On perturbations in the leading coefficient matrix of time-varying index-1 DAEs

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Perturbations in the leading coefficient of DAEs

$$E(t)\dot{x} = A(t)x, \quad E, A \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})$$

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[März 1991]: (E, A) is **index-1** $\Leftrightarrow$

$\exists\, Q \in \mathcal{C}^1 :\ Q(t)^2 = Q(t) \ \wedge\ \operatorname{im} Q(t) = \ker E(t)$, and $\exists\, D \in \mathcal{C}^0$:

$$E\dot{x} = Ax \quad \Leftrightarrow \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(Px) & = & (\dot{P} + PD)Px, \quad P = I - Q \\ \qquad Qx & = & QDPx \end{cases}$$

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$$\begin{aligned} \dot{x}_1 &= x_2 \\ 0 &= x_1 \end{aligned} \quad \not\Leftrightarrow \quad \begin{aligned} \dot{x}_1 &= D_{11}x_1 \\ x_2 &= D_{21}x_1 \end{aligned}$$

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$$\boxed{\dot{y} = (\dot{P} + PD)y, \quad y(t_0) = P(t_0)x(t_0)}$$

$$\overset{\text{uniqueness}}{\Longrightarrow} \quad y(t) = P(t)x(t)$$

crucial: $\boxed{E = EP}$

$$
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(A): $\Delta_E \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n\times n})$ s.t. $(E + \Delta_E, A)$ is index-1 and

$$\boxed{\ker E(t) = \ker(E(t) + \Delta_E(t))}$$

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$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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Example: Perturbation not in $\ker E$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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exp. stable

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$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{aligned} x_1(t) &= e^{-t} x_1^0 \\ x_2(t) &= e^{t/\varepsilon} x_2^0 \end{aligned}$$

not exp. stable

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Bohl exponent and perturbation operator

$$E(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\Phi(t,t_0) = A(t)\Phi(t,t_0), \quad P(t_0)(\Phi(t_0,t_0) - I) = 0.$$

(E,A) exp. stab. $\Leftrightarrow\ \exists\,\mu, M > 0\ \forall\, t \geq t_0:\ \|\Phi(t,t_0)\| \leq Me^{-\mu(t-t_0)}$

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$$k_B(E,A) = \inf \left\{ \rho \in \mathbb{R} \;\middle|\; \exists\, N_\rho > 0\ \forall\, t \ge s :\ \|\Phi(t,s)\| \le N_\rho e^{\rho(t-s)} \right\}$$

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Rem.: $k_B(E,A) < 0 \iff (E,A)$ exp. stable

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$$L_{t_0} : L^2([t_0,\infty);\mathbb{R}^n) \to L^2([t_0,\infty);\mathbb{R}^n),\ f(\cdot)\mapsto x(\cdot),$$
$$x \ \text{ solves }\ E(t)\dot{x} = A(t)x + f(t),\ \ P(t_0)x(t_0) = 0$$

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Lemma [Du et al. 2006]: (E,A) exp. stable and **(BC)** hold $\implies$ L_{t_0} is linear bd. operator and $t_0 \mapsto \|L_{t_0}\|$ is mon. nonincreasing

Theorem (Robustness of Bohl exponent)

(E, A) index-1, Q bounded, given $\varepsilon > 0$:

$$\begin{aligned} &\Delta_E \text{ satisfies } \textbf{(A)},\ \|\Delta_E\|_\infty \text{ suff. small} \\ \Longrightarrow\quad &\boxed{k_B(E + \Delta_E, A) \leq k_B(E, A) + \varepsilon} \end{aligned}$$

$$E^{\mathrm{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\mathrm{ex}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_E^{\mathrm{ex}} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}$$

sln. of $(E^{\mathrm{ex}} + \Delta_E^{\mathrm{ex}}, A^{\mathrm{ex}})$:

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$$\boxed{k_B(E^{\text{ex}}, A^{\text{ex}}) = -1} \qquad \boxed{k_B(E^{\text{ex}} + \Delta_E^{\text{ex}}, A^{\text{ex}}) = -\frac{1}{1+\delta}} \quad \delta > 0$$

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$$\delta \le \frac{\varepsilon}{1-\varepsilon} : \; k_B(E^{\mathrm{ex}} + \Delta_E^{\mathrm{ex}}, A^{\mathrm{ex}}) \le k_B(E^{\mathrm{ex}}, A^{\mathrm{ex}}) + \varepsilon$$

Theorem (Robustness via perturbation operator)

(E, A) index-1 and exp. stable, **(BC)** hold, Δ_E satisfies **(A)**:

$$\exists\, \kappa_i = \kappa_i(E, A, Q), i = 1, 2, 3, \quad \alpha := \min\left\{\lim_{t_0\to\infty} \|L_{t_0}\|^{-1}, \kappa_3\right\},$$

$$\lim_{t_0\to\infty} \left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty < \frac{\alpha}{\kappa_1 + \kappa_2\alpha}$$

$\Longrightarrow$ $(E + \Delta_E, A)$ is exponentially stable

$$E^{\text{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{\text{ex}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta_E^{\text{ex}} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}$$

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$$\begin{aligned} L_{t_0}^{\text{ex}} &: L^2([t_0,\infty);\mathbb{R}^3) \to L^2([t_0,\infty);\mathbb{R}^3), \\ &(f_1(\cdot), f_2(\cdot), f_3(\cdot)) \mapsto \Bigg(t \mapsto \operatorname{diag}\Bigg(\int_{t_0}^t e^{-(t-s)} f_1(s)\,\mathrm{d}s\,, \\ &\qquad \int_{t_0}^t e^{-(t-s)} f_2(s)\,\mathrm{d}s\,, -f_3(t)\Bigg)\Bigg) \end{aligned}$$

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$$\boxed{\|L_{t_0}^{\mathrm{ex}}\| = 1,\ t_0 \ge 0} \quad \kappa_1 = \kappa_2 = \kappa_3 = 1, \quad \|\Delta_E^{\mathrm{ex}}\| = \sqrt{2}|\delta|$$

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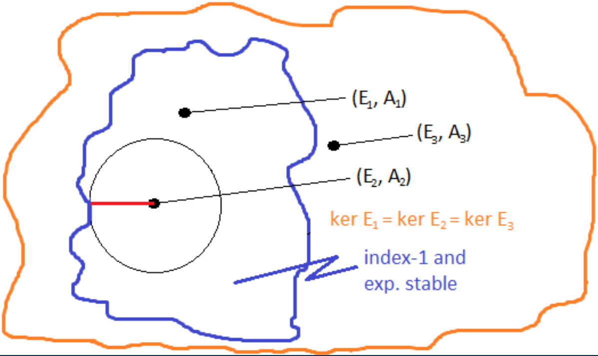
$$|\delta| < \frac{1}{2\sqrt{2}} :\ (E^{\mathrm{ex}} + \Delta_E^{\mathrm{ex}}, A^{\mathrm{ex}}) \text{ exp. stable}$$

Stability radius

$$\mathcal{P} := \left\{ [\Delta_E, \Delta_A] \in \mathcal{B}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times 2n}) \;\middle|\; \begin{array}{l} (E + \Delta_E, A + \Delta_A) \text{ is index-1,} \\ \ker E(t) = \ker(E(t) + \Delta_E(t)) \end{array} \right\},$$

$$\mathcal{S} := \left\{ (E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2 \;\middle|\; (E, A) \text{ is exponentially stable} \right\},$$

$$r(E, A) := \inf_{[\Delta_E, \Delta_A] \in \overline{\mathcal{P}}} \left\{ \|[\Delta_E, \Delta_A]\|_\infty \;\middle|\; \begin{array}{l} [\Delta_E, \Delta_A] \notin \mathcal{P} \text{ or} \\ (E + \Delta_E, A + \Delta_A) \notin \mathcal{S} \end{array} \right\}$$



Proposition (Properties of the stability radius)

- $r(E, A) = 0 \;\Leftrightarrow\; (E, A) \notin \mathcal{S}$
- $r(\alpha(E, A)) = r(\alpha E, \alpha A) = \alpha\, r(E, A)$ for all $\alpha \geq 0$
- $\mathcal{V}(t)$ time-varying subspace of $\mathbb{R}^n$ with constant dimension,

$$\mathcal{K}_{\mathcal{V}} := \left\{ [E, A] \in \mathcal{B}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times 2n}) \;\middle|\; \begin{array}{l} (E, A) \text{ is index-1,} \\ \ker E(t) = \mathcal{V}(t) \end{array} \right\},$$

$\implies \mathcal{K}_{\mathcal{V}} \ni [E, A] \mapsto r(E, A)$ is continuous

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$$\varepsilon = 0 : \qquad 0 = (-1 + \Delta_A(t))x$$

$$[\Delta_E, \Delta_A] = [0, 1 - 1/n] \in \mathcal{P}, n \in \mathbb{N} \implies \boxed{[\Delta_E, \Delta_A] = [0, 1] \in \overline{\mathcal{P}}}$$

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$$\varepsilon > 0: \qquad \dot{x} = \frac{-1 + \Delta_A(t)}{\varepsilon + \Delta_E(t)} x$$

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Theorem (Lower bound for the stability radius)

(E, A) index-1 and exp. stable, **(BC)** holds
$\Longrightarrow$

$$\exists\, \kappa_i = \kappa_i(E, A, Q), i = 1, 2, 3, \quad \alpha := \min \left\{ \lim_{t_0 \to \infty} \|L_{t_0}\|^{-1}, \kappa_3 \right\},$$

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Cor.: $\mathcal{V}(t)$ time-varying subspace of $\mathbb{R}^n$ with constant dimension,

$$\mathcal{S}_{\mathcal{V}} := \left\{ [E, A] \in \mathcal{B}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times 2n}) \;\middle|\; \begin{array}{l} (E, A) \text{ is index-1 and exp. stable,} \\ \ker E(t) = \mathcal{V}(t) \text{ for all } t \in \mathbb{R}_{\geq 0} \end{array} \right\}$$

$\Longrightarrow$ $\mathcal{S}_{\mathcal{V}}$ is open in $\overline{\mathcal{K}_{\mathcal{V}}}$