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Funnel control for electrical circuits

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System class

We consider passive linear time-invariant RLC circuits which are modelled by modified nodal analysis as a differential-algebraic system of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \qquad (1)$$

where

$$sE-A=\begin{bmatrix} sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top & A_{\mathcal{L}} & A_{\mathcal{V}} \\ -A_{\mathcal{L}}^\top & s\mathcal{L} & 0 \\ -A_{\mathcal{V}}^\top & 0 & 0 \end{bmatrix}, \quad B=C^\top=\begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I_{n_{\mathcal{V}}} \end{bmatrix},$$

$$x = (\eta^\top, i_{\mathcal{L}}^\top, i_{\mathcal{V}}^\top)^\top, \quad u = (i_{\mathcal{I}}^\top, v_{\mathcal{V}}^\top)^\top, \quad y = (-v_{\mathcal{I}}^\top, -i_{\mathcal{V}}^\top)^\top,$$

and

$\mathcal{C} \in \mathbb{R}^{n_{\mathcal{C}},n_{\mathcal{C}}}$, $\mathcal{G} \in \mathbb{R}^{n_{\mathcal{G}},n_{\mathcal{G}}}$, $\mathcal{L} \in \mathbb{R}^{n_{\mathcal{L}},n_{\mathcal{L}}}$,
$A_{\mathcal{C}} \in \mathbb{R}^{n_e,n_{\mathcal{C}}}$, $A_{\mathcal{R}} \in \mathbb{R}^{n_e,n_{\mathcal{G}}}$, $A_{\mathcal{L}} \in \mathbb{R}^{n_e,n_{\mathcal{L}}}$, $A_{\mathcal{V}} \in \mathbb{R}^{n_e,n_{\mathcal{V}}}$, $A_{\mathcal{I}} \in \mathbb{R}^{n_e,n_{\mathcal{I}}}$,
$n = n_e + n_{\mathcal{L}} + n_{\mathcal{V}}$, $\quad m = n_{\mathcal{I}} + n_{\mathcal{V}}$.

$A_{\mathcal{C}}, A_{\mathcal{R}}, A_{\mathcal{L}}, A_{\mathcal{V}}, A_{\mathcal{I}}$ — element-related incidence matrices
$\mathcal{C}, \mathcal{G}, \mathcal{L}$ — consecutive relations of capacitances, resistances and inductances
$\eta(t)$ — vector of node potentials
$i_{\mathcal{L}}(t), i_{\mathcal{V}}(t), i_{\mathcal{I}}(t)$ — vectors of currents through inductances, voltage and current sources
$v_{\mathcal{V}}(t), v_{\mathcal{I}}(t)$ — voltages of voltage and current sources

Passivity of the system implies

$$\mathcal{C} = \mathcal{C}^\top > 0, \mathcal{L} = \mathcal{L}^\top > 0, \mathcal{G} + \mathcal{G}^\top > 0.$$

Control objective

- $\varphi \in \mathcal{C}^\infty(\mathbb{R}_{\ge 0};\mathbb{R})$ s.t. $\varphi, \tfrac{\mathrm{d}}{\mathrm{d}t}\varphi$ are bounded, $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\liminf_{t\to\infty}\varphi(t) > 0$ (the set of these functions is denoted by Φ);
- $y_{\mathrm{ref}} \in \mathcal{B}^\infty(\mathbb{R}_{\ge 0};\mathcal{S})$, i.e., $y_{\mathrm{ref}} \in \mathcal{C}^\infty(\mathbb{R}_{\ge 0};\mathcal{S})$ and $y_{\mathrm{ref}}^{(k)}$ is bounded for all $k \ge 0$; $\mathcal{S} \subseteq \mathbb{R}^m$ is a subspace.

We seek that the tracking error $e = y - y_{\mathrm{ref}}$ evolves within the performance funnel $\mathcal{F}_\varphi := \{ (t,e) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1 \}$. To ensure error evolution within the funnel, we introduce the *funnel controller*:

$$u(t) = -k(t)\,e(t), \quad k(t) = \frac{1}{1-\varphi(t)^2\|e(t)\|^2}. \qquad (2)$$

We call $\lambda \in \mathbb{C}$ an *invariant zero* of (1) if

$$\mathrm{rk}_{\mathbb{C}}\begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} < \mathrm{rk}_{\mathbb{R}(s)}\begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}.$$

Main result

Theorem. Assume that all invariant zeros of (1) are in $\mathbb{C}_-$. Let $Z_{\mathcal{CRLI}}$ be a matrix with full column rank such that

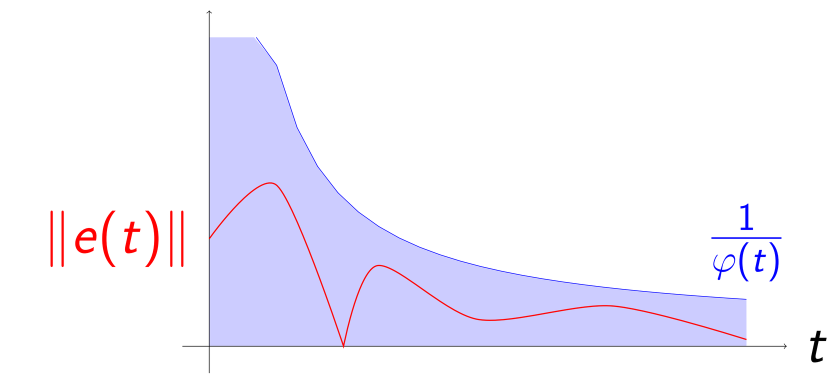
$$\mathrm{im}\, Z_{\mathcal{CRLI}} = \ker \left[A_{\mathcal{C}}\ A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{I}}\right]^\top$$

and let y_{ref} be a reference trajectory satisfying

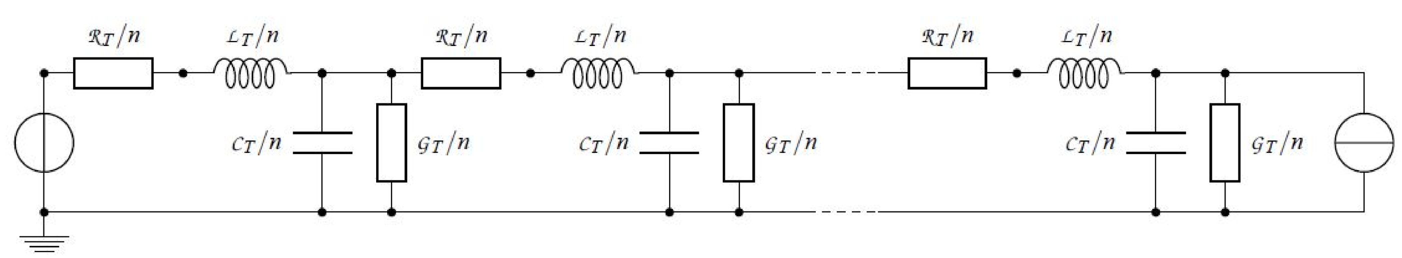
$$y_{\mathrm{ref}} \in \mathcal{B}^\infty\left(\mathbb{R}_{\ge 0}; \mathrm{im}\, A_{\mathcal{I}}^\top \times \ker Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}}\right).$$

Let $\varphi \in \Phi$ and $x^0 \in \mathbb{R}^n$ be a consistent initial value. Then (2) applied to (1) has a global solution $x \in L^\infty$, $k \in L^\infty$ such that

$$\exists\, \varepsilon > 0\ \forall\, t > 0: \quad \|e(t)\| \le \varphi(t)^{-1} - \varepsilon.$$

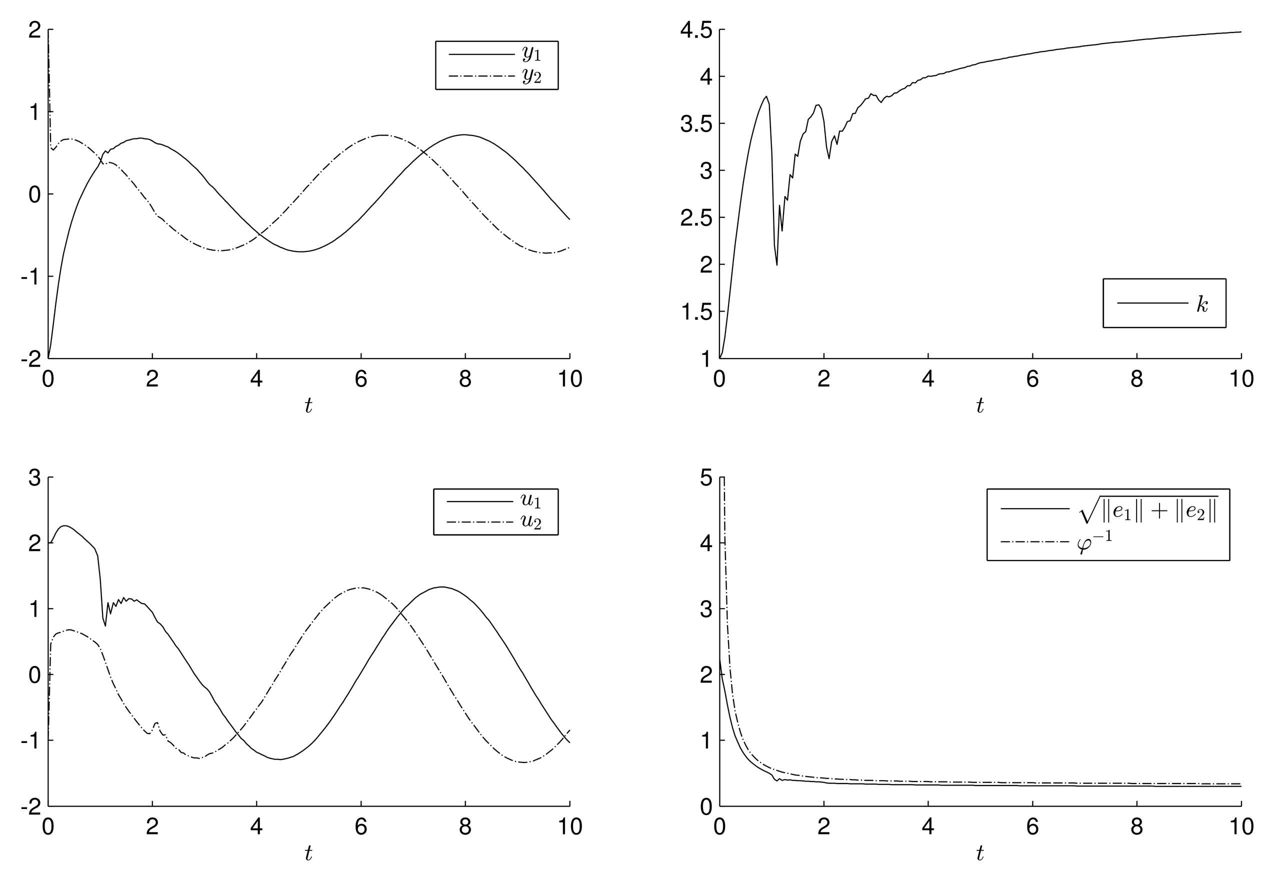


Simulation



$$n = 50, \quad \mathcal{C}_T = \mathcal{R}_T = \mathcal{G}_T = \mathcal{L}_T = 1, \quad y_{\mathrm{ref}} = (\sin, \cos)^\top$$

$$\varphi: \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}, \quad t \mapsto 0.5\ te^{-t} + 2\ \arctan t$$



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- Education
  - PhD in mathematics, TU Ilmenau
  - Research stay in London
  - PostDoc at U Hamburg
- Research Interests
  - Control of differential-algebraic equations
  - Analysis and control of electrical circuits

Reference: T. Berger and T. Reis, Zero dynamics and funnel control for linear electrical circuits, Hamburger Beiträge zur Angewandten Mathematik 2013-03
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