Zero dynamics and funnel control of linear DAEs

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Exeter, 17 November 2010

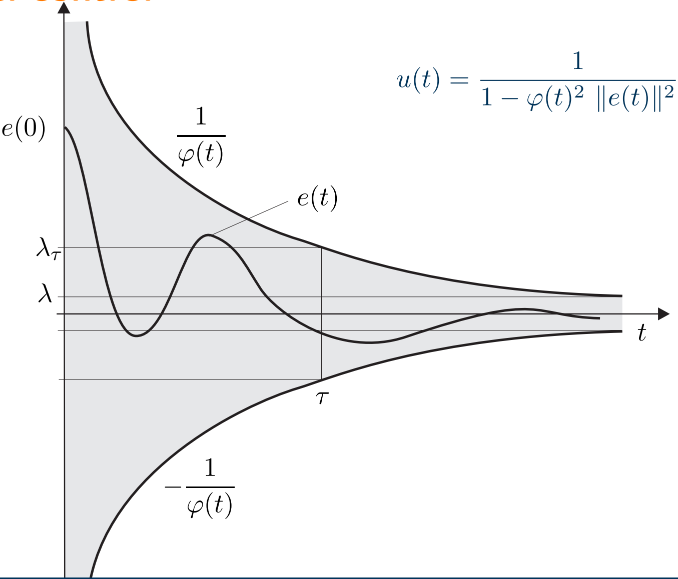
Systems class: (E, A, B, C)

$$
\boxed{\begin{aligned} E\ \dot{x}(t) &= A\ x(t) + B\ u(t) \\ y(t) &= C\ x(t) \end{aligned}}
\qquad
\begin{aligned} & E, A \in \mathbb{R}^{n \times n} \\ & B, C^\top \in \mathbb{R}^{n \times m} \end{aligned}
$$

Regularity: $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$

Solutions: $(x, u, y) \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m)$

Funnel control



$$u(t) = \frac{1}{1 - \varphi(t)^2 \ \|e(t)\|^2} \, e(t)$$

Zero dynamics

Def

$$\mathcal{ZD}_{(E,A,B,C)} := \left\{ (x,u,y) \;\middle|\; \begin{array}{l} (x,u,y) \text{ solves } \text{(E,A,B,C)} \\ \text{and} \quad y \equiv 0 \end{array} \right\}$$

Zero dynamics

Def

$$\mathcal{ZD}_{(E,A,B,C)} := \left\{ (x,u,y) \;\middle|\; \begin{array}{l} (x,u,y) \text{ solves } \text{(E,A,B,C)} \\ \text{and} \quad y \equiv 0 \end{array} \right\}$$

Transfer function $\quad G(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)^{m\times m}$

has proper inverse: $\quad \lim_{s\to\infty} G(s)^{-1} \in \mathbb{R}^{m\times m}$

Theorem: Zero Dynamics Form

$$\boxed{\begin{array}{rcl} E\,\dot x(t) &=& A\,x(t) + B\,u(t) \\ y(t) &=& C\,x(t) \end{array}} \quad \& \quad C(sE-A)^{-1}B \text{ has proper inverse}$$

$\Longrightarrow$

$$\exists\; S,T \in \mathbf{Gl}_n(\mathbb{R}) : \begin{pmatrix} y(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = T^{-1}\,x(t)$$

solves $(SET, SAT, SB, CT) \sim$

$$\boxed{\begin{array}{rcl} 0 &=& A_{11}\,y(t) + A_{12}\,x_2(t) + u(t) \\ \dot x_2(t) &=& Q\,x_2(t) + A_{21}\,y(t) \\ x_3(t) &=& \sum_{i=0}^{\nu-1} N_{33}^i\,E_{31}\,y^{(i+1)}(t) \\ x_4(t) &=& 0 \end{array}}$$

unique: $\dim x_i,\; A_{11}$

unique mod similarity: $Q,\; N_{33}$

Stable zero dynamics

Def

(E, A, B, C) *has stable zero dynamics* $:\Longleftrightarrow$

$$(x, u, y) \in \mathcal{ZD}_{(E,A,B,C)} \quad \Longrightarrow \quad (x(t), u(t)) \to 0$$

Prop

$$(E, A, B, C) \text{ has stable zero dynamics} \quad \Longleftrightarrow \quad \sigma(Q) \subset \mathbb{C}_-$$

Proof

$$\begin{array}{rcl} 0 &=& A_{11}\; y(t) + A_{12}\; x_2(t) + u(t) \\ \dot{x}_2(t) &=& Q\; x_2(t) + A_{21}\; y(t) \\ x_3(t) &=& \sum_{i=0}^{\nu-1} N_{33}^i\; E_{31}\; y^{(i+1)}(t) \\ x_4(t) &=& 0 \end{array}$$

Characterization of stable zero dynamics

Theorem

(E, A, B, C) has stable zero dynamics

$\iff$

$$\forall\, s \in \overline{\mathbb{C}}_+ : \ \det \begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix} \neq 0$$

$\iff$

(i) (E, A, B, C) is stabilizable $(\forall s \in \overline{\mathbb{C}}_+ : \ \mathrm{rk}[sE - A, B] = n)$

(ii) (E, A, B, C) is detectable $(\forall s \in \overline{\mathbb{C}}_+ : \ \mathrm{rk}[sE^\top - A^\top, C^\top] = n)$

(iii) $U(s)^{-1}\, C(sE - A)^{-1} B\, V(s)^{-1} = \mathrm{diag}\left\{ \frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_m(s)}{\psi_m(s)} \right\}$

$$\forall s \in \overline{\mathbb{C}}_+ : \ \epsilon_i(s) \neq 0$$

$\iff$

$\exists\, k^* \geq 0 \quad \forall\, k \in \mathbb{R}$ s.t. $|k| \geq k^*$: $\quad \lim_{t\to\infty} x(t) = 0$

where $x(\cdot)$ solves '$u(t) = ky(t)$ & (E, A, B, C)'

High-gain stabilization: sketch of proof

$$\begin{aligned} 0 &= A_{11}\ y(t) + A_{12}\ x_2(t) + u(t) \\ \dot{x}_2(t) &= Q\ x_2(t) + A_{21}\ y(t) \\ x_3(t) &= \textstyle\sum_{i=0}^{\nu-1} N_{33}^i\ E_{31}\ y^{(i+1)}(t) \end{aligned}$$

$$\overset{u(t) = k\,y(t)}{\Longrightarrow}$$

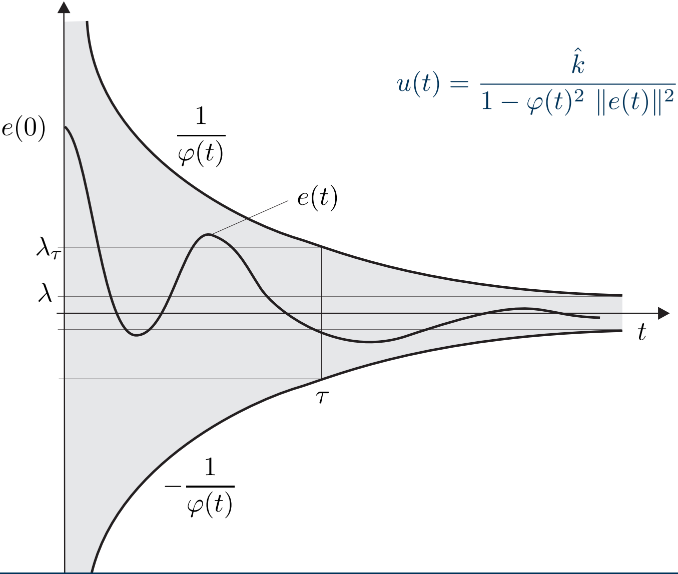
$$\begin{aligned} -(A_{11} + kI_m)\,y(t) &= A_{12}x_2(t) \\ \dot{x}_2(t) &= Qx_2(t) + A_{21}y(t) \\ x_3(t) &= \textstyle\sum_{i=0}^{\nu-1} N_{33}^i\ E_{31}\,y^{(i+1)}(t) \end{aligned}$$

$$\overset{|k| > \|A_{11}\|}{\Longrightarrow}$$

$$\begin{aligned} \dot{x}_2(t) &= \left[Q - A_{21}(A_{11} + kI_m)^{-1}A_{12}\right]\ x_2(t) \overset{|k|\gg 1}{\approx} Q\,x_2(t) \\ x_3(t) &= \textstyle\sum_{i=0}^{\nu-1} N_{33}^i\ E_{31}\,y^{(i+1)}(t) \end{aligned}$$

□

Funnel



$$u(t) = \frac{\hat{k}}{1 - \varphi(t)^2 \, \|e(t)\|^2} \, e(t)$$

Theorem: Funnel control

Suppose: (E, A, B, C) has stable zero dynamics
and $G(s) = C(sE - A)^{-1}B$ has proper inverse.

Then the funnel controller

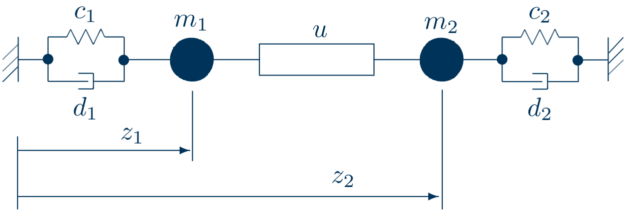
$$\begin{array}{|rcl|} \hline u(t) & = & k(t)\, e(t) \\ k(t) & = & \dfrac{\hat{k}}{1 - \varphi(t)^2 \; \|e(t)\|^2} \\ \hline \end{array}$$

$$\begin{aligned} e(t) &= y(t) - y_{\mathrm{ref}}(t) \\ |\hat{k}| &> \lim_{s\to\infty} \|G^{-1}(s)\| \end{aligned}$$

applied to (E, A, B, C) yields:

$$x(\cdot) \in L^\infty,\ k(\cdot) \in L^\infty \qquad \wedge \qquad \exists\, \varepsilon > 0\ \forall\, t \geq 0:\ \|e(t)\| < \frac{1}{\varphi(t)} - \varepsilon\,.$$

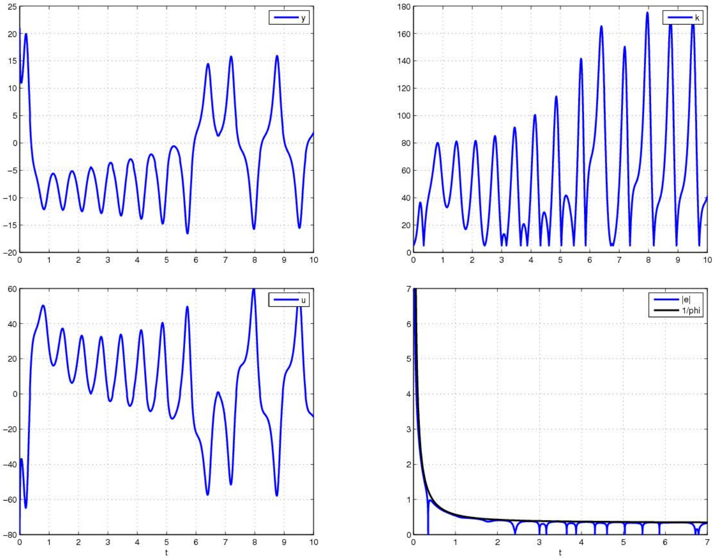
Mass-spring-damper system



$$
\begin{aligned}
m_1 \ddot{z}_1(t) + d_1 \dot{z}_1(t) + c_1 z_1(t) - \lambda(t) &= 0 \\
m_2 \ddot{z}_2(t) + d_2 \dot{z}_2(t) + c_2 z_2(t) + \lambda(t) &= 0 \\
z_2(t) - z_1(t) &= u(t) \\
y(t) &= z_2(t),
\end{aligned}
\tag{1}
$$

Defining $x(t) = (z_1(t), \dot{z}_1(t), z_2(t), \dot{z}_2(t), \lambda(t))^\top$, the model (1) may be rewritten as a linear differential-algebraic input-output system.

Simulations



Def: *non-positive strict relative degree*

$$\mathrm{sr\,deg}\, G(s) = \sup \left\{ k \in \mathbb{Z} \,\middle|\, \lim_{s\to\infty} s^k\, G(s) \in \mathbf{Gl}_m(\mathbb{R}) \right\} \le 0 \quad \text{exists}$$

Strict relative degree and proper inverse

Prop

$$\operatorname{sr\,deg} G(s) \le 0 \quad \underset{\text{i.g.}}{\overset{\Longrightarrow}{\not\Longleftarrow}} \quad G(s) \text{ has 'proper inverse'}$$

Proof

$\rho \le 0$ is largest integer : $\lim_{s\to\infty} s^\rho G(s) \in \mathbf{Gl}_m(\mathbb{R})$

$\Longrightarrow$

$$G(s) = P(s) + G_{\rm sp}(s), \qquad P(s) = \sum_{i=0}^N P_i s^i$$

$\Longrightarrow$

$$\operatorname{sr\,deg} G(s) = -\deg P(s), \quad P_N \in \mathbf{Gl}_m(\mathbb{R})$$

$\Longrightarrow$

$\exists\ P^{-1}(s)$

$\Longrightarrow$

Sherman-Morrison-Woodbury formula:

$$G^{-1}(s) = P^{-1}(s) - P^{-1}(s) G_{\rm sp}(s) \left[I + P^{-1}(s) G_{\rm sp}(s) \right]^{-1} P^{-1}(s)$$

$\Longrightarrow$

$\exists\ G^{-1}(s)$ and $G^{-1}(s)$ proper $\square$