - k_i\varphi_i(t)\|e_i(t)\| \right) \varphi_i(t)\|e_i(t)\|$$

Internal models

$\alpha(s) \in \mathbb{R}[s]$ monic polynomial such that

$$\forall\, \lambda \in \mathbb{C}:\ \alpha(\lambda) = 0 \implies \operatorname{rk}\begin{bmatrix} A - \lambda I_n & B \\ C & 0 \end{bmatrix} = n + m.$$

Class of reference signals y_{ref}:

$$\mathcal{R}(\alpha) := \left\{ w \in C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \;\middle|\; \alpha(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0 \right\}.$$

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Internal model: Choose $\beta(s)$ Hurwitz s.t. $\alpha(s)$ and $\beta(s)$ are coprime, $\deg \alpha(s) = \deg \beta(s)$ and $\lim_{s\to\infty} \frac{\beta(s)}{\alpha(s)} = 1$

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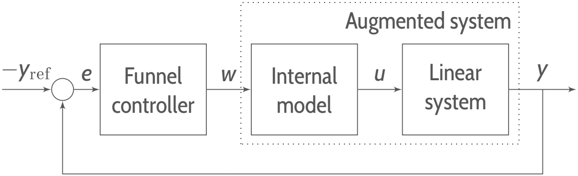
$\deg \alpha(s) = \deg \beta(s)$ and $\lim_{s\to\infty} \frac{\beta(s)}{\alpha(s)} = 1$
Find minimal realization $\frac{\beta(s)}{\alpha(s)} I_m$:

$$\dot{z}(t) = \tilde{A}z(t) + \tilde{B}w(t), \quad u(t) = \tilde{C}z(t) + I_m w(t)$$

Internal models



Lemma [B., Hackl, Trenn '24]

The interconnection of system and internal model,

$$\begin{pmatrix} \dot{x}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} A & B\tilde{C} \\ 0 & \tilde{A} \end{bmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix} + \begin{bmatrix} B \\ \tilde{B} \end{bmatrix} w(t),$$

$$y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix},$$

has relative degree r and is minimum phase.

Asymptotic funnel control with internal models

$$
\begin{aligned}
e_1(t) &= e(t), \qquad e(t) = y(t) - y_{\mathrm{ref}}(t),\\
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w(t) &= -k_r(t) e_r(t)\\
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Theorem [B., Hackl, Trenn '24]

$y_{\mathrm{ref}} \in \mathcal{R}(\alpha)$, internal model $(\tilde{A}, \tilde{B}, \tilde{C}, I_m)$, given φ_1 choose $\varphi_2, \dots, \varphi_r$ and $k_1, \dots, k_{r-1} > 0$ s.t. (K1) and (K2) hold

$$\Longrightarrow \quad x, z, u, k \in L^\infty \quad \text{and} \quad \varphi_i(t)\|e_i(t)\| \leq \varepsilon_i;$$

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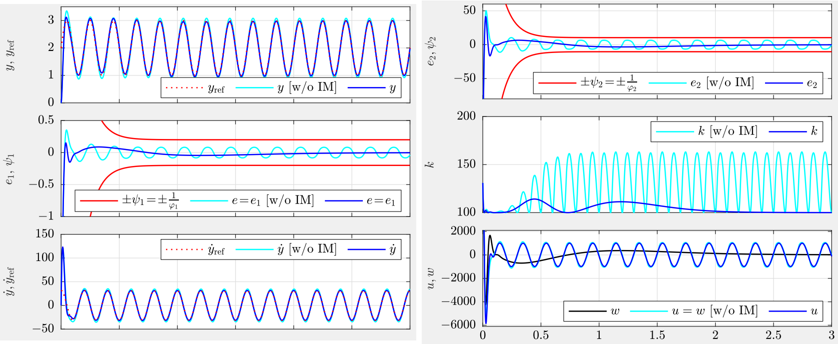
$$\implies \quad x, z, u, k \in L^\infty \quad \text{and} \quad \varphi_i(t)\|e_i(t)\| \le \varepsilon_i;$$

$\hat k > 0$ sufficiently large $\implies \lim_{t\to\infty} e^{(i)}(t) = 0, \quad i = 0, \ldots, r-1$

Simulation (third order, relative degree two)



[——] without internal model

[——] with internal model

Outlook

- gain adaptation for $\hat{k}$
- extension to nonlinear systems
- proof of robustness w.r.t. measurement noise

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Thanks for your attention!