Institut für Mathematik, Universität Paderborn

FUNNEL CONTROL FOR LINEAR NON-MINIMUM PHASE SYSTEMS

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Linear systems with relative degree $r \in \mathbb{N}$

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = x^0 \quad (\Sigma)$$

- $A \in \mathbb{R}^{n \times n}, B, C^\top \in \mathbb{R}^{n \times m}, x^0 \in \mathbb{R}^n$
- $CB = CAB = \ldots = CA^{r-2}B = 0, \quad CA^{r-1}B \in \mathbf{Gl}_n(\mathbb{R})$
- minimum phase $\iff$

$$\forall\, \lambda \in \mathbb{C}_- : \quad \operatorname{rk} \begin{bmatrix} A - \lambda I_n & B \\ C & 0 \end{bmatrix} = n + m$$

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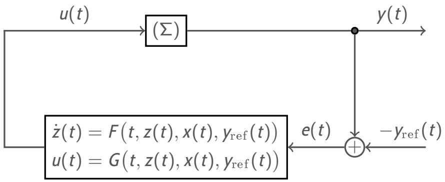
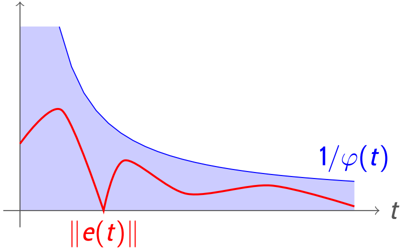
$$\forall\, \lambda \in \mathbb{C}_- : \quad \mathrm{rk} \begin{bmatrix} A - \lambda I_n & B \\ C & 0 \end{bmatrix} = n + m$$ **NOT required!**

Control objective





$$\Phi_r = \left\{ \varphi \in \mathcal{C}^r(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot\varphi, \ldots, \varphi^{(r)} \text{ bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau\to\infty} \varphi(\tau) > 0 \end{array} \right\}$$

Basic idea

1. Define a new output such that
   - the new system has a higher relative degree $r + \ell$
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2. Define a new reference signal s.t. it belongs to $\mathcal{W}^{r+\ell,\infty}$
3. Apply the available funnel controller to the system with new output and new reference signal
4. Show that the original tracking error satisfies the control objective

Definition of new output

Byrnes-Isidori form: $\exists\, U \in \mathbf{Gl}_n(\mathbb{R})$ s.t.
$Ux(t) = \left(y(t)^\top, \dot{y}(t)^\top, \ldots, y^{(r-1)}(t)^\top, \eta(t)^\top\right)^\top$ transforms (Σ) into

$$\begin{aligned} y^{(r)}(t) &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + S\eta(t) + \underbrace{\Gamma}_{=CA^{r-1}B} u(t), \\ \dot{\eta}(t) &= Py(t) + Q\eta(t) \end{aligned}$$

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Assumption: $TQT^{-1} = \begin{bmatrix} \hat{Q}_1 & \hat{Q}_2 \\ 0 & \tilde{Q} \end{bmatrix}, \quad TP = \begin{bmatrix} \hat{P} \\ \tilde{P} \end{bmatrix}$

$\tilde{Q} \in \mathbb{R}^{\ell m \times \ell m}$ and $\hat{Q}_1 \in \mathbb{R}^{k \times k}$ with $\sigma(\hat{Q}_1) \subseteq \mathbb{C}_-$, $k = n - rm - \ell m \geq 0$

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Definition of new output

$$
\begin{aligned}
y^{(r)}(t) &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + S_1\eta_1(t) + S_2\eta_2(t) + \Gamma u(t),\\
\dot\eta_1(t) &= \hat Q_1\eta_1(t) + \hat Q_2\eta_2(t) + \hat P y(t),\\
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K := [0,\dots,0,\Gamma^{-1}]\underbrace{[\tilde P, \tilde Q\tilde P, \dots, \tilde Q^{\ell-1}\tilde P]^{-1}}_{\textit{Assumption}}
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\begin{aligned}
\eta_2(t) &= \sum_{i=1}^{\ell} F_i\, y_{\mathrm{new}}^{(i-1)}(t),\\
y(t) &= \Gamma y_{\mathrm{new}}^{(\ell)}(t) + \sum_{i=1}^{\ell} \Gamma K\tilde Q^{\ell} F_i\, y_{\mathrm{new}}^{(i-1)}(t)
\end{aligned}
$$

Definition of new output

→ y_{new} is flat output for $\dot{\eta}_2(t) = \tilde{Q}\eta_2(t) + \tilde{P}y(t)$

Definition of new output

$\rightarrow$ y_{new} is flat output for $\dot\eta_2(t) = \tilde Q \eta_2(t) + \tilde P y(t)$

$$y_{\mathrm{new}}^{(r+\ell)}(t) = \sum_{i=1}^{r+\ell} \hat R_i \, y_{\mathrm{new}}^{(i-1)}(t) + S_1 \eta_1(t) + u(t),$$
$$\dot\eta_1(t) = \sum_{i=1}^{\ell+1} \hat P_i \, y_{\mathrm{new}}^{(i-1)}(t) + \hat Q_1 \eta_1(t)$$

Definition of new reference signal

$$\dot{\eta}_{2,\mathrm{ref}}(t) = \tilde{Q}\eta_{2,\mathrm{ref}}(t) + \tilde{P}y_{\mathrm{ref}}(t), \quad \eta_{2,\mathrm{ref}}(0) = \eta^0_{2,\mathrm{ref}}$$

$$\hat{y}_{\mathrm{ref}}(t) = K\eta_{2,\mathrm{ref}}(t)$$

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Lemma: We have $\hat{y}_{\mathrm{ref}} \in \mathcal{W}^{r+\ell,\infty}$, if

- $W\tilde{Q}W^{-1} = \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix}$ with $\sigma(Q_1) \subseteq \mathbb{C}_-$, $\sigma(Q_2) \subseteq \mathbb{C}_+$, $\sigma(Q_3) \subseteq \mathrm{i}\mathbb{R}$
- $\eta^0_{2,\mathrm{ref}} = W^{-1} \begin{bmatrix} 0_{k_1 \times k_2} \\ -I_{k_2} \\ 0_{k_3 \times k_2} \end{bmatrix} \int_0^\infty e^{-Q_2 s} P_2 y_{\mathrm{ref}}(s)\,\mathrm{d}s$
- $\dot{z}(t) = Q_3 z(t) + P_3 y_{\mathrm{ref}}(t),\ z(0) = 0$ has a bounded solution $z(\cdot)$

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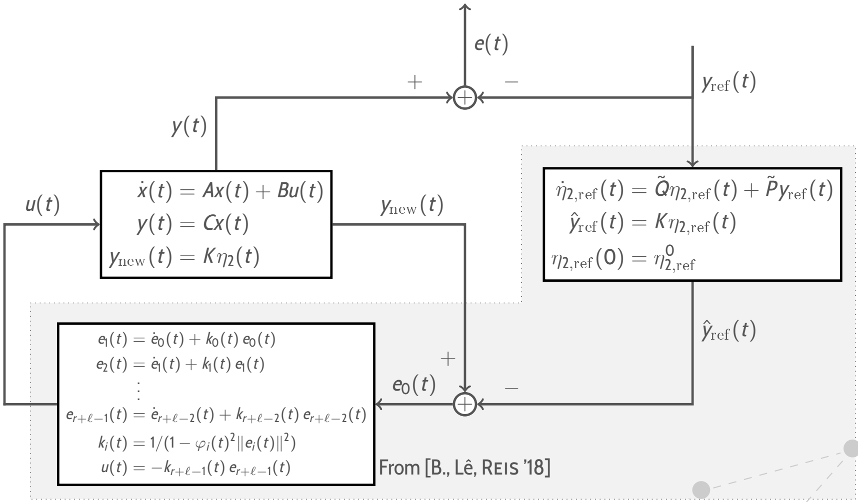
Controller structure



From [B., Lê, Reis '18]

Theorem

$y_{\mathrm{ref}} \in \mathcal{W}^{r-1,\infty}$, $\varphi_i \in \Phi_{r+\ell-i}$, $\varphi_i(0)\|e_i(0)\| < 1$ for $i = 0, \ldots, r+\ell-1$

$\implies \quad x, \eta_{2,\mathrm{ref}}, u, k_0, \ldots, k_{r+\ell-1} \in L^\infty$ and

$$\|e_i(t)\| \le \varphi_i(t)^{-1} - \varepsilon_i$$

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Original tracking error satisfies: $\|e(t)\| \le \Psi(t)$

- Ψ is *known a priori* and depends on φ_i and $e_i(0)$, $i = 0, \dots, r+\ell-1$
- $\varphi \in \Phi_r$ given $\to$ choose $\varphi_0, \dots, \varphi_{r+\ell-1}$ s.t. $\Psi(t) < \varphi(t)^{-1}$

Simulation

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad C = [1, 0, -3, 0], \quad x^0 = 0, \quad r = 2$$

$$y_{\mathrm{ref}}(t) = \begin{cases} (1 - \cos t), & t \in [0, 2\pi], \\ 0, & t > 2\pi. \end{cases}$$

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$$\begin{aligned} \ddot{y}(t) &= -18y(t) - 7\dot{y}(t) + \eta_1(t) - 8\eta_2(t) + 2u(t) \\ \dot{\eta}_1(t) &= -\eta_1(t) + \eta_2(t) \\ \dot{\eta}_2(t) &= \eta_2(t) + 3y(t) \end{aligned}$$

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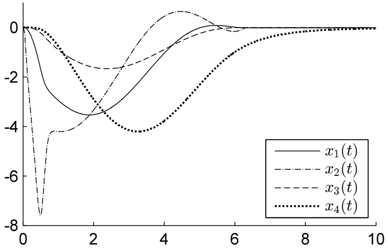
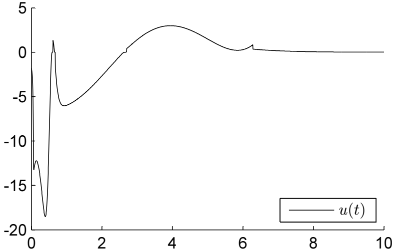
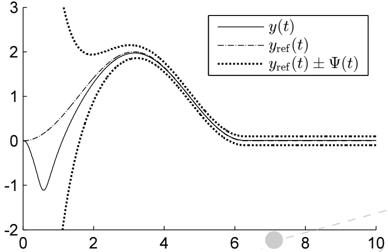
$$\ell = 1, \tilde{Q} = 1, \tilde{P} = 3, K = \frac{1}{6} \quad \rightarrow \quad y_{\mathrm{new}}(t) = \frac{1}{6}\eta_2(t) = \frac{1}{2}x_3(t)$$

Simulation

$$\eta_{2,\mathrm{ref}}^{0} = -\frac{1069}{714},$$
$$\varphi_0(t) = \left(e^{-2t} + 0.01\right)^{-1},$$
$$\varphi_1(t) = \left(2e^{-2t} + 0.01\right)^{-1},$$
$$\varphi_2(t) = \left(2e^{-10t} + 0.01\right)^{-1}$$







Outlook

1. features of funnel control are lost: not model-free, not robust
2. additional measurements required: $y_{\mathrm{new}}, \dot{y}_{\mathrm{new}}, \ldots, y_{\mathrm{new}}^{r+\ell-1}$
3. construction method for φ_i so that $\Psi(t) < \varphi(t)^{-1}$ not available yet