Die Quasi-Kronecker-Form für Matrizen-Büschel

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$sE - A \in \mathbb{K}^{m\times n}[s]$ regulär $\iff m = n$ und $\det(sE - A) \not\equiv 0$.

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Weierstraß-Normalform [Weierstraß 1868]:

$$\exists\, S, T \in \mathbb{C}^{n\times n} \text{ inv.}: \; S(sE - A)T = s\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}$$

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Quasi-Weierstraß-Form [B.,Ilchmann&Trenn 2011]:
$\longrightarrow$ **Wong-Sequenzen [Wong 1974]**

$$\mathcal{V}_0 := \mathbb{K}^n, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \qquad \mathcal{V}^* := \bigcap_{i \in \mathbb{N}} \mathcal{V}_i,$$

$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), \qquad \mathcal{W}^* := \bigcup_{i \in \mathbb{N}} \mathcal{W}_i.$$

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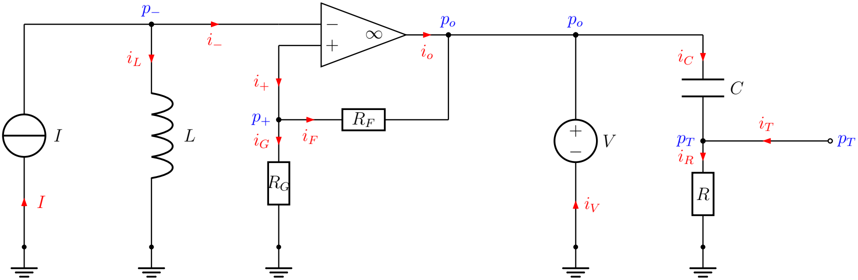
$\operatorname{im} V = \mathcal{V}^*$, $\operatorname{im} W = \mathcal{W}^*$:

$$[EV, AW]^{-1}(sE - A)[V, W] = s \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}$$

Beispiel: elektrischer Schaltkreis



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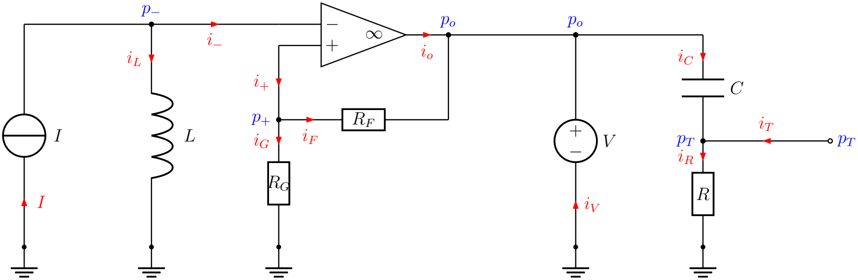


$$\longrightarrow \quad E\dot{x} = Ax + f$$

$$x = (p_+, p_-, p_o, p_T, i_L, i_p, i_m, i_G, i_F, i_R, i_o, i_V, i_C, i_T)^\top$$

$$f = Bu \text{ mit } u = (I, V)^\top$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -C & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \dot{x}
$$

$$
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & R_G & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & R_F & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} x +
\begin{bmatrix}
0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1
\end{bmatrix}
\begin{pmatrix} I \\ V \end{pmatrix}.
$$

Matrizen-Büschel $sE - A$

$$sE - A \in \mathbb{K}^{m\times n}[s], \quad \mathbb{K} = \mathbb{Q}, \mathbb{R} \text{ oder } \mathbb{C}$$

Kronecker Normalform [Kronecker 1890, Gantmacher 1959]:

$$S(sE-A)T = \operatorname{diag}(\mathcal{P}_1(s), \ldots, \mathcal{P}_P(s), \mathcal{J}_1(s), \ldots, \mathcal{J}_J(s), \\ \mathcal{N}_1(s), \ldots, \mathcal{N}_N(s), \mathcal{Q}_1(s), \ldots, \mathcal{Q}_Q(s))$$

$$\mathcal{P}_i(s) = s\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad \mathcal{J}_j(s) = sI - \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix},$$

$$\mathcal{N}_k(s) = s\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - I, \quad \mathcal{Q}_l(s) = s\begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} - \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix}.$$

$$\mathcal{P}_i(s) \leftrightarrow \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} x \quad \longrightarrow \quad \text{unterbestimmt}$$

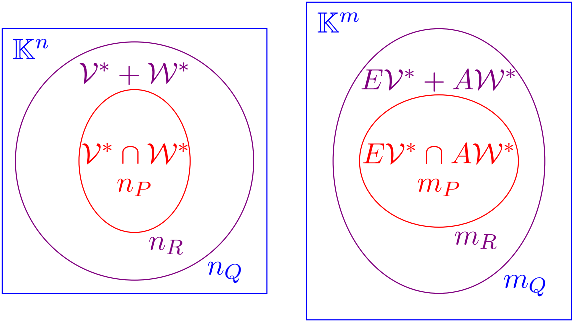
$$\left.\begin{aligned} &\mathcal{J}_j(s) \leftrightarrow \dot{x} = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} x \\ &\mathcal{N}_k(s) \leftrightarrow \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \dot{x} = x \end{aligned}\right\} \quad \longrightarrow \quad \text{regulär}$$

$$\mathcal{Q}_l(s) \leftrightarrow \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} x \quad \longrightarrow \quad \text{überbestimmt}$$

Ziel: Quasi-Kronecker-Form

$$S(sE-A)T = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccc} n_P & n_R & n_Q & \\ \left[\begin{matrix} sE_P - A_P & 0 & 0 \\ 0 & sE_R - A_R & 0 \\ 0 & 0 & sE_Q - A_Q \end{matrix}\right. & & \left.\right] & \begin{matrix} m_p \\ m_R \\ m_Q \end{matrix} \end{array}$$

Wong-Sequenzen [Wong 1974]



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$$\mathcal{W}_0 := \{0\}$$
$$\mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i)$$
$$\mathcal{W}^* := \bigcup_{i\in\mathbb{N}} \mathcal{W}_i$$

$$S(sE-A)T = \begin{array}{c} \\ \\ \\ \end{array} \overset{\begin{array}{ccc} n_P & \quad n_R & \quad n_Q \end{array}}{\begin{bmatrix} sE_P - A_P & 0 & 0 \\ 0 & sE_R - A_R & 0 \\ 0 & 0 & sE_Q - A_Q \end{bmatrix}} \begin{array}{c} m_p \\ m_R \\ m_Q \end{array}$$

Quasi-Kronecker-Dreiecks-Form [B.&Trenn 2011]

$$\begin{aligned} \operatorname{im} P_1 &= \mathcal{V}^* \cap \mathcal{W}^* & \operatorname{im} P_2 &= E\mathcal{V}^* \cap A\mathcal{W}^* \\ \mathcal{V}^* \cap \mathcal{W}^* \oplus \operatorname{im} R_1 &= \mathcal{V}^* + \mathcal{W}^* & E\mathcal{V}^* \cap A\mathcal{W}^* \oplus \operatorname{im} R_2 &= E\mathcal{V}^* + A\mathcal{W}^* \\ (\mathcal{V}^* + \mathcal{W}^*) \oplus \operatorname{im} Q_1 &= \mathbb{K}^n & (E\mathcal{V}^* + A\mathcal{W}^*) \oplus \operatorname{im} Q_2 &= \mathbb{K}^m \end{aligned}$$

$$[P_2, R_2, Q_2]^{-1}(sE - A)[P_1, R_1, Q_1] \\ = s \begin{bmatrix} E_P & E_{PR} & E_{PQ} \\ 0 & E_R & E_{RQ} \\ 0 & 0 & E_Q \end{bmatrix} - \begin{bmatrix} A_P & A_{PR} & A_{PQ} \\ 0 & A_R & A_{RQ} \\ 0 & 0 & A_Q \end{bmatrix}$$

- $E_P, A_P \in \mathbb{K}^{m_P \times n_P}$, $m_P < n_P$,
  $\forall\, \lambda \in \mathbb{C} \cup \{\infty\} :\ \operatorname{rk}_{\mathbb{C}}(\lambda E_P - A_P) = m_P$
- $E_R, A_R \in \mathbb{K}^{m_R \times n_R}$, $m_R = n_R$, $sE_R - A_R$ ist regulär
- $E_Q, A_Q \in \mathbb{K}^{m_Q \times n_Q}$, $m_Q > n_Q$,
  $\forall\, \lambda \in \mathbb{C} \cup \{\infty\} :\ \operatorname{rk}_{\mathbb{C}}(\lambda E_Q - A_Q) = n_Q$

$$[P_1, R_1, Q_1] = \left[\begin{array}{ccc|cccccccccc|c}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
R & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right], \qquad K := \frac{R_G + R_F}{R_G},$$

$$[P_2, R_2, Q_2] = \left[\begin{array}{c|cccccccccc|cc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -K & -\frac{R_F}{R_G} & 1 & 0 & 0 & -R_F & K & R_F & 0 & 0 & 0 & 1
\end{array}\right]$$

$$
sE - A \cong s \begin{bmatrix}
CR & 0 & 0 & 0 & 0 & -C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & LK
\end{bmatrix} - \begin{bmatrix}
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & R_G & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & R_F & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Quasi-Kronecker-~~Dreiecks~~-Form [B.&Trenn 2011]

$$S\left(s\begin{bmatrix}E_P & E_{PR} & E_{PQ}\\ 0 & E_R & E_{RQ}\\ 0 & 0 & E_Q\end{bmatrix}-\begin{bmatrix}A_P & A_{PR} & A_{PQ}\\ 0 & A_R & A_{RQ}\\ 0 & 0 & A_Q\end{bmatrix}\right)T$$

$$= s\begin{bmatrix}E_P & 0 & 0\\ 0 & E_R & 0\\ 0 & 0 & E_Q\end{bmatrix}-\begin{bmatrix}A_P & 0 & 0\\ 0 & A_R & 0\\ 0 & 0 & A_Q\end{bmatrix}$$

Quasi-Kronecker-~~Dreiecks~~-Form [B.&Trenn 2011]

$$S\left(s\begin{bmatrix}E_P & E_{PR} & E_{PQ}\\ 0 & E_R & E_{RQ}\\ 0 & 0 & E_Q\end{bmatrix} - \begin{bmatrix}A_P & A_{PR} & A_{PQ}\\ 0 & A_R & A_{RQ}\\ 0 & 0 & A_Q\end{bmatrix}\right)T$$

$$= s\begin{bmatrix}E_P & 0 & 0\\ 0 & E_R & 0\\ 0 & 0 & E_Q\end{bmatrix} - \begin{bmatrix}A_P & 0 & 0\\ 0 & A_R & 0\\ 0 & 0 & A_Q\end{bmatrix}$$

$$S := \begin{bmatrix}I & -G_2 & -H_2\\ 0 & I & -F_2\\ 0 & 0 & I\end{bmatrix}^{-1}$$

$$T := \begin{bmatrix}I & G_1 & H_1\\ 0 & I & F_1\\ 0 & 0 & I\end{bmatrix}$$

$$0 = E_{RQ} + E_R F_1 + F_2 E_Q$$
$$0 = A_{RQ} + A_R F_1 + F_2 A_Q$$

$$0 = E_{PR} + E_P G_1 + G_2 E_R$$
$$0 = A_{PR} + A_P G_1 + G_2 A_R$$

$$0 = (E_{PQ} + E_{PR} F_1) + E_P H_1 + H_2 E_Q$$
$$0 = (A_{PQ} + A_{PR} F_1) + A_P H_1 + H_2 A_Q$$

$$0 = M + PX + YQ$$
$$0 = R + SX + YT$$

$$0 = M + PX + YQ$$
$$0 = R + SX + YT$$

$$\begin{bmatrix} I \otimes P & Q^\top \otimes I \\ I \otimes S & T^\top \otimes I \end{bmatrix} \begin{pmatrix} \operatorname{vec}(X) \\ \operatorname{vec}(Y) \end{pmatrix} = - \begin{pmatrix} \operatorname{vec}(M) \\ \operatorname{vec}(R) \end{pmatrix}$$

$$F_1 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix},\quad F_2 = \begin{bmatrix} 0 & -\frac{1}{K}\\ 0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0 \end{bmatrix},\quad G_1 = \begin{bmatrix} 0&0&\frac{1}{R}&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix},\quad H_1 = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

$$G_2 = \begin{bmatrix} \frac{-1}{R_G} & \frac{-1}{R_G} & 0 & 0 & 0 & -1 & \frac{1}{R_G} & 1 & -1 & 0 \end{bmatrix},\quad H_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

$$T = \begin{bmatrix}
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
R&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
1&0&0&0&0&\frac{1}{R}&0&0&0&0&1&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&1&0&0&0&0&0&0&0&0&1&0&0\\
-1&1&-1&0&0&\frac{-1}{R}&0&0&0&0&0&0&1&0
\end{bmatrix},\quad S = \begin{bmatrix}
0&1&\frac{-1}{R_GK}&\frac{-1}{R_GK}&0&0&\frac{-1}{K}&0&0&\frac{1}{K}&-1&0&\frac{1}{R_GK}\\
0&0&\frac{-R_F}{R_GK}&\frac{1}{K}&0&0&\frac{-R_F}{K}&1&0&\frac{R_F}{K}&0&0&\frac{-1}{K}\\
0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0\\
K&0&\frac{R_F}{R_G}&-1&0&0&R_F&-K&0&-R_F&0&0&1
\end{bmatrix}.$$

$$
sE - A \cong s \begin{bmatrix}
CR & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & LK
\end{bmatrix}
- \begin{bmatrix}
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & R_G & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & R_F & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Char. der Lsg. der DAE [B.&Trenn 2011]

$$S(sE-A)T = \begin{bmatrix} sE_P - A_P & 0 & 0 & 0 \\ 0 & sI-J & 0 & 0 \\ 0 & 0 & sN-I & 0 \\ 0 & 0 & 0 & sE_Q - A_Q \end{bmatrix}$$

$$(sE_P - A_P)[M_P(s), K_P(s)] = [I, 0] \quad \text{und} \quad \begin{bmatrix} M_Q(s) \\ K_Q(s) \end{bmatrix} (sE_Q - A_Q) = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

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- $f \in \mathcal{C}^\infty$: $\exists$ Lsg. x von $E\dot{x} = Ax + f \iff K_Q(\frac{\mathsf{d}}{\mathsf{d}t})(f_Q) = 0,\ f_Q = [0,0,0,I]Sf$

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- $\exists$ Lsg. x des AWP $E\dot{x} = Ax + f$, $x(0) = x^0 \iff$

$$x_Q^0 = \left(M_Q(\tfrac{\mathrm{d}}{\mathrm{d}t})(f_Q)\right)(0) \quad \text{und} \quad x_N^0 = -\left(\sum_{k=0}^{n_N-1} N^k (\tfrac{\mathrm{d}}{\mathrm{d}t})^k (f_N)\right)(0)$$

Char. der Lsg. der DAE [B.&Trenn 2011]

$$S(sE-A)T = \begin{bmatrix} sE_P - A_P & 0 & 0 & 0 \\ 0 & sI - J & 0 & 0 \\ 0 & 0 & sN - I & 0 \\ 0 & 0 & 0 & sE_Q - A_Q \end{bmatrix}$$

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- $$T^{-1}x = \begin{pmatrix} x_P \\ x_J \\ x_N \\ x_Q \end{pmatrix} = \begin{pmatrix} M_P(\frac{\mathrm{d}}{\mathrm{d}t})(f_P) + K_P(\frac{\mathrm{d}}{\mathrm{d}t})(u_{x_P^0}) \\ e^{J\cdot} x_J^0 + e^{J\cdot} \int_0^{\cdot} e^{-Js} f_J(s)\,\mathrm{d}s \\ -\sum_{k=0}^{n_N-1} N^k (\frac{\mathrm{d}}{\mathrm{d}t})^k (f_N) \\ M_Q(\frac{\mathrm{d}}{\mathrm{d}t})(f_Q) \end{pmatrix}$$

$$[M_P(s), K_P(s)] = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & CRs & 1 \end{array}\right] \quad \text{und} \quad \begin{bmatrix} M_Q(s) \\ K_Q(s) \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ \hline -LKs & 1 \end{array}\right]$$

$f_P = \frac{V}{R_G + R_F}$, $f_J = []$,
$f_N = \frac{V}{K}[-1, -1, -K, 0, 0, -\frac{1}{R_G}, -\frac{1}{R_G}, 0, -\frac{1}{R_G}, \frac{1}{R_G}]^\top$, $f_Q = [I, V]^\top$

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- $\exists$ Lösungen $\iff$

$$0 = K_Q(\tfrac{\mathrm{d}}{\mathrm{d}t})(f_Q) = -LK\tfrac{\mathrm{d}}{\mathrm{d}t}I + V \quad \text{bzw.} \quad \boxed{V = LK\tfrac{\mathrm{d}}{\mathrm{d}t}I},$$

$$[M_P(s), K_P(s)] = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & CRs & 1 \end{array}\right] \quad \text{und} \quad \begin{bmatrix} M_Q(s) \\ K_Q(s) \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ \hline -LKs & 1 \end{array}\right]$$

$f_P = \frac{V}{R_G+R_F}$, $f_J = []$,
$f_N = \frac{V}{K}[-1,-1,-K,0,0,-\frac{1}{R_G},-\frac{1}{R_G},0,-\frac{1}{R_G},\frac{1}{R_G}]^\top$, $f_Q = [I,V]^\top$

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- x^0 ist konsistent $\iff$

$$x^0 = T[*,*,*,-f_N(0)^\top, M_Q(\tfrac{\mathrm{d}}{\mathrm{d}t})(f_Q)(0)]^\top,$$

$$[M_P(s), K_P(s)] = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & CRs & 1 \end{array}\right] \quad \text{und} \quad \begin{bmatrix} M_Q(s) \\ K_Q(s) \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ \hline -LKs & 1 \end{array}\right]$$

$f_P = \frac{V}{R_G+R_F}$, $f_J = []$,
$f_N = \frac{V}{K}[-1, -1, -K, 0, 0, -\frac{1}{R_G}, -\frac{1}{R_G}, 0, -\frac{1}{R_G}, \frac{1}{R_G}]^\top$, $f_Q = [I, V]^\top$

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- jede Lsg. x hat die Form

$$x = T[u_1, u_2, \tfrac{-V}{R_G+R_F} + RC\dot{u}_1 + u_2 \mid \tfrac{V}{K}, \tfrac{V}{K}, V, 0, 0, \tfrac{V}{R_F+R_G}, \tfrac{V}{R_F+R_G}, 0, \tfrac{V}{R_F+R_G}, \tfrac{-V}{R_F+R_G} \mid I]^\top$$