Single element changes in electrical networks

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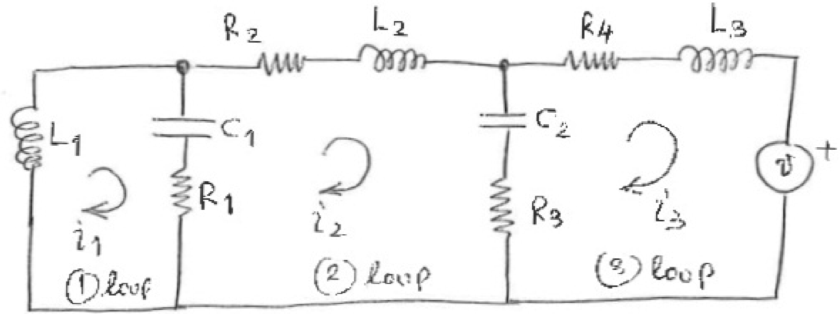
Example: electrical RLC network



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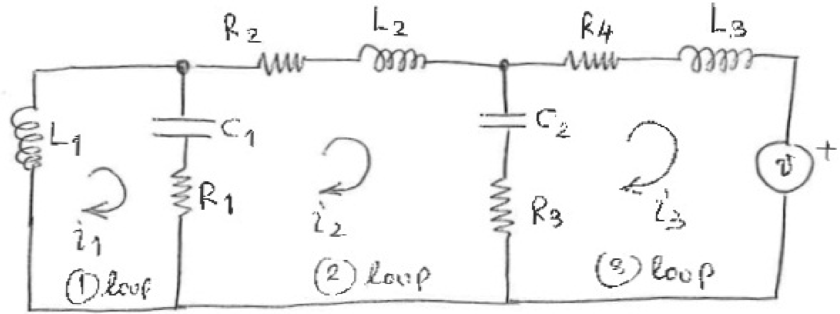

$$Z(s)i(s) = v_s(s),$$

$$Z(s) = s^{-1}\begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1}+\frac{1}{C_2} & -\frac{1}{C_2} \\ 0 & -\frac{1}{C_2} & \frac{1}{C_2} \end{bmatrix} + \begin{bmatrix} R_1 & -R_1 & 0 \\ -R_1 & R_1+R_2+R_3 & -R_3 \\ 0 & -R_3 & R_3+R_4 \end{bmatrix} + s\begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix}$$

RLC networks: impedance/admittance modeling

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impedance modeling (loop analysis):

- L: mass, spring, inductor
- C: inertor, capacitor
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- element present in i-th loop (node) $\Rightarrow$ its value is added to (i,i) position of the respective matrix
- element common to i-th and j-th loop (node) $\Rightarrow$ its value is added to (i,i) and (j,j) positions, substracted from (i,j) and (j,i) positions

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special cases: RC and RL networks $\rightarrow W(s) = sL + s^{-1}C + R$ becomes symmetric matrix pencil:

$$W(s) = \left\{ \begin{array}{l} \text{RL: } sL + R \\ \text{RC: } \hat{s}C + R,\ \hat{s} = s^{-1} \end{array} \right\} = sF + G, \quad F = F^\top, G = G^\top$$

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$$W(s) = s \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix} + \begin{bmatrix} R_1 & -R_1 & 0 \\ -R_1 & R_1+R_2+R_3 & -R_3 \\ 0 & -R_3 & R_3+R_4 \end{bmatrix}$$

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here: $\boxed{\det\left(s(F + xbb^\top) + G\right)}$ $\quad b = e_i$ or $b = e_i - e_j, \ i \neq j$

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$$\det W(s) = 0 \quad \iff \quad 1 + L_4 \frac{sz(s)}{p(s)} = 0$$

Reformulation as root locus problem [Binet-Cauchy-Theorem]

$$\begin{aligned}\det\left(s(F+xbb^\top)+G\right) &= \det\left([sF+G, I_k]\begin{bmatrix} I_k \\ sxbb^\top \end{bmatrix}\right) \\ &= g(s;F,G)^\top p(sx;b), \qquad x\in\mathbb{R}\end{aligned}$$

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root locus problem: $\boxed{1 + x\,\frac{s\, z(s; b)}{\det(sF + G)} = 0}$

Theorem (Properties of the root locus)

$$\boxed{1 + x\,\frac{s\,z(s;b)}{\det(sF+G)} = 0} \quad x \in \mathbb{R}, \qquad \textbf{Assumption}: G \text{ is invertible}$$

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- $\exists\, p < 0 \;\exists\, \mu \in \mathbb{N}, \mu \geq 2: \;\det(sF+G) = \eta(s)(s-p)^{\mu} \implies z(s;b) = \zeta(s)(s-p)^{\mu-1}$

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$$F + xbb^\top = \begin{bmatrix} C_1 + x & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1/R_1 & 0 \\ 0 & 1/R_2 \end{bmatrix}$$

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$$p(x) = -1/(R_1(C_1 + x)) :$$

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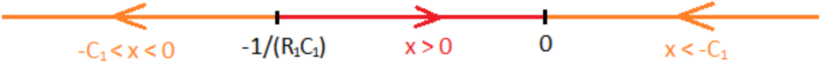


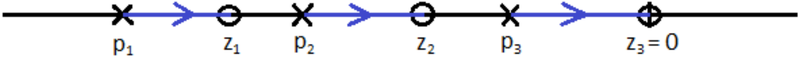
$$\lim_{x \searrow -C_1} p(x) = -\infty \qquad \lim_{x\to\infty} p(x) = 0 \qquad \lim_{x \nearrow -C_1} p(x) = \infty$$

"$p(-C_1) = \infty$" "$p(-C_1) = \infty$"

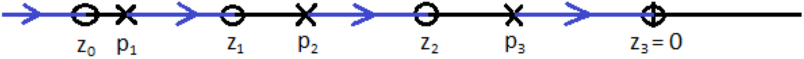
Theorem (Behaviour of the root locus)

$x > 0$:

- p pole of multiplicity $\mu \implies \mu$ or $\mu - 1$ of these poles do not change and at most one pole moves to the right
- $\exists$ pole $p(x)$ s.t. $\lim_{x\to\infty} p(x) = 0$
- if $\operatorname{rk}(F + bb^\top) = \operatorname{rk} F$:



- if $\operatorname{rk}(F + bb^\top) = \operatorname{rk} F + 1$, then an infinite pole becomes finite and moves to the right as x increases



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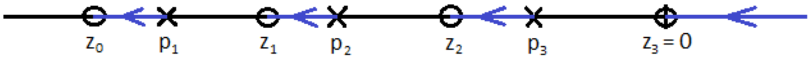
$$\det(sF+G) + x\, sz(s;b) = (sC_1 + 1/R_1)(sx + 1/R_2)$$

$$\det(sF+G) = 1/R_2(sC_1 + 1/R_1) \implies p_1 = -1/(C_1R_1)$$

$$sz(s) = sx(sC_1 + 1/R_1) \implies z_1 = -1/(C_1R_1),\ z_2 = 0$$

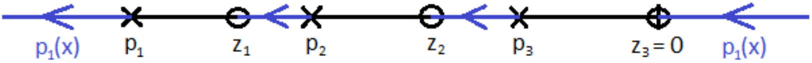
$x < 0$:

- p pole of multiplicity $\mu \implies \mu$ or $\mu - 1$ of these poles do not change and at most one pole moves to the left
- if $\operatorname{rk}(F + bb^\top) = \operatorname{rk} F + 1$, then an infinite pole becomes finite and moves to the left on the positive real axis as x decreases, reaching 0 for $x \to -\infty$



- if $\operatorname{rk}(F + bb^\top) = \operatorname{rk} F$, then the smallest pole $p_1(x)$ moves to the left towards $-\infty$ and

$$\exists\ \kappa > 0 : \quad \forall - \kappa < x < 0 : \ p_1(x) < p_1(0) \ \wedge \ \lim_{x \searrow -\kappa} p_1(x) = -\infty,$$
$$\text{“}p(-\kappa) = \infty\text{”},$$
$$\forall\, x < -\kappa : \ p_1(x) > 0 \ \wedge \ \lim_{x \to -\infty} p_1(x) = 0$$



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$x < 0$:
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- if $\mathrm{rk}(F + bb^\top) = \mathrm{rk}\, F + 1$, then an infinite pole becomes finite
- if $\mathrm{rk}(F + bb^\top) = \mathrm{rk}\, F$, then a finite pole becomes infinite for a single value of x