Universität Hamburg
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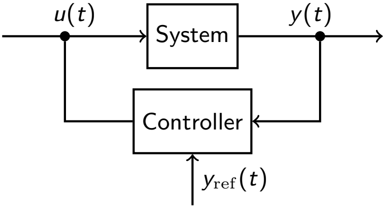
FAKULTÄT
FÜR MATHEMATIK, INFORMATIK
UND NATURWISSENSCHAFTEN

FACHBEREICH MATHEMATIK

Thomas Berger, Anna-Lena Rauert

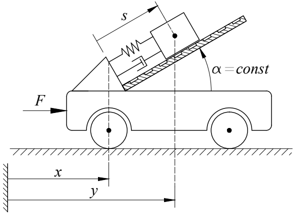
Funnel cruise control

Control systems



$$\begin{aligned}\dot{x}(t) &= f\big(x(t), u(t)\big), \quad x(0) = x^0 \\ y(t) &= h\big(x(t)\big)\end{aligned}$$

- no knowledge of system parameters, only "structural assumptions" on the model of the system
- **Aim:** design simple controller such that $y(t)$ "tracks" $y_{\mathrm{ref}}(t)$

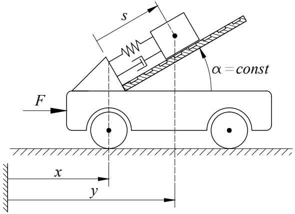


[SEIFRIED, BLAJER '13]

Angle: $0° < \alpha \leq 90°$

spring, damper with nonlinear characteristics: $K(z)$, $D(\dot{z})$

$$u(t) = F$$
$$y(t) = x(t) + s(t)\cos\alpha$$



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spring, damper with nonlinear characteristics: $K(z)$, $D(\dot{z})$

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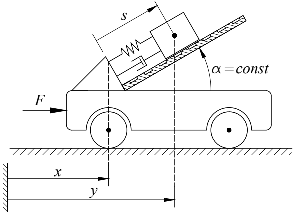
$$\begin{bmatrix} m_1 + m_2 & m_2 \cos\alpha \\ m_2 \cos\alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} u \\ -K(s) - D(\dot{s}) + m_2 g \sin\alpha \end{pmatrix}$$



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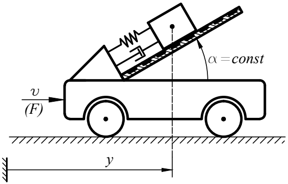
$$\dot{y} = \dot{x} + \dot{s}\cos\alpha$$

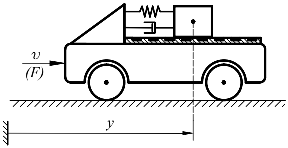
$$\ddot{y} = -c_1\big(K(s) + D(\dot{s}) - m_2 g\sin\alpha\big) + \frac{\sin^2\alpha}{m_1 + m_2\sin^2\alpha} u$$

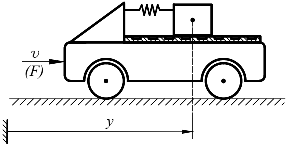


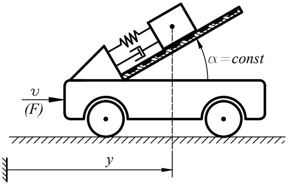
$0^\circ < \alpha \le 90^\circ$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

relative degree $= 2$







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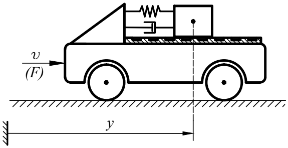
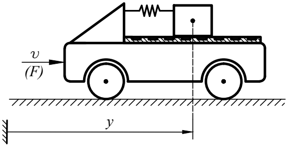
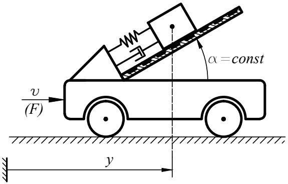


$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$

$$y^{(3)} = f_2(s, \dot{s}) + \frac{D'(\dot{s})}{m_1 m_2} u$$

relative degree $= 3$





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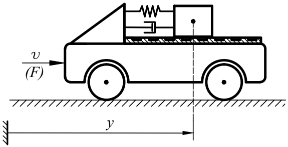

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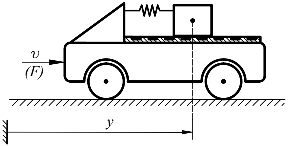
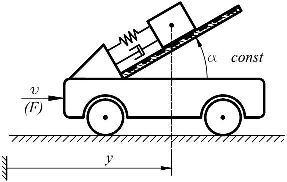


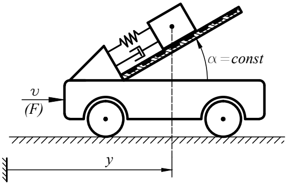
$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$

$$y^{(4)} = f_3(s, \dot{s}) + \frac{K'(s)}{m_1 m_2} u$$

relative degree $= 4$

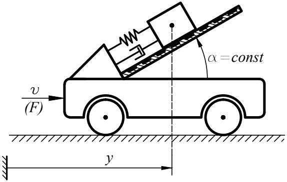


Internal dynamics: remaining dynamics when output is fixed



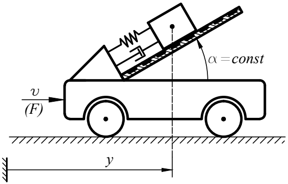
Internal dynamics: remaining dynamics when output is fixed

$$\ddot{\eta} = -c_3 K \left(\frac{\eta - y \cos \alpha}{\sin^2 \alpha} \right) - c_3 D \left(\frac{\dot{\eta} - \dot{y} \cos \alpha}{\sin^2 \alpha} \right) + c_4 g \sin \alpha$$



Internal dynamics: remaining dynamics when output is fixed

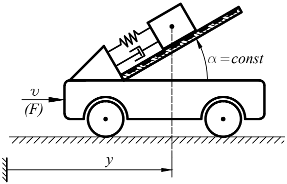
$$\alpha = 90^\circ, m_2 = 1: \qquad \ddot{s} = -K(s) - D(\dot{s}) + g$$



Internal dynamics: remaining dynamics when output is fixed

$$\alpha = 90^\circ, m_2 = 1 : \qquad \ddot{s} = -K(s) - D(\dot{s}) + g$$

- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s})\,\dot{s} \geq 0$



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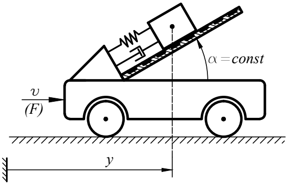
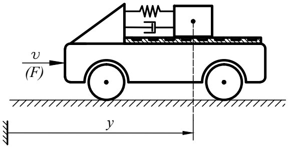
- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s})\,\dot{s} \geq 0$

$\Longrightarrow \; s, \dot{s} \in L^\infty$ **(stable internal dynamics)**



$0^\circ < \alpha \leq 90^\circ$

stable internal dynamics



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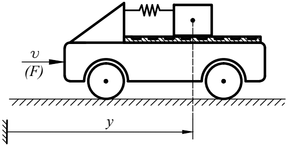
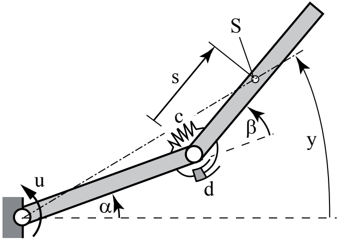
stable internal dynamics



$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$

no internal dynamics



[SEIFRIED, BLAJER '13]

Rotational Manipulator Arm

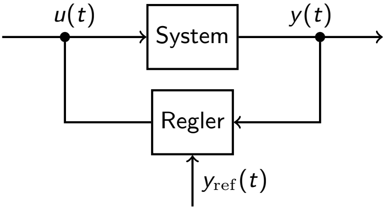
Input: angular velocity of first link

Output: position of S described by angle y

relative degree $= 1$

unstable internal dynamics

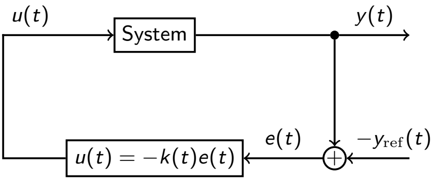
Reminder

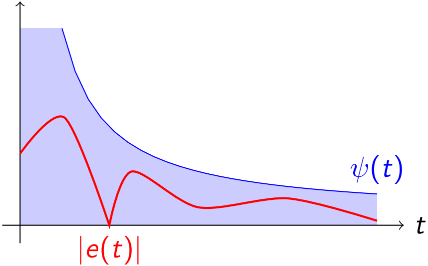


$$\begin{aligned} \dot{x}(t) &= f\big(x(t), u(t)\big), \quad x(0) = x^0 \\ y(t) &= h\big(x(t)\big) \end{aligned}$$

- no knowledge of system parameters, only: **known relative degree** and assumption of **stable internal dynamics**
- **Aim:** design simple controller such that $y(t)$ "tracks" $y_{\text{ref}}(t)$

Funnel Control



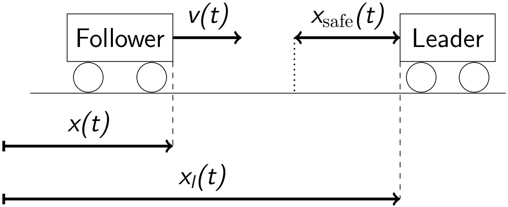


$$k(t) = \frac{1}{\psi(t) - |e(t)|}$$

[Ilchmann, Ryan, Sangwin '02]:
Works, if

- relative degree = 1
- stable internal dynamics

Vehicle following framework



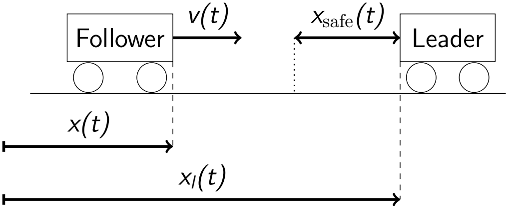
$$\dot{x}(t) = v(t), \quad m\dot{v}(t) = u(t) - F_g(t) - F_a\big(t, v(t)\big) - F_r\big(v(t)\big)$$

forces due to gravity: $F_g(t) = mg \sin\theta(t)$

aerodynamic drag : $F_a(t, v) = \frac{1}{2}\rho(t) C_d A v^2$

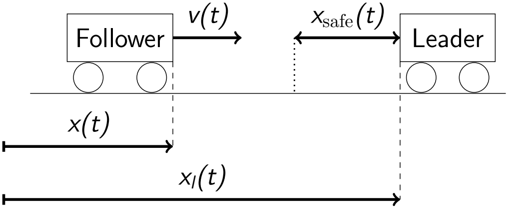
rolling friction : $F_r(v) = mgC_r \operatorname{sgn}(v)$

Control objective



given: favourite speed $v_{\text{ref}}(t)$, safety distance $x_{\text{safe}}(t) = \lambda_1 v(t) + \lambda_2$

Control objective



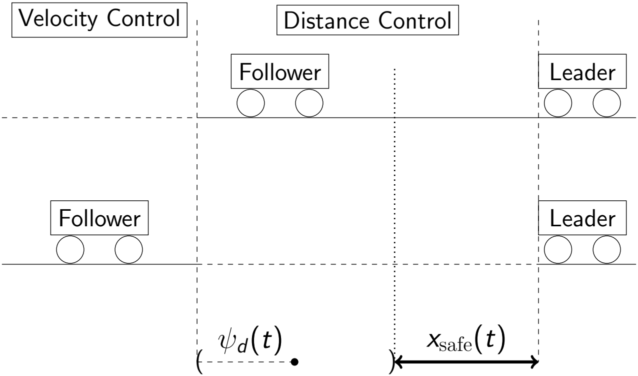
given: favourite speed $v_{\rm ref}(t)$, safety distance $x_{\rm safe}(t) = \lambda_1 v(t) + \lambda_2$

$$\textbf{Aim:} \quad u(t) = F\big(t, v(t), x_l(t) - x(t)\big) \quad \text{s.t.}$$

(O1) $x_l(t) - x(t) \geq x_{\rm safe}(t)$

(O2) $|v(t) - v_{\rm ref}(t)|$ is as small as possible such that (O1) is not violated

Funnel cruise control



Velocity funnel control

tracking error: $e_v(t) = v(t) - v_{\mathrm{ref}}(t) \in \mathcal{F}_v(t) := (-\psi_v(t), \psi_v(t))$

$\rightarrow$ treat $v(t)$ as output of the system

$$\dot{v}(t) = \tfrac{1}{m} u(t) + f_v\big(t, v(t)\big)$$

controller:

$$u_v(t) = -k_v(t) e_v(t),$$
$$k_v(t) = \frac{1}{\psi_v(t) - e_v(t)}$$

Distance funnel control

tracking error: $$e_d(t) = x(t) - x_l(t) + x_{\mathrm{safe}}(t) + \psi_d(t) \in \mathcal{F}_d(t) := (-\psi_d(t), \psi_d(t))$$

Distance funnel control

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output: $$y(t) = \lambda_1 v(t) + x(t),$$

reference signal: $$y_{\mathrm{ref}}(t) = x_l(t) - \lambda_2 - \psi_d(t)$$

Distance funnel control

tracking error: $$e_d(t) = x(t) - x_l(t) + x_{\mathrm{safe}}(t) + \psi_d(t) \in \mathcal{F}_d(t) := (-\psi_d(t), \psi_d(t))$$

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$$\dot{x}(t) = v(t) = -\frac{1}{\lambda_1}x(t) + \frac{1}{\lambda_1}y(t),$$
$$\dot{y}(t) = \frac{\lambda_1}{m}u(t) + f_d\big(t, x(t), y(t)\big)$$

controller: $$u_d(t) = -k_d(t)e_d(t),$$
$$k_d(t) = \frac{1}{\psi_d(t) - e_d(t)}$$

Final control design

Intuitive approach:

- if $e_d(t) = -\psi_d(t)$, then switch from u_v to u_d and vice versa
- while u_d is active it should be possible that $e_v(t) < -\psi_v(t)$, but not $e_v(t) \geq \psi_v(t)$

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Problem:

- when $e_d(t) \to -\psi_d(t)$ we have $k_d(t) \nearrow \infty$
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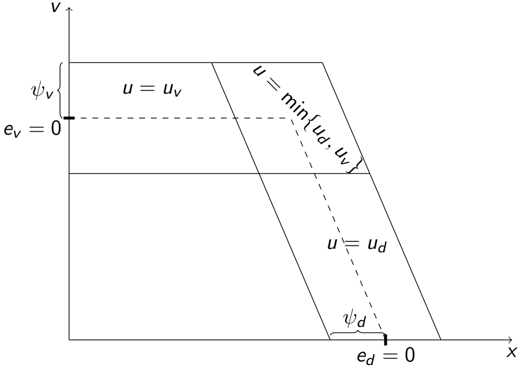

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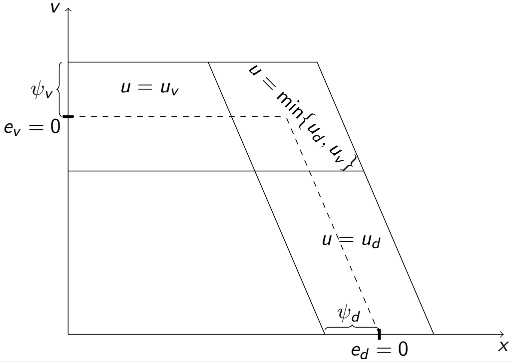
- when $e_d(t) \to -\psi_d(t)$ we have $k_d(t) \nearrow \infty$
- when $e_v(t) \to -\psi_v(t)$ we have $k_v(t) \nearrow \infty$

Final control design

$$u(t) = \begin{cases} u_v(t), & e_d(t) \le -\psi_d(t) \ \wedge \ e_v(t) \in \mathcal{F}_v(t), \\ u_d(t), & e_v(t) \le -\psi_v(t) \ \wedge \ e_d(t) \in \mathcal{F}_d(t), \\ \min\{u_v(t), u_d(t)\}, & e_v(t) \in \mathcal{F}_v(t) \ \wedge \ e_d(t) \in \mathcal{F}_d(t) \end{cases}$$



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Theorem [B., Rauert '18]

$v_{\mathrm{ref}}, x_l \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R})$, $v_{\mathrm{ref}} \ge 0$, $x_{\mathrm{safe}} = \lambda_1 v(t) + \lambda_2$
$\implies x, v, u \in L^\infty$ and

$$\psi_v(t) - e_v(t) \ge \varepsilon, \quad \psi_d(t) - e_d(t) \ge \varepsilon,$$
$$\max\{0, \psi_v(t) + e_v(t)\} + \max\{0, \psi_d(t) + e_d(t)\} \ge \varepsilon$$

Sketch of the Proof

- existence of a local solution of the closed-loop system on $[0, \omega)$ (continuity of u)

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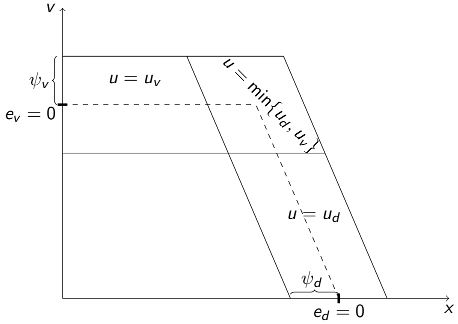
- boundedness of x and v on $[0, \omega)$ using the sets $M_d = \{t \mid u(t) = u_d(t)\}$ and $M_v = \{t \mid u(t) = u_v(t)\}$

Sketch of the Proof

- existence of a local solution of the closed-loop system on $[0, \omega)$ (continuity of u)
- boundedness of x and v on $[0, \omega)$ using the sets $M_d = \{t \mid u(t) = u_d(t)\}$ and $M_v = \{t \mid u(t) = u_v(t)\}$
- show $\psi_v(t) - e_v(t) \geq \varepsilon$ using that $u(t) \leq u_v(t)$ when $e_v(t) \geq 0$



Sketch of the Proof - Part 2

- show $\psi_d(t) - e_d(t) \geq \varepsilon$ using that $u(t) \leq u_d(t)$ when $e_d(t) \geq 0$

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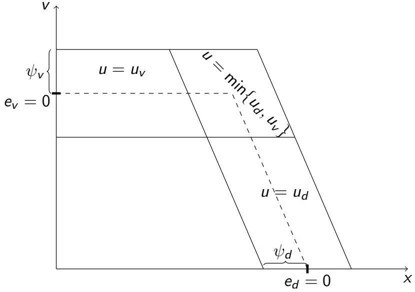
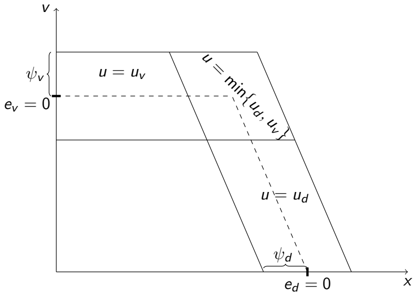
- show $\max\{0, \psi_v(t) + e_v(t)\} + \max\{0, \psi_d(t) + e_d(t)\} \geq \varepsilon$ by using that if $e_d(t^*) = -\psi_d(t^*)$, then $u(t) = u_v(t)$ in a neighborhood of t^*; if $e_v(t^*) = -\psi_v(t^*)$, then $u(t) = u_d(t)$ in a neighborhood of t^*



Sketch of the Proof - Part 2

- show $\psi_d(t) - e_d(t) \geq \varepsilon$ using that $u(t) \leq u_d(t)$ when $e_d(t) \geq 0$
- show $\max\{0, \psi_v(t) + e_v(t)\} + \max\{0, \psi_d(t) + e_d(t)\} \geq \varepsilon$ by using that if $e_d(t^*) = -\psi_d(t^*)$, then $u(t) = u_v(t)$ in a neighborhood of t^*; if $e_v(t^*) = -\psi_v(t^*)$, then $u(t) = u_d(t)$ in a neighborhood of t^*



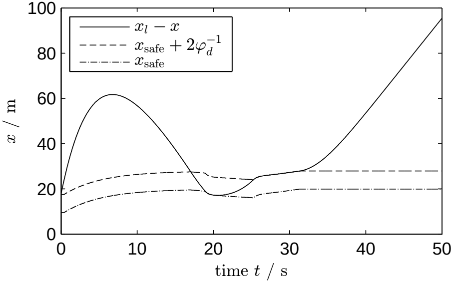
- boundedness of u and $\omega = \infty$

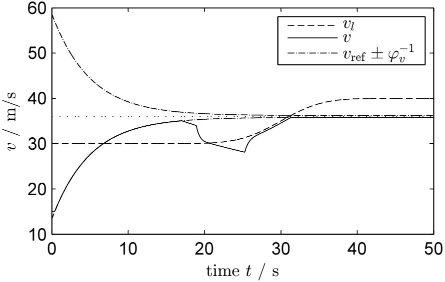
Simulation

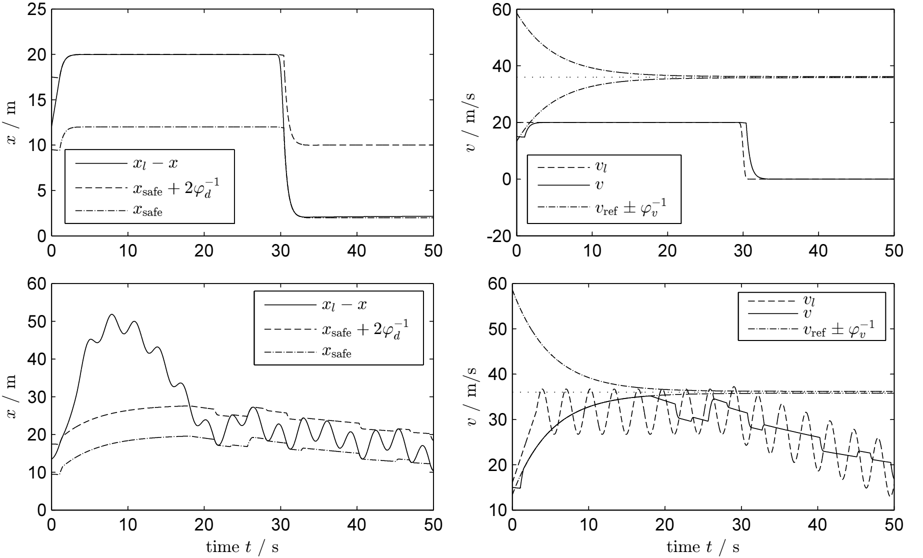
m	$\theta(t)$	$\rho(t)$	C_d	C_r	A
1300 kg	0 rad	1.3 kg/m^3	0.32	0.01	2.4 m^2

x^0	v^0	λ_1	λ_2	v_{ref}
0 m	15 m s^{-1}	0.5 s	2 m	36 m s^{-1}

$$\psi_v(t) = 22.5e^{-0.2t} + 0.2, \quad \psi_d(t) = 4.$$






25
20
15
10
5
0
0 10 20 30 40 50
x / m
$x_l - x$
$x_{\text{safe}} + 2\varphi_d^{-1}$
x_{safe}
60
40
20
0
-20
0 10 20 30 40 50
v / m/s
v_l
v
$v_{\text{ref}} \pm \varphi_v^{-1}$
60
50
40
30
20
10
0
0 10 20 30 40 50
x / m
$x_l - x$
$x_{\text{safe}} + 2\varphi_d^{-1}$
x_{safe}
time t / s
60
50
40
30
20
10
0 10 20 30 40 50
v / m/s
v_l
v
$v_{\text{ref}} \pm \varphi_v^{-1}$
time t / s