

On perturbations in the leading coefficient matrix of time-varying index-1 DAEs

Thomas Berger

Institute of Mathematics, Ilmenau University of Technology

London, February 14, 2012

Perturbations in the leading coefficient of DAEs

$$E(t)\dot{x} = A(t)x, \quad E, A \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})$$

Perturbations in the leading coefficient of DAEs

$$E(t)\dot{x} = A(t)x, \quad E, A \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})$$

[März 1991]: (E, A) is **index-1** \Leftrightarrow

$\exists Q \in \mathcal{C}^1 : Q(t)^2 = Q(t) \wedge \text{im } Q(t) = \ker E(t), \text{ and } \exists D \in \mathcal{C}^0:$

$$E\dot{x} = Ax \quad \Leftrightarrow \quad \begin{cases} \frac{d}{dt}(Px) &= (\dot{P} + PD)Px, & P = I - Q \\ Qx &= QDPx \end{cases}$$

Perturbations in the leading coefficient of DAEs

$$E(t)\dot{x} = A(t)x, \quad E, A \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})$$

[März 1991]: (E, A) is **index-1** \Leftrightarrow

$\exists Q \in \mathcal{C}^1 : Q(t)^2 = Q(t) \wedge \text{im } Q(t) = \ker E(t)$, and $\exists D \in \mathcal{C}^0$:

$$E\dot{x} = Ax \Leftrightarrow \begin{cases} \frac{d}{dt}(Px) = (\dot{P} + PD)Px, & P = I - Q \\ Qx = QDPx \end{cases}$$

$$\begin{array}{lcl} \dot{x}_1 & = & x_2 \\ 0 & = & x_1 \end{array} \not\Leftrightarrow \begin{array}{lcl} \dot{x}_1 & = & D_{11}x_1 \\ x_2 & = & D_{21}x_1 \end{array}$$

$$E\dot{x} = Ax \Leftrightarrow \begin{cases} \frac{d}{dt}(Px) = (\dot{P} + PD)Px, \\ Qx = QDPx \end{cases}$$

$$E\dot{x} = Ax \Leftrightarrow \begin{cases} \frac{d}{dt}(Px) = (\dot{P} + PD)Px, \\ Qx = QDPx \end{cases}$$

$$E\dot{x} = Ax \Rightarrow \begin{cases} x = Px + Qx = (I + QD)Px, \\ \frac{d}{dt}(Px) = (\dot{P} + PD)Px \end{cases}$$

$$E\dot{x} = Ax \Leftrightarrow \begin{cases} \frac{d}{dt}(Px) = (\dot{P} + PD)Px, \\ Qx = QDPx \end{cases}$$

$$E\dot{x} = Ax \Rightarrow \begin{cases} x = Px + Qx = (I + QD)Px, \\ \frac{d}{dt}(Px) = (\dot{P} + PD)Px \end{cases}$$

$$\dot{y} = (\dot{P} + PD)y, \quad y(t_0) = P(t_0)x(t_0)$$

$$\xrightarrow{\text{uniqueness}} y(t) = P(t)x(t)$$

crucial:

$$E = EP$$

$$E\dot{x} = Ax$$

$$\Leftrightarrow EP\dot{x} = Ax$$

$$\Leftrightarrow \begin{cases} \frac{d}{dt}(Px) = (\dot{P} + PD)Px, \\ Qx = QDPx \end{cases}$$

crucial:

$$E = EP$$

$$E\dot{x} = (A + \Delta_A)x$$

$$\Leftrightarrow EP\dot{x} = (A + \Delta_A)x$$

$$\Leftrightarrow \begin{cases} \frac{d}{dt}(Px) &= (\dot{P} + PD)Px + PG\Delta_Ax, \\ Qx &= QDPx + QG\Delta_Ax \end{cases} \quad \exists G^{-1}$$

crucial:

$$E = EP$$

$$(E + \Delta_E)\dot{x} = Ax$$

i.g.
 \nRightarrow

$$(E + \Delta_E)P\dot{x} = Ax$$

crucial:

$$E = EP$$

$$(E + \Delta_E)\dot{x} = Ax$$

i.g.



$$(E + \Delta_E)P\dot{x} = Ax$$

(A): $\Delta_E \in C(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})$ s.t. $(E + \Delta_E, A)$ is index-1 and

$$\ker E(t) = \ker(E(t) + \Delta_E(t))$$

crucial:

$$E = EP$$

$$(E + \Delta_E)\dot{x} = Ax$$

i.g.



$$(E + \Delta_E)P\dot{x} = Ax$$

(A): $\Delta_E \in C(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})$ s.t. $(E + \Delta_E, A)$ is index-1 and

$$\ker E(t) = \ker(E(t) + \Delta_E(t))$$

$$(E + \Delta_E)\dot{x} = Ax \Leftrightarrow$$

$$\begin{aligned} \frac{d}{dt}(Px) &= (\dot{P} + PD)Px + PG\Delta Px, \\ Qx &= QDPx + QG\Delta Px \end{aligned}$$

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{bmatrix} I_{n_1} & 0 \\ \Delta(t) & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$0 = (A_{21} - \Delta A_{11})(t)x_1 + (A_{22} - \Delta A_{12})(t)x_2$$

$$\begin{bmatrix} I_{n_1} & 0 \\ \Delta(t) & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$0 = (A_{21} - \Delta A_{11})(t)x_1 + (A_{22} - \Delta A_{12})(t)x_2$$

$$E^{\text{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\text{ex}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_E^{\text{ex}} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}, \quad \delta \neq -1$$

$$\begin{bmatrix} I_{n_1} & 0 \\ \Delta(t) & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$0 = (A_{21} - \Delta A_{11})(t)x_1 + (A_{22} - \Delta A_{12})(t)x_2$$

$$E^{\text{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\text{ex}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_E^{\text{ex}} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}, \quad \delta \neq -1$$

sln. of $(E^{\text{ex}}, A^{\text{ex}})$: $x_1(t) = c_1 e^{-t}$, $x_2(t) = c_2 e^{-t}$, $x_3(t) = 0$

$$\begin{bmatrix} I_{n_1} & 0 \\ \Delta(t) & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$0 = (A_{21} - \Delta A_{11})(t)x_1 + (A_{22} - \Delta A_{12})(t)x_2$$

$$E^{\text{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\text{ex}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_E^{\text{ex}} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}, \quad \delta \neq -1$$

sln. of $(E^{\text{ex}}, A^{\text{ex}})$: $x_1(t) = c_1 e^{-t}$, $x_2(t) = c_2 e^{-t}$, $x_3(t) = 0$

sln. of $(E^{\text{ex}} + \Delta_E^{\text{ex}}, A^{\text{ex}})$: $x_1(t) = (c_1 - c_2)e^{-t} + c_2 e^{-\frac{1}{1+\delta}t}$,
 $x_2(t) = c_2 e^{-\frac{1}{1+\delta}t}$,
 $x_3(t) = -\delta(c_1 - c_2)e^{-t} - \frac{\delta c_2}{1+\delta} e^{-\frac{1}{1+\delta}t}$

Example: Perturbation not in $\ker E$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1(t) = e^{-t}x_1^0$$

$$x_2(t) = 0$$

exp. stable

Example: Perturbation not in $\ker E$

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1(t) = e^{-t} x_1^0$$

$$x_2(t) = e^{t/\varepsilon} x_2^0$$

not exp. stable

Bohl exponent and perturbation operator

$$E(t) \frac{d}{dt} \Phi(t, t_0) = A(t) \Phi(t, t_0), \quad P(t_0)(\Phi(t_0, t_0) - I) = 0.$$

$$(E, A) \text{ exp. stab.} \Leftrightarrow \exists \mu, M > 0 \forall t \geq t_0 : \|\Phi(t, t_0)\| \leq M e^{-\mu(t-t_0)}$$

Bohl exponent and perturbation operator

$$E(t) \frac{d}{dt} \Phi(t, t_0) = A(t) \Phi(t, t_0), \quad P(t_0)(\Phi(t_0, t_0) - I) = 0.$$

$$(E, A) \text{ exp. stab.} \Leftrightarrow \exists \mu, M > 0 \forall t \geq t_0 : \|\Phi(t, t_0)\| \leq M e^{-\mu(t-t_0)}$$

$$k_B(E, A) = \inf \left\{ \rho \in \mathbb{R} \mid \exists N_\rho > 0 \forall t \geq s : \|\Phi(t, s)\| \leq N_\rho e^{\rho(t-s)} \right\}$$

Bohl exponent and perturbation operator

$$E(t) \frac{d}{dt} \Phi(t, t_0) = A(t) \Phi(t, t_0), \quad P(t_0)(\Phi(t_0, t_0) - I) = 0.$$

$$(E, A) \text{ exp. stab.} \Leftrightarrow \exists \mu, M > 0 \forall t \geq t_0 : \|\Phi(t, t_0)\| \leq M e^{-\mu(t-t_0)}$$

$$k_B(E, A) = \inf \left\{ \rho \in \mathbb{R} \mid \exists N_\rho > 0 \forall t \geq s : \|\Phi(t, s)\| \leq N_\rho e^{\rho(t-s)} \right\}$$

Rem.: $k_B(E, A) < 0 \iff (E, A) \text{ exp. stable}$

Bohl exponent and perturbation operator

$$E(t) \frac{d}{dt} \Phi(t, t_0) = A(t) \Phi(t, t_0), \quad P(t_0)(\Phi(t_0, t_0) - I) = 0.$$

$$(E, A) \text{ exp. stab.} \Leftrightarrow \exists \mu, M > 0 \forall t \geq t_0 : \|\Phi(t, t_0)\| \leq M e^{-\mu(t-t_0)}$$

$$k_B(E, A) = \inf \left\{ \rho \in \mathbb{R} \mid \exists N_\rho > 0 \forall t \geq s : \|\Phi(t, s)\| \leq N_\rho e^{\rho(t-s)} \right\}$$

Rem.: $k_B(E, A) < 0 \iff (E, A) \text{ exp. stable}$

$$L_{t_0} : L^2([t_0, \infty); \mathbb{R}^n) \rightarrow L^2([t_0, \infty); \mathbb{R}^n), \quad f(\cdot) \mapsto x(\cdot),$$
$$x \text{ solves } E(t)\dot{x} = A(t)x + f(t), \quad P(t_0)x(t_0) = 0$$

Bohl exponent and perturbation operator

$$E(t) \frac{d}{dt} \Phi(t, t_0) = A(t) \Phi(t, t_0), \quad P(t_0)(\Phi(t_0, t_0) - I) = 0.$$

$$(E, A) \text{ exp. stab.} \Leftrightarrow \exists \mu, M > 0 \forall t \geq t_0 : \|\Phi(t, t_0)\| \leq M e^{-\mu(t-t_0)}$$

$$k_B(E, A) = \inf \left\{ \rho \in \mathbb{R} \mid \exists N_\rho > 0 \forall t \geq s : \|\Phi(t, s)\| \leq N_\rho e^{\rho(t-s)} \right\}$$

Rem.: $k_B(E, A) < 0 \iff (E, A) \text{ exp. stable}$

$$L_{t_0} : L^2([t_0, \infty); \mathbb{R}^n) \rightarrow L^2([t_0, \infty); \mathbb{R}^n), \quad f(\cdot) \mapsto x(\cdot),$$
$$x \text{ solves } E(t)\dot{x} = A(t)x + f(t), \quad P(t_0)x(t_0) = 0$$

Lemma [Du et al. 2006]: $(E, A) \text{ exp. stable and (BC) hold} \implies L_{t_0}$ is linear bd. operator and $t_0 \mapsto \|L_{t_0}\|$ is mon. nonincreasing

Theorem (Robustness of Bohl exponent)

(E, A) index-1, Q bounded, given $\varepsilon > 0$:

Δ_E satisfies **(A)**, $\|\Delta_E\|_\infty$ suff. small

$$\implies \boxed{k_B(E + \Delta_E, A) \leq k_B(E, A) + \varepsilon}$$

$$E^{\text{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\text{ex}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_E^{\text{ex}} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}$$

sln. of $(E^{\text{ex}} + \Delta_E^{\text{ex}}, A^{\text{ex}})$:

$$\begin{aligned} x_1(t) &= (c_1 - c_2)e^{-t} + c_2 e^{-\frac{1}{1+\delta}t}, \\ x_2(t) &= c_2 e^{-\frac{1}{1+\delta}t}, \\ x_3(t) &= -\delta(c_1 - c_2)e^{-t} - \frac{\delta c_2}{1+\delta} e^{-\frac{1}{1+\delta}t} \end{aligned}$$

$$E^{\text{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\text{ex}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_E^{\text{ex}} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}$$

sln. of $(E^{\text{ex}} + \Delta_E^{\text{ex}}, A^{\text{ex}})$:

$$\begin{aligned} x_1(t) &= (c_1 - c_2)e^{-t} + c_2 e^{-\frac{1}{1+\delta}t}, \\ x_2(t) &= c_2 e^{-\frac{1}{1+\delta}t}, \\ x_3(t) &= -\delta(c_1 - c_2)e^{-t} - \frac{\delta c_2}{1+\delta} e^{-\frac{1}{1+\delta}t} \end{aligned}$$

$$k_B(E^{\text{ex}}, A^{\text{ex}}) = -1$$

$$k_B(E^{\text{ex}} + \Delta_E^{\text{ex}}, A^{\text{ex}}) = -\frac{1}{1+\delta} \quad \delta > 0$$

$$E^{\text{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\text{ex}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_E^{\text{ex}} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}$$

sln. of $(E^{\text{ex}} + \Delta_E^{\text{ex}}, A^{\text{ex}})$:

$$\begin{aligned} x_1(t) &= (c_1 - c_2)e^{-t} + c_2 e^{-\frac{1}{1+\delta}t}, \\ x_2(t) &= c_2 e^{-\frac{1}{1+\delta}t}, \\ x_3(t) &= -\delta(c_1 - c_2)e^{-t} - \frac{\delta c_2}{1+\delta} e^{-\frac{1}{1+\delta}t} \end{aligned}$$

$$k_B(E^{\text{ex}}, A^{\text{ex}}) = -1$$

$$k_B(E^{\text{ex}} + \Delta_E^{\text{ex}}, A^{\text{ex}}) = -\frac{1}{1+\delta} \quad \delta > 0$$

$$\delta \leq \frac{\varepsilon}{1-\varepsilon} : k_B(E^{\text{ex}} + \Delta_E^{\text{ex}}, A^{\text{ex}}) \leq k_B(E^{\text{ex}}, A^{\text{ex}}) + \varepsilon$$

Theorem (Robustness via perturbation operator)

(E, A) index-1 and exp. stable, **(BC)** hold, Δ_E satisfies **(A)**:

$$\exists \kappa_i = \kappa_i(E, A, Q), i = 1, 2, 3, \quad \alpha := \min \left\{ \lim_{t_0 \rightarrow \infty} \|L_{t_0}\|^{-1}, \kappa_3 \right\},$$

$$\lim_{t_0 \rightarrow \infty} \left\| \Delta_E|_{[t_0, \infty)} \right\|_{\infty} < \frac{\alpha}{\kappa_1 + \kappa_2 \alpha}$$

$\implies (E + \Delta_E, A)$ is exponentially stable

$$E^{\text{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\text{ex}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_E^{\text{ex}} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}$$

$$E^{\text{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\text{ex}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_E^{\text{ex}} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}$$

$$L_{t_0}^{\text{ex}} : L^2([t_0, \infty); \mathbb{R}^3) \rightarrow L^2([t_0, \infty); \mathbb{R}^3),$$

$$(f_1(\cdot), f_2(\cdot), f_3(\cdot)) \mapsto \left(t \mapsto \text{diag} \left(\int_{t_0}^t e^{-(t-s)} f_1(s) \, ds, \int_{t_0}^t e^{-(t-s)} f_2(s) \, ds, -f_3(t) \right) \right)$$

$$E^{\text{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\text{ex}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_E^{\text{ex}} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}$$

$$L_{t_0}^{\text{ex}} : L^2([t_0, \infty); \mathbb{R}^3) \rightarrow L^2([t_0, \infty); \mathbb{R}^3),$$

$$(f_1(\cdot), f_2(\cdot), f_3(\cdot)) \mapsto \left(t \mapsto \text{diag} \left(\int_{t_0}^t e^{-(t-s)} f_1(s) \, ds, \int_{t_0}^t e^{-(t-s)} f_2(s) \, ds, -f_3(t) \right) \right)$$

$$\boxed{\|L_{t_0}^{\text{ex}}\| = 1, \quad t_0 \geq 0} \quad \kappa_1 = \kappa_2 = \kappa_3 = 1, \quad \|\Delta_E^{\text{ex}}\| = \sqrt{2}|\delta|$$

$$E^{\text{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\text{ex}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_E^{\text{ex}} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}$$

$$L_{t_0}^{\text{ex}} : L^2([t_0, \infty); \mathbb{R}^3) \rightarrow L^2([t_0, \infty); \mathbb{R}^3),$$

$$(f_1(\cdot), f_2(\cdot), f_3(\cdot)) \mapsto \left(t \mapsto \text{diag} \left(\int_{t_0}^t e^{-(t-s)} f_1(s) \, ds, \int_{t_0}^t e^{-(t-s)} f_2(s) \, ds, -f_3(t) \right) \right)$$

$$\boxed{\|L_{t_0}^{\text{ex}}\| = 1, t_0 \geq 0} \quad \kappa_1 = \kappa_2 = \kappa_3 = 1, \quad \|\Delta_E^{\text{ex}}\| = \sqrt{2}|\delta|$$

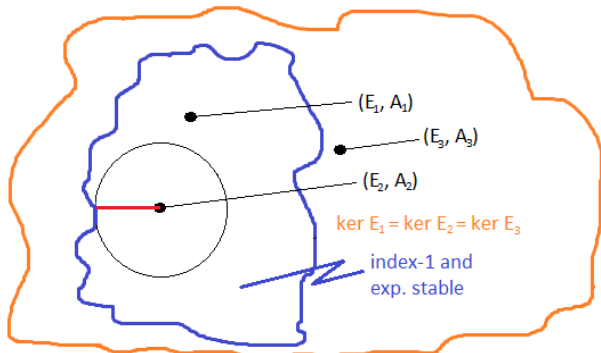
$$|\delta| < \frac{1}{2\sqrt{2}} : (E^{\text{ex}} + \Delta_E^{\text{ex}}, A^{\text{ex}}) \text{ exp. stable}$$

Stability radius

$$\mathcal{P} := \left\{ [\Delta_E, \Delta_A] \in \mathcal{B}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times 2n}) \mid \begin{array}{l} (E + \Delta_E, A + \Delta_A) \text{ is index-1,} \\ \ker E(t) = \ker(E(t) + \Delta_E(t)) \end{array} \right\},$$

$$\mathcal{S} := \left\{ (E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2 \mid (E, A) \text{ is exponentially stable} \right\},$$

$$r(E, A) := \inf_{[\Delta_E, \Delta_A] \in \overline{\mathcal{P}}} \left\{ \|\Delta_E, \Delta_A\|_{\infty} \mid \begin{array}{l} [\Delta_E, \Delta_A] \notin \mathcal{P} \text{ or} \\ (E + \Delta_E, A + \Delta_A) \notin \mathcal{S} \end{array} \right\}$$



Proposition (Properties of the stability radius)

- $r(E, A) = 0 \Leftrightarrow (E, A) \notin \mathcal{S}$
- $r(\alpha(E, A)) = r(\alpha E, \alpha A) = \alpha r(E, A)$ for all $\alpha \geq 0$
- $\mathcal{V}(t)$ time-varying subspace of \mathbb{R}^n with constant dimension,

$$\mathcal{K}_{\mathcal{V}} := \left\{ [E, A] \in \mathcal{B}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times 2n}) \mid \begin{array}{l} (E, A) \text{ is index-1,} \\ \ker E(t) = \mathcal{V}(t) \end{array} \right\},$$

$\Rightarrow \mathcal{K}_{\mathcal{V}} \ni [E, A] \mapsto r(E, A)$ is continuous

Example: $\varepsilon \geq 0$: $E = \varepsilon$, $A = -1$

Example: $\varepsilon \geq 0$: $E = \varepsilon$, $A = -1$

$$\varepsilon > 0: \quad \dot{x} = \frac{-1 + \Delta_A(t)}{\varepsilon + \Delta_E(t)} x$$

Example: $\varepsilon \geq 0$: $E = \varepsilon$, $A = -1$

$$\varepsilon > 0: \quad \dot{x} = \frac{-1 + \Delta_A(t)}{\varepsilon + \Delta_E(t)} x$$

$$0 \leq r(\varepsilon, -1) \leq \varepsilon, \quad \lim_{\varepsilon \rightarrow 0} r(\varepsilon, -1) = 0$$

Example: $\varepsilon \geq 0$: $E = \varepsilon$, $A = -1$

$$\varepsilon > 0: \quad \dot{x} = \frac{-1 + \Delta_A(t)}{\varepsilon + \Delta_E(t)} x$$

$$0 \leq r(\varepsilon, -1) \leq \varepsilon, \quad \lim_{\varepsilon \rightarrow 0} r(\varepsilon, -1) = 0$$

$$\varepsilon = 0: \quad 0 = (-1 + \Delta_A(t))x$$

$$[\Delta_E, \Delta_A] = [0, 1 - 1/n] \in \mathcal{P}, n \in \mathbb{N} \implies \boxed{[\Delta_E, \Delta_A] = [0, 1] \in \overline{\mathcal{P}}}$$

Example: $\varepsilon \geq 0$: $E = \varepsilon$, $A = -1$

$$\varepsilon > 0: \quad \dot{x} = \frac{-1 + \Delta_A(t)}{\varepsilon + \Delta_E(t)} x$$

$$0 \leq r(\varepsilon, -1) \leq \varepsilon, \quad \lim_{\varepsilon \rightarrow 0} r(\varepsilon, -1) = 0$$

$$\varepsilon = 0: \quad 0 = (-1 + \Delta_A(t))x$$

$$[\Delta_E, \Delta_A] = [0, 1 - 1/n] \in \mathcal{P}, n \in \mathbb{N} \implies \boxed{[\Delta_E, \Delta_A] = [0, 1] \in \overline{\mathcal{P}}}$$

$$r(0, -1) \leq 1; \quad \|\Delta_A\|_\infty < 1 \implies 0 = (-1 + \Delta_A(t))x \text{ is exp. stable}$$

Example: $\varepsilon \geq 0$: $E = \varepsilon$, $A = -1$

$$\varepsilon > 0: \quad \dot{x} = \frac{-1 + \Delta_A(t)}{\varepsilon + \Delta_E(t)} x$$

$$0 \leq r(\varepsilon, -1) \leq \varepsilon, \quad \lim_{\varepsilon \rightarrow 0} r(\varepsilon, -1) = 0$$

$$\varepsilon = 0: \quad 0 = (-1 + \Delta_A(t))x$$

$$[\Delta_E, \Delta_A] = [0, 1 - 1/n] \in \mathcal{P}, n \in \mathbb{N} \implies \boxed{[\Delta_E, \Delta_A] = [0, 1] \in \overline{\mathcal{P}}}$$

$r(0, -1) \leq 1$; $\|\Delta_A\|_\infty < 1 \implies 0 = (-1 + \Delta_A(t))x$ is exp. stable

$$\boxed{\lim_{\varepsilon \rightarrow 0} r(\varepsilon, -1) = 0 \neq 1 = r(0, -1)}$$

Theorem (Lower bound for the stability radius)

(E, A) index-1 and exp. stable, **(BC)** holds

\implies

$$\exists \kappa_i = \kappa_i(E, A, Q), i = 1, 2, 3, \quad \alpha := \min \left\{ \lim_{t_0 \rightarrow \infty} \|L_{t_0}\|^{-1}, \kappa_3 \right\},$$

$$\frac{\alpha}{\kappa_1 + \kappa_2 \alpha} \leq r(E, A)$$

Theorem (Lower bound for the stability radius)

(E, A) index-1 and exp. stable, **(BC)** holds

\implies

$$\exists \kappa_i = \kappa_i(E, A, Q), i = 1, 2, 3, \quad \alpha := \min \left\{ \lim_{t_0 \rightarrow \infty} \|L_{t_0}\|^{-1}, \kappa_3 \right\},$$

$$\boxed{\frac{\alpha}{\kappa_1 + \kappa_2 \alpha} \leq r(E, A)}$$

Cor.: $\mathcal{V}(t)$ time-varying subspace of \mathbb{R}^n with constant dimension,

$$\mathcal{S}_{\mathcal{V}} := \left\{ [E, A] \in \mathcal{B}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times 2n}) \mid \begin{array}{l} (E, A) \text{ is index-1 and exp. stable,} \\ \ker E(t) = \mathcal{V}(t) \text{ for all } t \in \mathbb{R}_{\geq 0} \end{array} \right\}$$

$\implies \mathcal{S}_{\mathcal{V}}$ is open in $\overline{\mathcal{K}_{\mathcal{V}}}$