Lyapunov equations for time-varying DAEs

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Definition (solution)

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Equivalence relation

$S \in C,\ T \in C^1,\ \det S(t),\ \det T(t) \neq 0,\ t \in \mathbb{R}$:

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$\rightarrow (E_1, A_1) \sim (E_2, A_2)$.

Definition (SCF)

(E,A) is called **transferable into standard canonical form (SCF)** $:\Longleftrightarrow \; \exists\, n_1, n_2 \in \mathbb{N}$:

$$(E, A) \sim \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right),$$

$J : \mathbb{R} \to \mathbb{R}^{n_1 \times n_1}$, $N : \mathbb{R} \to \mathbb{R}^{n_2 \times n_2}$, and $N(t) = \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{bmatrix}$

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$$E(t)\dot{x} = A(t)x \quad \overset{x=Ty}{\sim} \quad \begin{aligned} \dot{y}_1 &= J(t)y_1 \\ N(t)\dot{y}_2 &= y_2 \end{aligned}$$

Definition (consistent initial values)

$$\mathcal{V} := \{ (t^0, x^0) \mid \exists \text{ solution to (E,A)}, \ x(t^0) = x^0 \}$$

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- $\forall\, t^0 \in \mathbb{R}: \ \mathcal{V}(t^0)$ is a linear subspace of $\mathbb{R}^n$
- $x : \mathcal{J} \to \mathbb{R}^n$ solution to (E,A) $\Rightarrow x(t) \in \mathcal{V}(t)$ for all $t \in \mathcal{J}$

Theorem

(E,A) transferable into SCF via S, T:

$$(t^0, x^0) \in \mathcal{V} \iff x^0 \in \operatorname{im} T(t^0) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}.$$

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$$x : \mathbb{R} \to \mathbb{R}^n,\ t \mapsto \underbrace{T(t) \begin{bmatrix} \Phi_J(t, t^0) & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1}}_{=: U(t,t^0)} x^0,$$

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$U(\cdot\,,\cdot)$ is the **generalized transition matrix** of the system (E,A); it is well-defined.

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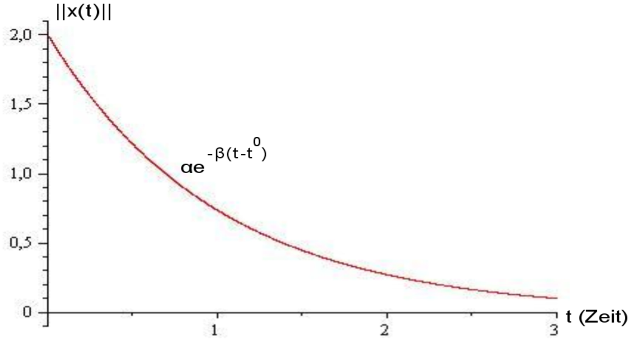
- $U(t,t)^2 = U(t,t)$
- $\forall\, x \in \mathcal{V}(t):\ U(t,t)x = x$

exponential stability:



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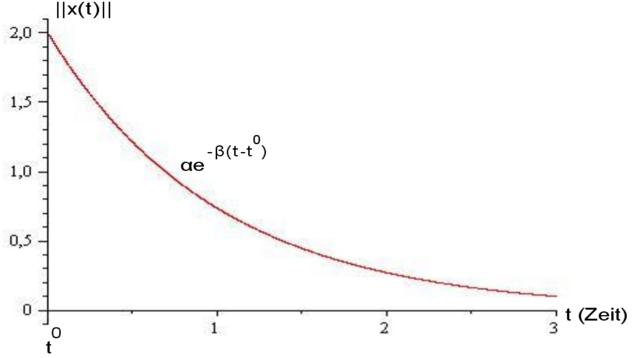

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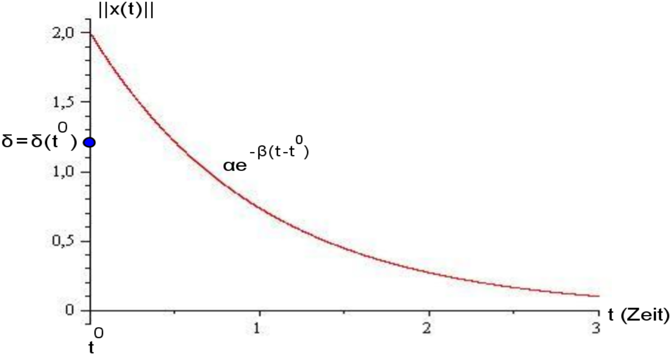

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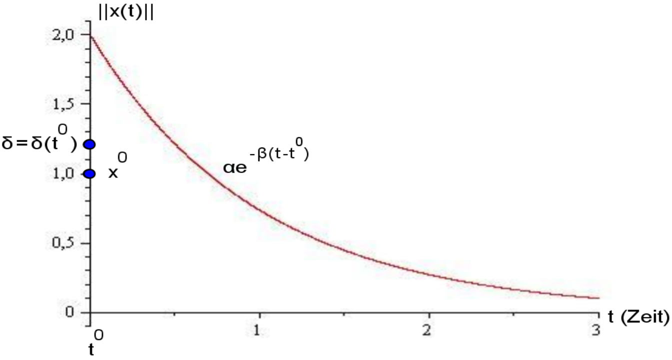

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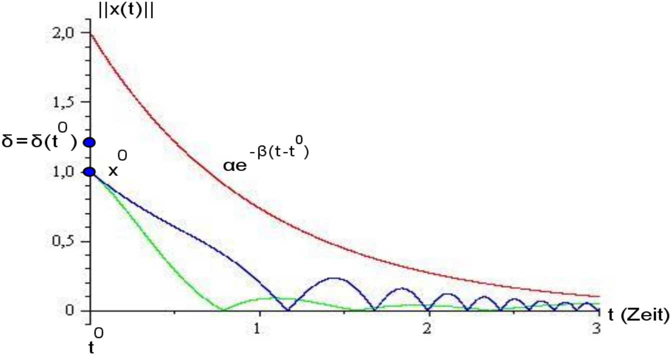

$$\exists\,\alpha,\beta>0\;\forall\,t^0\in\mathbb{R}\;\exists\,\delta>0\;\forall\,x^0\in\mathcal{B}_\delta(0)\;\forall\,x(\cdot)\in\mathcal{S}(t^0,x^0):$$

$$[t^0,\infty)\subseteq\operatorname{dom}x\quad\wedge\quad\forall\,t\geq t^0:\|x(t)\|\leq\alpha e^{-\beta(t-t^0)}\|x^0\|.$$

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$$A(\cdot)^\top P(\cdot)E(\cdot) + E(\cdot)^\top P(\cdot)A(\cdot) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(\cdot)^\top P(\cdot)E(\cdot)\big) =_{\mathcal{G}} -Q(\cdot).$$

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Theorem

$\exists\, Q(\cdot) \in \mathcal{P}_{\mathcal{G}}, P(\cdot) \in C$ with $E(\cdot)^\top P(\cdot)E(\cdot) \in \mathcal{P}_{\mathcal{G}} \cap\ C^1$ and (PGTVLE) holds $\Longrightarrow$

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$$\exists\, c > 0\ \forall\, t \geq t^0 :\ \ \tfrac{\mathrm{d}}{\mathrm{d}t}V(t, x(t)) \leq -cV(t, x(t))$$

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2. $E(\cdot), N(\cdot) \in C^1$; $E(\cdot)^\top E(\cdot), Q(\cdot) \in \mathcal{P}_{\mathcal{G}}$; $E(\cdot)$, $\big(\dot{E}(\cdot) + A(\cdot)\big)$ bounded.
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$$P : \mathbb{R} \to \mathbb{R}^{n\times n},\ t \mapsto S(t)^\top T(t)^\top \int_t^\infty U(s,t)^\top Q(s)U(s,t)\,\mathrm{d}s\ T(t)S(t)$$

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$$Q(\cdot) \equiv I \;\Rightarrow\; P(t) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & p(t) \end{bmatrix}, p(\cdot) \in C \text{ is a solution}$$

Proposition (uniqueness)

(E, A) transferable into SCF and exponentially stable, $Q(\cdot) \in C$;
$P_1(\cdot), P_2(\cdot) \in C$ solutions to (PGTVLE) with $E(\cdot)^\top P_i(\cdot) E(\cdot) \in C^1$, $i = 1, 2$ and

$$\forall\, i = 1, 2\ \exists\, \alpha_i, \beta_i > 0 :\ \alpha_i I_n \leq_{\mathcal{G}} E(\cdot)^\top P_i(\cdot) E(\cdot) \leq_{\mathcal{G}} \beta_i I_n.$$

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Sketch of proof: Let $s \in \mathbb{R}$ and consider

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Show $\dot{M}(t) = 0$ and $\lim_{t \to \infty} M(t) = 0 \Longrightarrow M(s) = 0$