

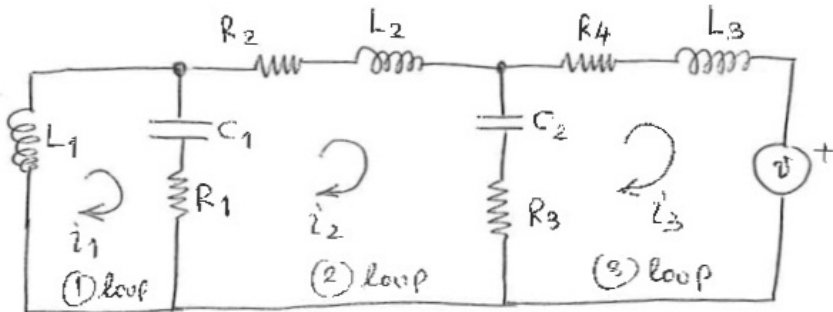
# Single element changes in electrical networks

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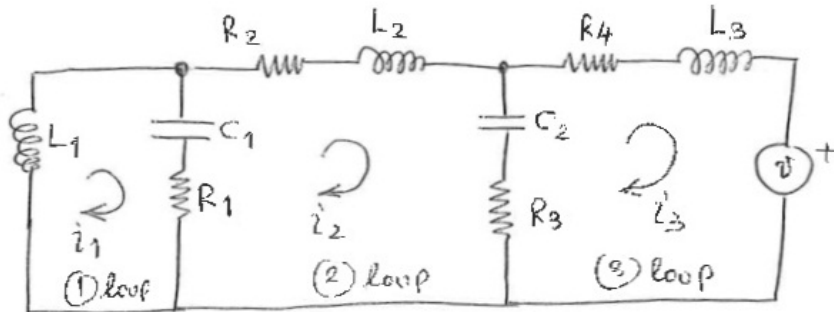
Institute of Mathematics, Ilmenau University of Technology

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## Example: electrical RLC network



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$$Z(s)i(s) = v_s(s),$$

$$Z(s) = s^{-1} \begin{bmatrix} \frac{1}{C_1} & -\frac{1}{C_1} & 0 \\ -\frac{1}{C_1} & \frac{1}{C_1} + \frac{1}{C_2} & -\frac{1}{C_2} \\ 0 & -\frac{1}{C_2} & \frac{1}{C_2} \end{bmatrix} + \begin{bmatrix} R_1 & -R_1 & 0 \\ -R_1 & R_1 + R_2 + R_3 & -R_3 \\ 0 & -R_3 & R_3 + R_4 \end{bmatrix} + s \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix}$$

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impedance modeling (loop analysis):

$L$ : spring, inductor

$C$ : mass, inertor, capacitor

$R$ : damper, resistor

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- element present in  $i$ -th loop (node)  $\Rightarrow$  its value is added to  $(i, i)$  position of the respective matrix
- element common to  $i$ -th and  $j$ -th loop (node)  $\Rightarrow$  its value is added to  $(i, i)$  and  $(j, j)$  positions, subtracted from  $(i, j)$  and  $(j, i)$  positions

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**special cases:** RC and RL networks  $\rightarrow W(s) = sL + s^{-1}C + R$   
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$$W(s) = \left\{ \begin{array}{l} \text{RL: } sL + R \\ \text{RC: } \hat{s}C + R, \hat{s} = s^{-1} \end{array} \right\} = sF + G, \quad F = F^T, G = G^T$$



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$$W(s) = s \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix} + \begin{bmatrix} R_1 & -R_1 & 0 \\ -R_1 & R_1+R_2+R_3 & -R_3 \\ 0 & -R_3 & R_3+R_4 \end{bmatrix}$$

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here:  $\boxed{\det(s(F + xbb^\top) + G)}$   $b = e_i$  or  $b = e_i - e_j, i \neq j$

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$$\det W(s) = 0 \iff 1 + L_4 \frac{sz(s)}{p(s)} = 0$$



# Reformulation as root locus problem

## [Binet-Cauchy-Theorem]

$$\begin{aligned}\det (s(F + xbb^\top) + G) &= \det \left( [sF + G, I_k] \begin{bmatrix} I_k \\ sxbb^\top \end{bmatrix} \right) \\ &= g(s; F, G)^\top p(sx; b), \quad x \in \mathbb{R}\end{aligned}$$

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$$\det(s(F + xbb^\top) + G) = \det(sF + G) + sx z(s; b)$$

root locus problem:

$$1 + x \frac{sz(s; b)}{\det(sF + G)} = 0$$

## Theorem (Properties of the root locus)

$$1 + x \frac{s z(s; b)}{\det(sF + G)} = 0 \quad x \in \mathbb{R}, \quad \text{Assumption: } G \text{ is invertible}$$

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$$F + xbb^T = \begin{bmatrix} C_1 + x & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1/R_1 & 0 \\ 0 & 1/R_2 \end{bmatrix}$$

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$$\det(sF + G) + x \, sz(s; b) = 1/R_2(s(C_1 + x) + 1/R_1)$$

$$p(x) = -1/(R_1(C_1 + x)) :$$



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$$\lim_{x \searrow -C_1} p(x) = -\infty$$

$$"p(-C_1) = \infty"$$

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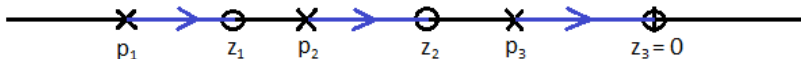
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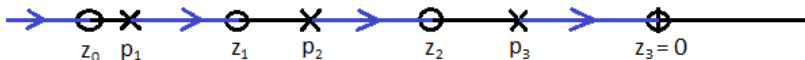
# Theorem (Behaviour of the root locus)

$x > 0$ :

- $p$  pole of multiplicity  $\mu \implies \mu$  or  $\mu - 1$  of these poles do not change and at most one pole moves to the right
- $\exists$  pole  $p(x)$  s.t.  $\lim_{x \rightarrow \infty} p(x) = 0$
- if  $\text{rk}(F + bb^\top) = \text{rk } F$ :



- if  $\text{rk}(F + bb^\top) = \text{rk } F + 1$ , then an infinite pole becomes finite and moves to the right as  $x$  increases



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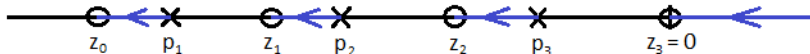
$$\det(sF + G) = 1/R_2(sC_1 + 1/R_1) \implies p_1 = -1/(C_1 R_1)$$

$$sz(s) = sx(sC_1 + 1/R_1) \implies z_1 = -1/(C_1 R_1), \quad z_2 = 0$$



$x < 0$ :

- $p$  pole of multiplicity  $\mu \implies \mu$  or  $\mu - 1$  of these poles do not change and at most one pole moves to the left
- if  $\text{rk}(F + bb^\top) = \text{rk } F + 1$ , then an infinite pole becomes finite and moves to the left on the positive real axis as  $x$  decreases, reaching 0 for  $x \rightarrow -\infty$

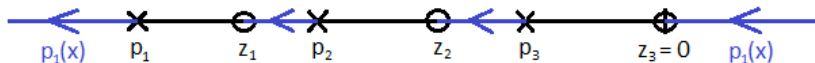


- if  $\text{rk}(F + bb^\top) = \text{rk } F$ , then the smallest pole  $p_1(x)$  moves to the left towards  $-\infty$  and

$$\exists \kappa > 0 : \forall -\kappa < x < 0 : p_1(x) < p_1(0) \wedge \lim_{x \searrow -\kappa} p_1(x) = -\infty,$$

“ $p(-\kappa) = \infty$ ”,

$$\forall x < -\kappa : p_1(x) > 0 \wedge \lim_{x \rightarrow -\infty} p_1(x) = 0$$



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single element changes  $\rightarrow$  root locus problem  $1 + x \frac{s z(s; b)}{\det(sF + G)} = 0$

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- all poles move to the right
  - if  $\text{rk}(F + bb^\top) = \text{rk } F + 1$ , then an infinite pole becomes finite
- $x < 0$ :
- all poles move to the left
  - if  $\text{rk}(F + bb^\top) = \text{rk } F + 1$ , then an infinite pole becomes finite
  - if  $\text{rk}(F + bb^\top) = \text{rk } F$ , then a finite pole becomes infinite for a single value of  $x$