



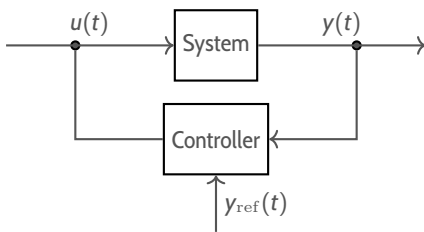
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INPUT-CONSTRAINED FUNNEL CONTROL

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Stuttgart, April 1, 2022

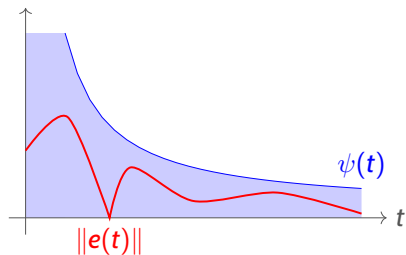
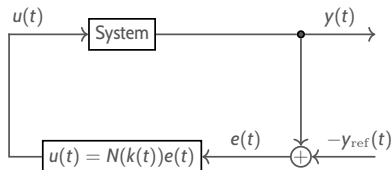
Control objective



$$\begin{aligned}\dot{x}(t) &= f(t, x(t), u(t)), & x(0) &= x^0 \in \mathbb{R}^n \\ y(t) &= h(x(t))\end{aligned}$$

- **Goal:** simple controller, so that “ $y(t)$ tracks $y_{\text{ref}}(t)$ ”
- only uses $y(t)$, no knowledge of x^0 or system parameters
- only: **known relative degree and stable internal dynamics**

Funnel control



$$k(t) = \frac{1}{1 - \|e(t)\|^2 / \psi(t)^2},$$

$N \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$ a surjection
(e.g. $N(s) = s \sin s$)

[ILCHMANN, RYAN, SANGWIN '02]:
Works, if

- relative degree = 1
- stable internal dynamics
- high-gain property

Funnel control for systems with arbitrary relative degree

$$e_1(t) = e(t) = y(t) - y_{\text{ref}}(t),$$

$$e_2(t) = \dot{e}(t) + k_1(t)e_1(t),$$

$$e_3(t) = \ddot{e}(t) + k_2(t)e_2(t),$$

$$\vdots$$

$$e_r(t) = e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t),$$

$$u(t) = N(k_r(t))e_r(t)$$

$$k_i(t) = 1/(1 - \|e_i(t)\|^2/\psi_i(t)^2), \quad i = 1, \dots, r$$

Theorem [B., LÊ, REIS '18] & [B., ILCHMANN, RYAN '21]

$y_{\text{ref}} \in W^{r, \infty} \implies u, k_i, y^{(i)} \in L^\infty$ and $\|e(t)\| \leq \psi(t) - \varepsilon$

Funnel control under input constraints

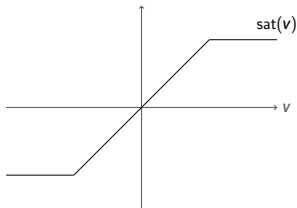
$$u(t) = \text{sat}(v(t))$$

such that for some $M > 0$

- $\text{sat} \in C(\mathbb{R}^m, \mathbb{R}^m)$ bounded
 - $\forall v \in \mathbb{R}^m : \|v\| \leq M \implies \text{sat}(v) = v$
- } (sat_M)

- sat with (sat_M) is assumed arbitrary but *known*

○



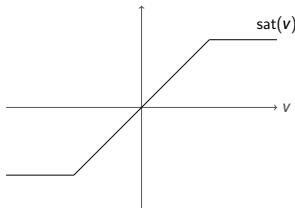
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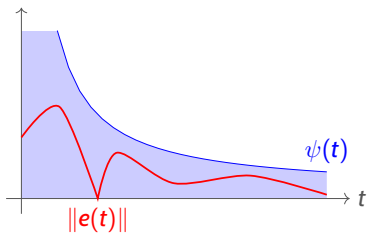
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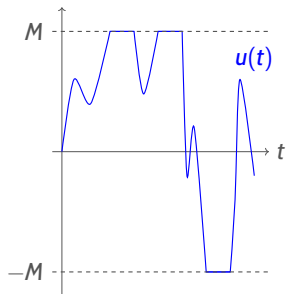
- sat with (sat_M) is assumed arbitrary but *known*
- input constraints and output constraints are conflicting!



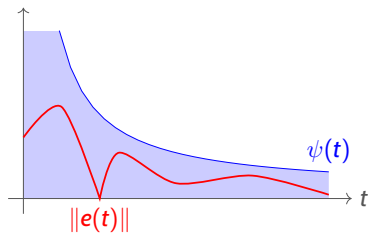
Funnel control under input constraints



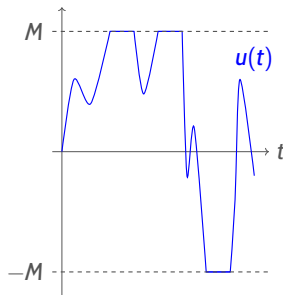
trade-off
↔



Funnel control under input constraints



trade-off
↔



- design controller which “negotiates” this trade-off
- here: input constraints are strict, output constraints are flexible

Funnel control under input constraints – known results

so far only: output constraints are strict, input constraints are flexible
→ funnel control works, if M in (sat_M) suff. large

[HOPFE, ILCHMANN, RYAN '10 A & B]: relative degree 1

[HACKL, HOPFE, ILCHMANN, MÜLLER, TRENN '13]: relative degree 2

[LIBERZON & TRENN '13] (bang-bang funnel control): arbitrary relative degree

Funnel control under input constraints – controller design

$$e_1(t) = e(t) = y(t) - y_{\text{ref}}(t),$$

$$e_2(t) = \dot{e}(t) + k_1(t)e_1(t),$$

$$\vdots$$

$$e_r(t) = e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t),$$

$$k_i(t) = 1/(1 - \|e_i(t)\|^2/\psi_i(t)^2),$$

$$v(t) = N(k_r(t))e_r(t)$$

$$\dot{\psi}_1(t) = -\alpha_1\psi_1(t) + \beta_1$$

$$+ \rho_1 \left(\psi_2(t) - \frac{\beta_2}{\alpha_2} \right),$$

$$\vdots$$

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$$+ \rho_{r-1} \left(\psi_r(t) - \frac{\beta_r}{\alpha_r} \right),$$

$$\dot{\psi}_r(t) = -\alpha_r\psi_r(t) + \beta_r + \psi_r(t) \frac{\kappa(v(t))}{\|e_r(t)\|},$$

$$\kappa(v(t)) = \|v(t) - \text{sat}(v(t))\|$$

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- $\alpha_1 > \alpha_2 > \dots > \alpha_r > 0$, $p_i > 1$, $\beta_i > 0$, $\psi_i(0) > \frac{\beta_i}{\alpha_i}$
- Note: $\|e_r(t)\| \ll 1 \implies \kappa(v(t)) = 0$
- ψ_1, \dots, ψ_r are *not* bounded in general

Lipschitz condition

sat with (sat_M) arbitrary \rightarrow global solutions do not necessarily exist

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Ex.: $\dot{y}(t) = y(t)^2 + u(t), y(0) = 1, u(t) = \text{sat}(v(t))$

Lipschitz condition

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Ex.: $\dot{y}(t) = y(t)^2 + u(t), y(0) = 1, u(t) = \text{sat}(v(t))$

$$\implies \dot{y}(t) \geq y(t)^2 - M \implies y(t) \geq z(t),$$

$$z(t) = \sqrt{M} \frac{\sqrt{M} + 1 + (1 - \sqrt{M})e^{2\sqrt{M}t}}{\sqrt{M} + 1 - (1 - \sqrt{M})e^{2\sqrt{M}t}}$$

$M \geq 1 \implies z(\cdot)$ globally defined

$M < 1 \implies z(\cdot)$ blows up at $\omega = \frac{1}{2\sqrt{M}} \ln \left(\frac{1 + \sqrt{M}}{1 - \sqrt{M}} \right)$

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\rightarrow some kind of Lipschitz condition required

Funnel control under input constraints – results

Theorem 1 [B. '22]

- relative degree r
- Lipschitz condition
- $y_{\text{ref}} \in \mathcal{C}^r$

$\implies \exists$ global solution $(y, \psi_1, \dots, \psi_r)$ with $\|e_i(t)\| \leq \varepsilon_i \psi_i(t)$ for $i = 1, \dots, r - 1$ and $\|e_r(t)\| < \psi_r(t)$

\implies if $v(t) = \text{sat}(v(t))$ for $t \in [t_0, t_1]$, then

$$\psi_i(t) \leq \frac{\beta_i}{\alpha_i} + \sum_{j=i}^r c_{ij}(t_0) e^{-\alpha_j(t-t_0)}$$

Funnel control under input constraints – results

Theorem 2 [B. '22]

- relative degree r
- **Lipschitz condition** stable internal dynamics and high-gain property
- ~~$y_{\text{ref}} \in \mathbb{R}^r$~~ $y_{\text{ref}} \in W^{r,\infty}$

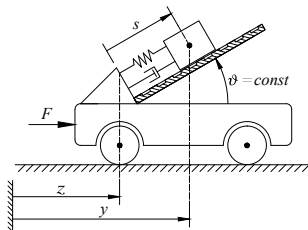
$\implies \forall \varepsilon \in (0, 1) \forall K > 0 \exists M = M(\varepsilon, K)$ such that

- \forall sat with (sat_M) ,
- $\forall y^0$ with $\|e_i(0)\| \leq \varepsilon \psi_i(0), i = 1, \dots, r$,
- $\forall y_{\text{ref}} \in W^{r,\infty}$ with $\|y_{\text{ref}}^{(i)}\|_{\infty} \leq K, i = 0, \dots, r$,

\exists global solution $(y, \psi_1, \dots, \psi_r)$ with $y \in W^{r,\infty}, \psi_i, k_i \in L^{\infty}$

and $\|v(t)\| \leq M$

Simulation



angle: $\vartheta = \pi/4$

$m_1 = 4, m_2 = 1, k = 2, d = 1$

$z(0) = s(0) = \dot{z}(0) = \dot{s}(0) = 0$

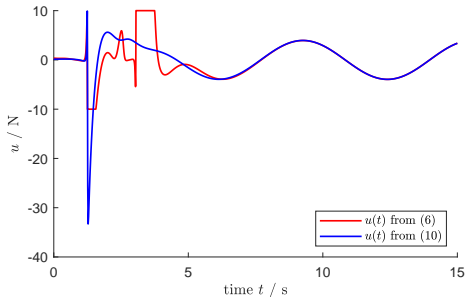
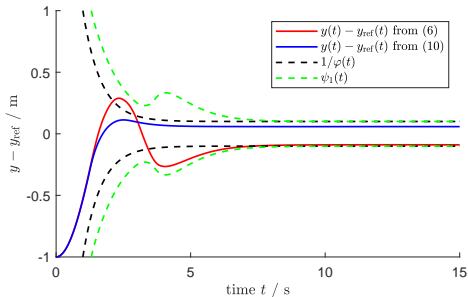
$y_{\text{ref}}(t) = \cos t,$

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos \vartheta \\ m_2 \cos \vartheta & m_2 \end{bmatrix} \begin{pmatrix} \ddot{z}(t) \\ \ddot{s}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ ks(t) + d\dot{s}(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix},$$

$$y(t) = z(t) + s(t) \cos \vartheta$$

Controller parameters: $\alpha_1 = 1.5, \alpha_2 = 0.9 \cdot \alpha_1, \beta_1 = 0.15, \beta_2 = 0.5 \cdot \alpha_2,$
 $p_1 = 1.1, \psi_1^0 = 4.1, \psi_2^0 = 2, N(s) = -s^2 \cos s$

Simulation



blue curve: funnel controller from [B., ILCHMANN, RYAN '21] with

$$\varphi(t) = (4e^{-3t/2} + 0.1)^{-1}$$

red curve: input-constrained funnel controller from [B. '22]