Funnel MPC for nonlinear systems with arbitrary relative degree[☆]

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Abstract

The Model Predictive Control (MPC) scheme Funnel MPC enables output tracking of smooth reference signals with prescribed error bounds for nonlinear multi-input multi-output systems with stable internal dynamics. Earlier works achieved the control objective for system with relative degree restricted to one or incorporated additional feasibility constraints in the optimal control problem. Here we resolve these limitations by introducing a modified stage cost function relying on a weighted sum of the tracking error derivatives. The weights need to be sufficiently large and we state explicit lower bounds. Under these assumptions we are able to prove initial and recursive feasibility of the novel Funnel MPC scheme for systems with arbitrary relative degree – without requiring any terminal conditions, a sufficiently long prediction horizon or additional output constraints.

Keywords: model predictive control, funnel control, reference tracking, nonlinear systems, initial feasibility, recursive feasibility

1. Introduction

Model Predictive Control (MPC) is a nowadays widely used control technique which has seen various applications, see e.g. [20]. It is applicable to nonlinear multi-input multi-output system and able to take state and control constraints directly into account. MPC relies on the iterative solution of finite horizon Optimal Control Problems (OCP), see e.g. [21, 14].

Solvability of the OCP at any particular time instance is essential for the successful application of MPC. Incorporating suitably designed terminal conditions (costs and constraints) in the optimization problem is an often used method to guarantee *initial and recursive feasibility*, meaning guaranteeing that the solvability of the OCP at a particular instance in time automatically implies that the OCP can be solved at the successor time instance. However, the computational effort for solving the OCP and finding initially feasible control signals becomes significantly more complicated by the introduction of such (artificial) terminal conditions. Thus, the domain of admissible controls for MPC might shrink substantially, see e.g. [10, 13]. Alternative methods relying on controllability conditions, e.g. cost controllability [11], require a sufficiently long prediction horizon, see e.g. [8, 12]. Especially in the presence of time-varying state and output constraints these techniques are considerably more involved, see e.g. [19].

Funnel MPC (FMPC) was proposed in [5] to overcome these restrictions. It allows for output reference tracking such that the tracking error evolves within predefined (time-varying) performance bounds. While in [5] output constraints were incorporated in the OCP, it was shown in the successor work [2] that for a class of systems with relative degree one and, in a certain sense, input-to-state stable internal dynamics, these constraints are superfluous. Utilizing a "funnel-like" stage cost, which penalizes the tracking error and becomes infinite when approaching predefined boundaries, guarantees initial and recursive feasibility – without the necessity to impose additional terminal conditions or requirements on the length of the prediction horizon.

FMPC is inspired by funnel control which is an adaptive feedback control technique of high-gain type first proposed in [16], see also the recent work [4] for a comprehensive literature overview. The funnel controller is inherently robust and allows for output tracking with prescribed performance guarantees for a fairly large class of systems solely invoking structural assumptions. In contrast to MPC, funnel control does not use a model of the system. The control input signal is solely determined by the instantaneous values of the system output. The controller therefore cannot "plan ahead". This often results in unnecessary high control values and a rapidly changing control signal with peaks. Compared to this, by utilizing a system model, FMPC exhibits a significantly better controller performance in numerical simulations, see [2, 5]. A direct combination of both control techniques which allows for the application of FMPC in the presence of disturbances and even a structural plant-model mismatch was recently proposed in [3]. This approach was further extended in [17] by a learning component which realizes online learning of the model to allow for a steady improvement of the controller performance over time.

Nevertheless, the results of [2, 3, 17] are still restricted to the case of systems with relative degree one. Utilizing so-called feasibility constraints in the optimization problem and restricting the class of admissible funnel functions, the case of arbitrary relative degree was considered in [1]. Like in previous results no terminal conditions nor requirements on the length of the prediction horizon are imposed. But then again these feasibility constraints lead to an in-

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creased computational effort and they depend on a number of design parameters which are not easy to determine. Furthermore, the cost functional used in [1] is rather complex (using several auxiliary error variables). In the present paper, we resolve these problems and propose a novel cost functional to extend FMPC to systems with arbitrary relative degree. We further enlarge the considered system class considered in previous works to encompass systems with nonlinear time delays and potentially infinite-dimensional internal dynamics. Similar to FMPC for relative degree one systems, only the distance of one error variable to the funnel boundary is penalized and no feasibility constraints are required.

1.1. Nomenclature

$\mathbb{N}$ and $\mathbb{R}$ denote natural and real numbers, respectively. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_{\geq 0} := [0,\infty)$. $\|x\| := \sqrt{\langle x, x\rangle}$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. $\|A\|$ denotes the induced operator norm $\|A\| := \sup_{\|x\|=1} \|Ax\|$ for $A \in \mathbb{R}^{n\times n}$. $\mathrm{GL}_n(\mathbb{R})$ is the group of invertible $\mathbb{R}^{n\times n}$ matrices. $\mathcal{C}^p(V, \mathbb{R}^n)$ is the linear space of p-times continuously differentiable functions $f : V \to \mathbb{R}^n$, where $V \subset \mathbb{R}^m$ and $p \in \mathbb{N}_0 \cup \{\infty\}$. $\mathcal{C}(V, \mathbb{R}^n) := \mathcal{C}^0(V, \mathbb{R}^n)$. On an interval $I \subset \mathbb{R}$, $L^\infty(I, \mathbb{R}^n)$ denotes the space of measurable and essentially bounded functions $f : I \to \mathbb{R}^n$ with norm $\|f\|_\infty := \operatorname{ess\,sup}_{t\in I} \|f(t)\|$, $L^\infty_{\mathrm{loc}}(I, \mathbb{R}^n)$ the set of measurable and locally essentially bounded functions, and $L^p(I, \mathbb{R}^n)$ the space of measurable and p-integrable functions with norm $\|\cdot\|_{L^p}$ and with $p \geq 1$. Furthermore, $W^{k,\infty}(I, \mathbb{R}^n)$ is the Sobolev space of all k-times weakly differentiable functions $f : I \to \mathbb{R}^n$ such that $f, \ldots, f^{(k)} \in L^\infty(I, \mathbb{R}^n)$.

1.2. System class

We consider nonlinear control affine multi-input multi-output systems of the form

$$\begin{aligned} y^{(r)}(t) &= f\big(\mathbf{T}(y, \ldots, y^{(r-1)})(t)\big) \\ &\quad + g\big(\mathbf{T}(y, \ldots, y^{(r-1)})(t)\big)u(t), \\ &\left.\begin{aligned} y|_{[t_0-\sigma,t_0]} &= y^0 \in \mathcal{C}^{r-1}([t_0-\sigma, t_0], \mathbb{R}^m), &&\text{if } \sigma > 0, \\ \big(y(0), \ldots, y^{(r-1)}(0)\big) &= y^0 \in \mathbb{R}^{rm}, &&\text{if } \sigma = 0, \end{aligned}\right\} \end{aligned} \tag{1}$$

with $t_0 \geq 0$, "memory" $\sigma \geq 0$, functions $f \in \mathcal{C}(\mathbb{R}^q, \mathbb{R}^m)$, $g \in \mathcal{C}(\mathbb{R}^q, \mathbb{R}^{m\times m})$, and an operator $\mathbf{T}$. The operator $\mathbf{T}$ is causal, locally Lipschitz and satisfies a bounded-input bounded-output property and is characterised in detail in the following definition.

Definition 1.1. For $n, q \in \mathbb{N}$ and $\sigma \geq 0$, the set $\mathcal{T}^{n,q}_\sigma$ denotes the class of operators $\mathbf{T} : \mathcal{C}([-\sigma,\infty), \mathbb{R}^n) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^q)$ for which the following properties hold:

- *Causality*: $\forall\, y_1, y_2 \in \mathcal{C}([-\sigma,\infty), \mathbb{R}^n)\ \ \forall\, t \geq 0$:
$$y_1|_{[-\sigma,t]} = y_2|_{[-\sigma,t]} \quad\Longrightarrow\quad \mathbf{T}(y_1)|_{[0,t]} = \mathbf{T}(y_2)|_{[0,t]}.$$
- *Local Lipschitz*: $\forall\, t \geq 0\ \ \forall\, y \in \mathcal{C}([-\sigma, t]; \mathbb{R}^n)$ $\exists\, \Delta, \delta, c > 0\ \ \forall\, y_1, y_2 \in \mathcal{C}([-\sigma,\infty); \mathbb{R}^n)$ with $y_1|_{[-\sigma,t]} = y_2|_{[-\sigma,t]} = y$ and $\|y_1(s) - y(t)\| < \delta$, $\|y_2(s) - y(t)\| < \delta$ for all $s \in [t, t+\Delta]$:
$$\operatorname*{ess\,sup}_{s\in[t,t+\Delta]} \|\mathbf{T}(y_1)(s) - \mathbf{T}(y_2)(s)\| \leq c \sup_{s\in[t,t+\Delta]} \|y_1(s) - y_2(s)\|.$$
- *Bounded-input bounded-output (BIBO)*: $\forall\, c_0 > 0$ $\exists\, c_1 > 0\ \ \forall\, y \in \mathcal{C}([-\sigma,\infty), \mathbb{R}^n)$:
$$\sup_{t\in[-\sigma,\infty)} \|y(t)\| \leq c_0 \implies \sup_{t\in[0,\infty)} \|\mathbf{T}(y)(t)\| \leq c_1.$$

We note that many physical phenomena such as *backlash* and *relay hysteresis*, and *nonlinear time delays* can be modelled by means of the operator $\mathbf{T}$ where σ corresponds to the initial delay, cf. [4, Sec. 1.2]. Moreover, systems with infinite-dimensional internal dynamics can be represented by (1), see [6]. For a practically relevant example of infinite-dimensional internal dynamics appear (modelled by an operator $\mathbf{T}$) we refer to the moving water tank system considered in [7].

For $t_0 \geq 0$ and a control $u \in L^\infty_{\mathrm{loc}}([t_0,\infty), \mathbb{R}^m)$, a function $x = (x_1, \ldots, x_r)$ with $x_i : [t_0 - \sigma, \omega) \to \mathbb{R}^m$, $\omega \in (t_0, \infty]$, $i = 1, \ldots, r$, is called a solution of (1) (in the sense of *Carathéodory*), if

$$\begin{cases} x|_{[t_0-\sigma,t_0]} = (y^0, \dot{y}^0, \ldots, (y^0)^{(r-1)}), & \text{if } \sigma > 0, \\ x(0) = \big(x_1(0), \ldots, x_r(0)\big) = (y^0_1, \ldots, y^0_r) = y^0, & \text{if } \sigma = 0, \end{cases}$$

and $x|_{[t_0,\omega)}$ is absolutely continuous such that $\dot{x}_i(t) = x_{i+1}(t)$ for $i = 1, \ldots, r-1$, and $\dot{x}_r(t) = f(\mathbf{T}(x(t))) + g(\mathbf{T}(x(t)))u(t)$ for almost all $t \in [t_0, \omega)$. A solution x is said to be *maximal*, if it has no right extension that is also a solution. This maximal solution is called the *response* associated with u and denoted by $x(\cdot; t_0, y^0, u)$. Its first component x_1 is denoted by $y(\cdot; t_0, y^0, u)$. We summarize our assumptions and define the general system class under consideration.

Definition 1.2 (System class)**.** We say that the system (1) belongs to the *system class* $\mathcal{N}^{m,r}$ for $m, r \in \mathbb{N}$, written $(f, g, \mathbf{T}) \in \mathcal{N}^{m,r}$, if, for some $q \in \mathbb{N}$ and $\sigma \geq 0$, the following holds: $f \in \mathcal{C}(\mathbb{R}^q, \mathbb{R}^m)$, $g \in \mathcal{C}(\mathbb{R}^q, \mathbb{R}^{m\times m})$ satisfies $g(x) \in \mathrm{GL}_m(\mathbb{R})$ for all $x \in \mathbb{R}^q$, and $\mathbf{T} \in \mathcal{T}^{rm,q}_\sigma$.

1.3. Control objective

The objective is to design a control strategy which allows tracking of a given reference trajectory $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ within pre-specified error bounds. To be more precise, the tracking error $t \mapsto e(t) := y(t) - y_{\mathrm{ref}}(t)$ shall evolve within the prescribed performance funnel

$$\mathcal{F}_\psi = \{\ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m\ \mid\ \|e\| < \psi(t)\ \}.$$

This funnel is determined by the choice of the function ψ belonging to

$$\mathcal{G} := \left\{\ \psi \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})\ \left|\ \begin{array}{l} \inf_{t\geq 0} \psi(t) > 0, \\ \exists\, \alpha, \beta > 0\ \forall\, t \geq 0 : \\ \dot{\psi}(t) \geq -\alpha\psi(t) + \beta \end{array}\right.\right\},$$

see also Figure 1. Note that the evolution in $\mathcal{F}_\psi$ does not force the tracking error to converge to zero asymptotically. Furthermore, the funnel boundary is not necessarily monotonically decreasing and there are situations, like in the presence of periodic disturbances, where widening the funnel over some later time interval might be beneficial. The specific application usually dictates the constraints on the tracking error and thus indicates suitable choices for ψ.

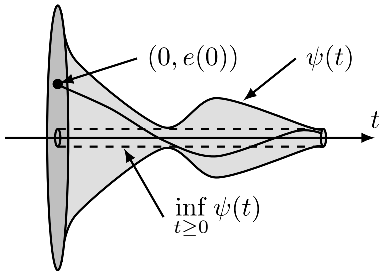


Figure 1: Error evolution in a funnel $\mathcal{F}_\psi$ with boundary $\psi(t)$.

To achieve the control objective, we introduce auxiliary error variables. Define, for parameters $k_1, \ldots, k_{r-1} \in \mathbb{R}_{\geq 0}$, the functions $e_i : \mathbb{R}^{rm} \to \mathbb{R}^m$, $i = 1, \ldots, r-1$, by

$$\begin{aligned} e_1 &: (\xi_1, \ldots, \xi_r) \mapsto \xi_1, \\ e_{i+1} &: (\xi_1, \ldots, \xi_r) \mapsto e_i(S(\xi)) + k_i e_i(\xi), \end{aligned} \tag{2}$$

for $i = 1, \ldots, r-1$, where

$$S : \mathbb{R}^{rm} \to \mathbb{R}^{rm}, \ (\xi_1, \ldots, \xi_r) \mapsto (\xi_2, \ldots, \xi_r, 0)$$

is the left shift operator.

Remark 1.3. Using the shorthand notation

$$\chi(\zeta)(t) := (\zeta(t), \dot{\zeta}(t), \ldots, \zeta^{(r-1)}(t)) \in \mathbb{R}^{rm}$$

for a function $\zeta \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and $t \in \mathbb{R}_{\geq 0}$, we get

$$\begin{aligned} e_1(\chi(\zeta)(t)) &= \zeta(t), \\ e_{i+1}(\chi(\zeta)(t)) &= \tfrac{\mathrm{d}}{\mathrm{d}t} e_i(\chi(\zeta)(t)) + k_i e_i(\chi(\zeta)(t)) \end{aligned} \tag{3}$$

for $i = 1, \ldots, r-1$. Furthermore, using the polynomials $p_i(s) = \prod_{j=1}^{i} (s + k_j) \in \mathbb{R}[s]$, the function $e_{i+1}(\chi(\zeta)(t))$ can be represented as

$$e_{i+1}(\chi(\zeta)(t)) = p_i(\tfrac{\mathrm{d}}{\mathrm{d}t})\zeta(t)$$

for $i = 1, \ldots, r-1$.

2. Funnel MPC

We propose for $\theta \in \mathcal{G}$, design parameter $\lambda_u \in \mathbb{R}_{\geq 0}$, and functions $e_1, \ldots, e_r$ as defined in (2) with parameters $k_i > 0$ for $i = 1, \ldots, r-1$, the *stage cost function* $\ell_\theta : \mathbb{R}_{\geq 0} \times \mathbb{R}^{rm} \times \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ defined by

$$\ell_\theta(t, \xi, u) = \begin{cases} \dfrac{\|e_r(\xi)\|^2}{\theta(t)^2 - \|e_r(\xi)\|^2} + \lambda_u \|u\|^2, & \|e_r(\xi)\| \neq \theta(t) \\ \infty, & \text{else.} \end{cases} \tag{4}$$

Algorithm 2.1 (Funnel MPC)**.**
Given: System (1), reference signal $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, funnel function $\theta \in \mathcal{G}$, input saturation level $M > 0$, initial data $y^0 \in \mathcal{C}^{r-1}([t_0 - \sigma, t_0], \mathbb{R}^m)$ if $\sigma > 0$ or $y^0 \in \mathbb{R}^{rm}$ if $\sigma = 0$, and stage cost function ℓ_θ as in (4).
Set the time shift $\delta > 0$, the prediction horizon $T \geq \delta$, and initialize the current time $\hat{t} := t_0$.
Steps:

(a) Obtain a measurement of the output y of (1) on the interval $[\hat{t} - \sigma, \hat{t}]$ and set $\hat{y} := y|_{[\hat{t}-\sigma,\hat{t}]}$ if $\sigma > 0$ and $\hat{y} := \big(y(\hat{t}), \ldots, y^{(r-1)}(\hat{t})\big)$ if $\sigma = 0$.

(b) Compute a solution $u^\star \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)$ of

$$\underset{\substack{u \in L^\infty([\hat{t},\hat{t}+T],\mathbb{R}^m), \\ \|u\|_\infty \leq M}}{\text{minimize}} \int_{\hat{t}}^{\hat{t}+T} \ell_\theta(t, \chi(y(\cdot; \hat{t}, \hat{y}, u) - y_{\mathrm{ref}})(t), u(t)) \, \mathrm{d}t. \tag{5}$$

(c) Apply the time-varying feedback law $\mu : [\hat{t}, \hat{t}+\delta) \times \mathcal{C}^{r-1}([\hat{t} - \sigma, \hat{t}], \mathbb{R}^m) \to \mathbb{R}^m$ (or, $\mu : [\hat{t}, \hat{t}+\delta) \times \mathbb{R}^{rm} \to \mathbb{R}^m$ if $\sigma = 0$), defined by

$$\mu(t, \hat{y}) = u^\star(t), \tag{6}$$

to system (1). Increase $\hat{t}$ by δ and go to Step (a).

Remark 2.2. For a nonlinear system of the form

$$\begin{aligned} \dot{x}(t) &= \tilde{f}(x(t)) + \tilde{g}(x(t))u(t), \quad x(t^0) = x^0 \in \mathbb{R}^n, \\ y(t) &= h(x(t)), \end{aligned} \tag{7}$$

with nonlinear functions $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$, $\tilde{g} : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$, there exists, under assumptions provided in [9, Cor. 5.6], a coordinate transformation induced by a diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ which puts the system in the form (1) with $\sigma = 0$, appropriate functions f and g and an operator $\mathbf{T}$, which is the solution operator of the internal dynamics of the transformed system. Assuming the existence of the diffeomorphism Φ, the Funnel MPC Algorithm 2.1 can be directly applied to the system (7) without computing Φ. In this case, the output derivatives $\dot{y}, \ldots, y^{(r-1)}$ required in the OCP (5) can be determined as functions of the state; e.g., $\dot{y}(t) = h'(x(t))\tilde{f}(x(t))$ if $r = 1$. All results presented in this paper can also be expressed for the system (7) using Φ. Concrete knowledge about the coordinate transformation however is not required for the design and application of the controller – it is merely used as a tool for the proofs.

In the following main result we show that for a funnel function $\psi \in \mathcal{G}$, a reference signal y_{ref} and sufficiently large $k_1, \ldots, k_{r-1}$ (depending on the choice of ψ, y_{ref} and the initial data y^0) there exists a sufficiently large saturation level $M > 0$ such that the FMPC Algorithm 2.1 (with a suitable function $\theta \in \mathcal{G}$) is initially and recursively feasible for every prediction horizon $T > 0$ and that it guarantees the evolution of the tracking error within the performance funnel $\mathcal{F}_\psi$.

Theorem 2.3. *Consider system* (1) *with* $(f, g, \mathbf{T}) \in \mathcal{N}^{m,r}$ *and initial data* $y^0 \in \mathcal{C}^{r-1}([t_0 - \sigma, t_0], \mathbb{R}^m)$ *if* $\sigma > 0$ *or* $y^0 \in \mathbb{R}^{rm}$ *if* $\sigma = 0$*. If* $\sigma = 0$*, then, for brevity, we identify* $(y^0)^{(i-1)}(t_0) = y_i^0$ *for* $i = 1, \ldots, r$ *in the following. Let* $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ *and choose* $\psi \in \mathcal{G}$ *with associated constants* $\alpha, \beta > 0$ *such that*

$$\exists\, \gamma \in (0,1) : \ \left\| y^0(t_0) - y_{\mathrm{ref}}(t_0) \right\| \leq \gamma^r \psi(t_0).$$

Furthermore, choose parameters $k_1, \ldots, k_{r-1}$ *such that for all* $i = 2, \ldots, r-1$ *we have*

$$\begin{aligned} k_1 &\geq \frac{2\left\|(\dot{y}^0 - \dot{y}_{\mathrm{ref}})(t_0)\right\|}{\gamma^{r-1}(1-\gamma)\psi(t_0)} + \frac{2\left(\alpha + \frac{1}{\gamma^{r-1}}\right)}{1-\gamma}, \\ k_i &\geq \frac{2\gamma \left\| \frac{\mathrm{d}}{\mathrm{d}t} e_i(\chi(y^0 - y_{\mathrm{ref}})(t_0)) \right\|}{(1-\gamma)\left(\|e_i(\chi(y^0 - y_{\mathrm{ref}})(t_0))\| + \frac{\beta}{\alpha\gamma^{i-2}} \right)} + \frac{2(1+\alpha)}{1-\gamma}. \end{aligned} \tag{8}$$

Then there exists $M > 0$ such that the FMPC Algorithm 2.1 with prediction horizon $T > 0$, time shift $\delta > 0$, and stage cost function ℓ_θ with

$$\theta(t) := \frac{1}{\gamma}\big(\left\| \tfrac{\mathrm{d}}{\mathrm{d}t} e_{r-1}(\chi(y^0 - y_{\mathrm{ref}})(t_0)) \right\| + k_{r-1} \left\| e_{r-1}(\chi(y^0 - y_{\mathrm{ref}})(t_0)) \right\| \big) e^{-\alpha(t-t_0)} + \frac{\beta}{\alpha\gamma^{r-1}} \tag{9}$$

is initially and recursively feasible, i.e., at time $\hat{t} = t_0$ and at each successor time $\hat{t} \in t_0 + \delta\mathbb{N}$ the OCP (5) has a solution. In particular, the closed-loop system consisting of (1) and the FMPC feedback (6) has a (not necessarily unique) global solution $x : [t_0 - \sigma, \infty) \to \mathbb{R}^{rm}$ with corresponding output $y = x_1$ and the corresponding input is given by

$$u_{\mathrm{FMPC}}(t) = \begin{cases} \mu(t, y|_{[\hat{t}-\sigma,\hat{t}]}), & \text{if } \sigma > 0, \\ \mu\big(t, \big(y(\hat{t}), \ldots, y^{(r-1)}(\hat{t})\big)\big), & \text{if } \sigma = 0, \end{cases}$$

for $t \in [\hat{t}, \hat{t} + \delta)$ and $\hat{t} \in t^0 + \delta\mathbb{N}$. Furthermore, each global solution x with corresponding output y and input u_{FMPC} satisfies:

(i) $\forall\, t \geq t^0 : \quad \|u_{\mathrm{FMPC}}(t)\| \leq M$,

(ii) $\forall\, t \geq t_0 : \quad \|y(t) - y_{\mathrm{ref}}(t)\| < \psi(t)$.

3. Proof of the main result

Throughout this section, let the assumptions of Theorem 2.3 hold. Then set $e_i^0 := e_i(\chi(y^0 - y_{\mathrm{ref}})(t_0))$ and $\dot{e}_i^0 := \frac{\mathrm{d}}{\mathrm{d}t} e_i(\chi(y^0 - y_{\mathrm{ref}})(t_0))$ for $i = 1, \ldots, r-1$ and we define $\psi_1 := \psi$ and $\psi_2, \ldots, \psi_r$ as follows:

$$\psi_{i+1}(t) := \frac{1}{\gamma^{r-i}}\big(\left\| \dot{e}_i^0 \right\| + k_i \left\| e_i^0 \right\| \big) \mathrm{e}^{-\alpha(t-t_0)} + \frac{\beta}{\alpha\gamma^{r-1}} \tag{10}$$

for $t \geq t_0$ and $i = 1, \ldots, r-1$. Note that $\psi_i \in \mathcal{G}$ for all $i = 1, \ldots, r$. Further note that $\psi_r = \theta$ as in (9). In order to achieve that the tracking error $e = y - y_{\mathrm{ref}}$ evolves within the funnel $\mathcal{F}_\psi$, we address the problem of ensuring that, for all $t \geq t_0$, $\chi(e)(t)$ is an element of the set

$$\mathcal{D}_t^r := \left\{\, \xi \in \mathbb{R}^{rm} \;\middle|\; \|e_i(\xi)\| < \psi_i(t),\ i = 1, \ldots, r \,\right\}. \tag{11}$$

By construction of ψ_i and (3) we have

$$\|e_i(t_0)\| \leq \|\dot{e}_{i-1}(t_0)\| + k_{i-1}\|e_{i-1}(t_0)\| < \psi_i(t_0)$$

for all $i = 2, \ldots, r$, and by assumption we have

$$\|e(t_0)\| \leq \gamma^r \psi(t_0) < \psi_1(t_0).$$

Therefore, $\chi(y^0 - y_{\mathrm{ref}})(t_0) \in \mathcal{D}_{t_0}^r$.

We define the set of all functions $\zeta \in \mathcal{C}^{r-1}([t_0 - \sigma, \infty), \mathbb{R}^m)$ which coincide with y^0 on the interval $[t_0 - \sigma, t_0]$ and for which $\chi(\zeta - y_{\mathrm{ref}})(t) \in \mathcal{D}_t^r$ on the interval $I_{t_0}^\tau := [t_0, \tau)$ for some $\tau \in (t_0, \infty]$ as follows; recall that if $\sigma = 0$, then we identify $(y^0)^{(i-1)}(t_0) = y_i^0$ for $i = 1, \ldots, r$.

$$\mathcal{Y}_\tau^r := \left\{ \zeta \in \mathcal{C}^{r-1}(I_{t_0-\sigma}^\infty, \mathbb{R}^m) \;\middle|\; \begin{array}{l} \chi(\zeta|_{[t_0-\sigma,t_0]}) = \chi(y^0), \\ \forall\, t \in I_{t_0}^\tau : \chi(\zeta - y_{\mathrm{ref}})(t) \in \mathcal{D}_t^r \end{array} \right\}.$$

Lemma 3.1. *Consider the system (1) with $(f, g, \mathbf{T}) \in \mathcal{N}^{m,r}$. Let $\psi_i \in \mathcal{G}$, for $i = 1, \ldots, r$ with parameters $k_i > 0$ for $i = 1, \ldots, r-1$. Further, let $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and $y^0 \in \mathcal{C}^{r-1}([t_0 - \sigma, t_0], \mathbb{R}^m)$ if $\sigma > 0$ or $y^0 \in \mathbb{R}^{rm}$ if $\sigma = 0$, with $\chi(y^0 - y_{\mathrm{ref}})(t_0) \in \mathcal{D}_{t_0}^r$, where we identify $(y^0)^{(i-1)}(t_0) = y_i^0$ for $i = 1, \ldots, r$ if $\sigma = 0$. Then, there exist constants $f_{\max}, g_{\max} > 0$ such that for all $\tau \in (t_0, \infty]$, $\zeta \in \mathcal{Y}_\tau^r$, and $t \in [t_0, \tau)$*

$$f_{\max} \geq \left\| f(\mathbf{T}(\chi(\zeta))|_{[t_0,\tau)}) \right\|_\infty,$$
$$g_{\max} \geq \left\| g(\mathbf{T}(\chi(\zeta))|_{[t_0,\tau)})^{-1} \right\|_\infty.$$

Proof. We prove the Lemma by adapting the proof of [18, Lem. 1.2] to the given setting. By definition of $\mathcal{Y}_\infty^r$ and $\mathcal{D}_t^r$, we have for all $i = 1, \ldots, r$

$$\forall\, \zeta \in \mathcal{Y}_\infty^r\ \forall\, t \geq t_0 : \quad \|e_i(\chi(\zeta - y_{\mathrm{ref}})(t))\| < \psi_i(t).$$

Due to the definition of the error variables e_i there exists an invertible matrix $S \in \mathbb{R}^{rm \times rm}$ such that

$$\begin{pmatrix} e_1(\chi(\zeta - y_{\mathrm{ref}})) \\ \vdots \\ e_r(\chi(\zeta - y_{\mathrm{ref}})) \end{pmatrix} = S\chi(\zeta - y_{\mathrm{ref}}). \tag{12}$$

Hence, by boundedness of ψ_i and $y_{\mathrm{ref}}^{(i)}$ for all $i = 1, \ldots, r$, there exists a compact set $K \subset \mathbb{R}^{rm}$ with

$$\forall\, \zeta \in \mathcal{Y}_\infty^r\ \forall\, t \geq t_0 : \quad \chi(\zeta)(t) \in K.$$

Invoking the BIBO property of the operator $\mathbf{T}$, there exists a compact set $K_q \subset \mathbb{R}^q$ with $\mathbf{T}(\xi)([t_0, \infty)) \subset K_q$ for all $\xi \in \mathcal{C}([t_0, \infty), \mathbb{R}^{rm})$ with $\xi([t_0, \infty)) \subset K$. For arbitrary $\tau \in (t_0, \infty)$ and $\zeta \in \mathcal{Y}_\tau^r$, we have $\chi(\zeta)(t) \in K$ for all $t \in [t_0, \tau)$. For every element $\zeta \in \mathcal{Y}_\tau^r$ the function $\chi(\zeta)|_{[t_0 - \sigma}, \tau)$ can be smoothly extended to a function $\tilde{\zeta} \in \mathcal{C}([t_0 - \sigma, \infty), \mathbb{R}^m)^r$ with $\tilde{\zeta}(t) \in K$ for all $t \in [t_0, \infty)$. We have $\mathbf{T}(\tilde{\zeta})(t) \in K_q$ for all $t \in \mathbb{R}_{\geq 0}$ because of the BIBO property of the operator $\mathbf{T}$. This implies $\mathbf{T}(\chi(\zeta))|_{[t_0,\tau)} \in K_q$ for all $t \in [t_0, \tau)$ and $\zeta \in \mathcal{Y}_\tau^r$ since $\mathbf{T}$ is causal. Since $f(\cdot)$ and $g(\cdot)^{-1}$ are continuous, the constants $f_{\max} = \max_{x \in K_q} \|f(x)\|$ and $g_{\max} = \max_{x \in K_q} \left\|g(x)^{-1}\right\|$ are well-defined. For all $\tau \in (t_0, \infty]$ and $\zeta \in \mathcal{Y}_\tau^r$ we have

$$\forall\, t \in [t_0, \tau) : \ \mathbf{T}(\chi(\zeta))(t) \in K_q,$$

which proves the assertion. □

Lemma 3.2. *Under the assumptions of Theorem 2.3, consider the functions $\psi_2, \ldots, \psi_r \in \mathcal{G}$ defined in (10). Let $\hat{t} \geq t_0$ and $\zeta \in \mathcal{C}^{r-1}([\hat{t}, \infty), \mathbb{R}^m)$ be such that $\chi(\zeta)(\hat{t}) \in \mathcal{D}_{\hat{t}}^r$. If $\|e_r(\chi(\zeta)(t))\| < \psi_r(t)$ for all $t \in [\hat{t}, \tau)$ for some $\tau > \hat{t}$, then $\chi(\zeta)(t) \in \mathcal{D}_t^r$ for all $t \in [\hat{t}, \tau)$.*

Proof. Seeking a contradiction, we assume that for at least one $i \in \{1, \ldots, r-1\}$ there exists $t \in (\hat{t}, \tau)$ such that $\|e_i(\chi(\zeta)(t))\| \geq \psi_i(t)$. W.l.o.g. let i be the largest index with this property. In the following we use the shorthand notation $e_i(t) := e_i(\chi(\zeta)(t))$ and $e(t) = e_1(t)$. However, we like to emphasize that $e_i(t_0) \neq e_i^0$ (if e_i is defined at t_0) in general, since $\chi(\zeta)(t_0) \neq \chi(y^0)(t_0)$ is possible. Invoking $\left\|e_i(\hat{t})\right\| < \psi_i(\hat{t})$ and continuity of the involved functions, define $t^\star := \min\left\{\, t \in [\hat{t}, \tau) \;\middle|\; \|e_i(t)\| = \psi_i(t) \,\right\}$.

Set $\varepsilon := \sqrt{\frac{1}{2}(1+\gamma)} \in (0,1)$. Due to continuity there exists $t_\star := \max\left\{ t \in [\hat{t}, t^\star) \;\middle|\; \left\|\frac{e_i(t)}{\psi_t(t)}\right\| = \varepsilon \right\}$, hence we have that $\varepsilon \le \left\|\frac{e_i(t)}{\psi_t(t)}\right\| \le 1$ for all $t \in [t_\star, t^\star]$. Utilizing (3) and omitting the dependency on t, we calculate for $t \in [t_\star, t^\star]$:

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{e_i}{\psi_i}\right\|^2 &= \left\langle \frac{e_i}{\psi_i}, \frac{\dot{e}_i\psi_i - e_i\dot{\psi}_i}{\psi_i^2} \right\rangle \\
&= \left\langle \frac{e_i}{\psi_i}, -\left(k_i + \frac{\dot{\psi}_i}{\psi_i}\right)\frac{e_i}{\psi_i} + \frac{e_{i+1}}{\psi_i} \right\rangle \\
&\le -\left(k_i + \frac{\dot{\psi}_i}{\psi_i}\right)\left\|\frac{e_i}{\psi_i}\right\|^2 + \left\|\frac{e_i}{\psi_i}\right\| \frac{\|e_{i+1}\|}{\psi_i} \\
&\le -\left(k_i + \frac{\dot{\psi}_i}{\psi_i}\right)\varepsilon^2 + \frac{\psi_{i+1}}{\psi_i},
\end{aligned}$$

where we used $\|e_{i+1}(t)\| \le \psi_{i+1}(t)$ due to the maximality of i. Now we distinguish the two cases $i = 1$ and $i > 1$. For $i = 1$ we find that $\psi_1 = \psi$ and by properties of $\mathcal{G}$ it follows

$$-\frac{\dot{\psi}(t)}{\psi(t)} \le \frac{\alpha\psi(t) - \beta}{\psi(t)} \le \alpha.$$

Furthermore, we have that $\psi(t) \ge \psi(t_0)\mathrm{e}^{-\alpha(t-t_0)} + \frac{\beta}{\alpha}$ for all $t \ge t_0$. Therefore,

$$\begin{aligned}
\frac{\psi_2(t)}{\psi(t)} &\le \frac{1}{\gamma^{r-1}} \frac{\left(\|\dot{e}_1^0\| + k_1\|e_1^0\|\right)\mathrm{e}^{-\alpha(t-t_0)}}{\psi(t_0)\mathrm{e}^{-\alpha(t-t_0)} + \frac{\beta}{\alpha}} \\
&\quad + \frac{\beta}{\alpha\gamma^{r-1}\left(\psi(t_0)\mathrm{e}^{-\alpha(t-t_0)} + \frac{\beta}{\alpha}\right)} \\
&\le \frac{1}{\gamma^{r-1}} \frac{\|\dot{e}_1^0\| + k_1\|e_1^0\|}{\psi(t_0)} + \frac{1}{\gamma^{r-1}} \le \gamma k_1 + \frac{\|\dot{e}_1^0\|}{\gamma^{r-1}\psi(t_0)} + \frac{1}{\gamma^{r-1}}
\end{aligned}$$

for all $t \ge t_0$, where we have use that $\|e(t_0)\| \le \gamma^r\psi(t_0)$. Hence we obtain that

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{e}{\psi}\right\|^2 &\le -\tfrac{1}{2}(k_1 - \alpha)(1+\gamma) + \gamma k_1 + \frac{\|\dot{e}_1^0\|}{\gamma^{r-1}\psi(t_0)} + \frac{1}{\gamma^{r-1}} \\
&\le -\tfrac{1}{2}(1-\gamma)k_1 + \alpha + \frac{\|\dot{e}_1^0\|}{\gamma^{r-1}\psi(t_0)} + \frac{1}{\gamma^{r-1}} \le 0
\end{aligned}$$

for all $t \in [t_\star, t^\star]$, where the last inequality follows from (8). Now consider the case $i > 1$. Then we have $-\frac{\dot{\psi}_i(t)}{\psi_i(t)} \le \alpha$ for all $t \ge 0$ and, invoking that by (3)

$$\|e_i^0\| \le \|\dot{e}_{i-1}^0\| + k_{i-1}\|e_{i-1}^0\|,$$

we find that

$$\begin{aligned}
\frac{\psi_{i+1}(t)}{\psi_i(t)} &= \frac{\frac{1}{\gamma^{r-i}}\left(\|\dot{e}_i^0\| + k_i\|e_i^0\|\right)\mathrm{e}^{-\alpha(t-t_0)} + \frac{\beta}{\alpha\gamma^{r-1}}}{\frac{1}{\gamma^{r-i+1}}\left(\|\dot{e}_{i-1}^0\| + k_{i-1}\|e_{i-1}^0\|\right)\mathrm{e}^{-\alpha(t-t_0)} + \frac{\beta}{\alpha\gamma^{r-1}}} \\
&\le \gamma \frac{\|\dot{e}_i^0\| + k_i\|e_i^0\|}{\|\dot{e}_{i-1}^0\| + k_{i-1}\|e_{i-1}^0\| + \frac{\beta}{\alpha\gamma^{i-2}}} + 1 \\
&\le \gamma k_i + \gamma\frac{\|\dot{e}_i^0\|}{\|e_i^0\| + \frac{\beta}{\alpha\gamma^{i-2}}} + 1
\end{aligned}$$

for all $t \ge t_0$. Hence we obtain that

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{e_i}{\psi_i}\right\|^2 &\le -\tfrac{1}{2}(k_i - \alpha)(1+\gamma) + \gamma k_i + \gamma\frac{\|\dot{e}_i^0\|}{\|e_i^0\| + \frac{\beta}{\alpha\gamma^{i-2}}} + 1 \\
&\le -\tfrac{1}{2}(1-\gamma)k_i + \alpha + \gamma\frac{\|\dot{e}_i^0\|}{\|e_i^0\| + \frac{\beta}{\alpha\gamma^{i-2}}} + 1 \le 0
\end{aligned}$$

for all $t \in [t_\star, t^\star]$, where the last inequality follows from (8). Summarizing, in each case the contradiction

$$1 \le \|e_i(t^*)/\psi_i(t^*)\|^2 \le \|e_i(t_*)/\psi_i(t_*)\|^2 = \varepsilon^2 < 1$$

arises, which completes the proof. □

For $\hat{t} \ge t_0$, $M > 0$, $T > 0$ and $\hat{y} \in \mathcal{C}^{r-1}([\hat{t} - \sigma, \hat{t}], \mathbb{R}^m)$ if $\sigma > 0$ or $\hat{y} \in \mathbb{R}^{rm}$ if $\sigma = 0$ we denote by $\mathcal{U}_T(M, \hat{t}, \hat{y})$ the set

$$\left\{ u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m) \;\middle|\; \begin{array}{l} x(t; \hat{t}, \hat{y}, u) - \chi(y_{\mathrm{ref}})(t) \in \mathcal{D}_t^r \\ \text{for all } t \in [\hat{t}, \hat{t}+T], \|u\|_\infty \le M \end{array} \right\}. \tag{13}$$

This is the set of all L^∞-controls u bounded by M which, if applied to the system (1), guarantee that the error signals $e_i(x(t; \hat{t}, \hat{y}, u) - \chi(y_{\mathrm{ref}})(t))$ evolve within their respective funnels defined by ψ_i on the interval $[\hat{t}, \hat{t} + T]$. We note that the conditions in (13) implicitly require the solution $x(\cdot; \hat{t}, \hat{y}, u)$ to exist on the interval $[\hat{t}, \hat{t} + T]$.

Lemma 3.3. *Under the assumptions of Theorem 2.3, consider the functions $\psi_2, \ldots, \psi_r \in \mathcal{G}$ defined in (10). Further, let $\hat{t} \ge t_0$ and $\hat{y} \in \mathcal{C}^{r-1}([\hat{t} - \sigma, \hat{t}], \mathbb{R}^m)$ if $\sigma > 0$ or $\hat{y} \in \mathbb{R}^{rm}$ if $\sigma = 0$. If $\chi(\hat{y} - y_{\mathrm{ref}})(\hat{t}) \in \mathcal{D}_{\hat{t}}^r$ (where we identify $(\hat{y})^{(i-1)}(\hat{t}) = \hat{y}_i$ for $i = 1, \ldots, r$ if $\sigma = 0$), then there exists $M > 0$ such that for all $T > 0$ we have*

$$\mathcal{U}_T(M, \hat{t}, \hat{y}) \ne \emptyset.$$

Furthermore,

$$\begin{aligned}
&\forall\, T_1, T_2 > 0 \;\forall\, u \in \mathcal{U}_{T_1}(M, \hat{t}, \hat{y}) \;\forall\, t \in [\hat{t}, \hat{t} + T_1]: \\
&\begin{cases} \mathcal{U}_{T_2}(M, t, y(\cdot; \hat{t}, \hat{y}, u)|_{[t-\sigma, t]}) \ne \emptyset, & \text{if } \sigma > 0, \\ \mathcal{U}_{T_2}(M, t, y(t; \hat{t}, \hat{y}, u)) \ne \emptyset, & \text{if } \sigma = 0. \end{cases}
\end{aligned} \tag{14}$$

Proof. *Step 1*: We define $M > 0$. To that end, define, for $i = 1, \ldots, r$ and $j = 0, \ldots, r - i - 1$

$$\mu_i^0 := \|\psi_i\|_\infty, \quad \mu_i^{j+1} := \mu_{i+1}^j + k_i\mu_i^j. \tag{15}$$

Using the constants $f_{\max}$ and $g_{\max}$ from Lemma 3.1, define

$$M := g_{\max}\left(f_{\max} + \left\|y_{\mathrm{ref}}^{(r)}\right\|_\infty + \sum_{j=1}^{r-1} k_j\mu_j^{r-j} + \left\|\dot{\psi}_r\right\|_\infty \right).$$

Step 2: Let $T > 0$ be arbitrary. We construct a control function u and show that $u \in \mathcal{U}_T(M, \hat{t}, \hat{y})$. To this end, for some $u \in L^\infty([\hat{t}, \hat{t} + T], \mathbb{R}^m)$, we use the shorthand notation $y(t) := y(t; \hat{t}, \hat{y}, u)$, $e(t) := y(t) - y_{\mathrm{ref}}(t)$ and $e_i(t) := e_i(\chi(y - y_{\mathrm{ref}})(t))$ for $i = 1, \ldots, r$. The application of the output feedback

$$\begin{aligned}
u(t) := g(\mathbf{T}(\chi(y))(t))^{-1}\Big(&-f(\mathbf{T}(\chi(y))(t)) + y_{\mathrm{ref}}^{(r)}(t) \\
&- \sum_{j=1}^{r-1} k_j e_j^{(r-j)}(t) + e_r(t)\tfrac{\dot{\psi}_r(t)}{\psi_r(t)}\Big)
\end{aligned} \tag{16}$$

to the system (1) leads to a closed-loop system. If this initial value problem is considered on the interval $[\hat t, \hat t+T]$ with initial conditions

$$\begin{aligned} y|_{[\hat t-\sigma,\hat t]} &= \hat y, && \text{if } \sigma>0,\\ \big(y(\hat t),\dots,y^{(r-1)}(\hat t)\big) &= \hat y, && \text{if } \sigma=0, \end{aligned}$$

then an application of a variant (a straightforward modification tailored to the current context) of [15, Thm. B.1] yields the existence of a maximal solution $x : [\hat t-\sigma,\omega) \to \mathbb{R}^{rm}$. If x is bounded, then $\omega=\infty$, so the solution exists on $[\hat t-\sigma, \hat t+T]$. Utilizing (3) one can show that

$$e_r(t) = e^{(r-1)}(t) + \sum_{j=1}^{r-1} k_j e_j^{(r-j-1)}(t). \tag{17}$$

Omitting the dependency on t, we calculate for $t\in[\hat t,\omega)$:

$$\begin{aligned} &\frac{\dot e_r\psi_r - e_r\dot\psi_r}{\psi_r} = e^{(r)} + \sum_{j=1}^{r-1} k_j e_j^{(r-j)} - e_r\frac{\dot\psi_r}{\psi_r}\\ &= f(\mathbf{T}(\chi(y))) + g(\mathbf{T}(\chi(y)))u - y_{\mathrm{ref}}^{(r)} + \sum_{j=1}^{r-1} k_j e_j^{(r-j)} - e_r\frac{\dot\psi_r}{\psi_r} = 0. \end{aligned}$$

Therefore,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac12\left\|\frac{e_r}{\psi_r}\right\|^2 = \left\langle \frac{e_r}{\psi_r}, \frac{\dot e_r\psi_r - e_r\dot\psi_r}{\psi_r^2}\right\rangle = 0.$$

Since $\left\|\frac{e_r(\hat t)}{\psi_r(\hat t)}\right\| < 1$ by the assumption $\chi(\hat y - y_{\mathrm{ref}})(\hat t) \in \mathcal{D}^r_{\hat t}$, this yields $\left\|\frac{e_r(t)}{\psi_r(t)}\right\| < 1$ for all $t\in[\hat t,\omega)$. This implies, according to Lemma 3.2, $\chi(y-y_{\mathrm{ref}})(t) \in \mathcal{D}^r_t$ for all $t\in[\hat t,\omega)$, i.e., $\|e_i(t)\| < \psi_i(t)$ for all $i=1,\dots,r$. Thus, $\|e_i(t)\| \le \mu_i^0$ for all $i=1,\dots,r$. Invoking boundedness of $y_{\mathrm{ref}}^{(i)}$, $i=0,\dots,r$, and the relation in (12), we may infer that $x=\chi(y)$ is bounded on $[\hat t,\omega)$. Hence, $\omega=\infty$. Furthermore, $\|f(\mathbf{T}(\chi(y))(t))\| \le f_{\max}$ and $\left\|g(\mathbf{T}(\chi(y))(t))^{-1}\right\| \le g_{\max}$ for all $t\in[\hat t,\hat t+T]$ according to Lemma 3.1. Finally, using (3) and the definition of μ_i^j it follows that

$$\left\|e_i^{(j+1)}(t)\right\| = \left\|e_{i+1}^{(j)}(t) - k_i e_i^{(j)}(t)\right\| \le \mu_{i+1}^j + k_i\mu_i^j = \mu_i^{j+1} \tag{18}$$

inductively for all $i=1,\dots,r$ and $j=0,\dots,r-i-1$. Thus, by definition of u and M we have $\|u\|_\infty \le M$ and hence $u\in\mathcal{U}_T(M,\hat t,\hat y)$.

Step 3: We show implication (14). If, for any $T_1>0$ an arbitrary but fixed control $\hat u \in \mathcal{U}_{T_1}(M,\hat t,\hat y)$ is applied to the system (1), then $x(t;\hat t,\hat y,u) - \chi(y_{\mathrm{ref}})(t) \in \mathcal{D}^r_t$ for all $t\in[\hat t,\hat t+T_1]$. If for any $\tilde t\in[\hat t,\hat t+T]$, the system is considered on the interval $[\tilde t,\tilde t+T_2]$ with $T_2>0$ and initial data $\tilde y := y(\cdot;\hat t,\hat y,u)|_{[\tilde t-\sigma,\tilde t]}$ if $\sigma>0$ or $\tilde y := x(\tilde t;\hat t,\hat y,u)$ if $\sigma=0$, then one can show by a repetition of the arguments in Step 2 that the application of the control $\tilde u \in L^\infty([\tilde t,\tilde t+T_2],\mathbb{R}^m)$ as in (16), *mutatis mutandis*, guarantees $x(t;\tilde t,\tilde y,\tilde u) - \chi(y_{\mathrm{ref}})(t) \in \mathcal{D}^r_t$ for all $t\in[\tilde t,\tilde t+T_2]$. Since the prerequisites for Lemmata 3.1 and 3.2 are still satisfied, the control $\tilde u$ is bounded by M as constructed in Step 1. Thus, $\tilde u \in \mathcal{U}_{T_2}(M,\tilde t,\tilde y) \neq \emptyset$. □

Lemma 3.4. *Under the assumptions of Theorem 2.3, consider the functions $\psi_2,\dots,\psi_r\in\mathcal{G}$ defined in (10). Further, let $T>0$ $M>0$, $\hat t\ge t_0$, and $\hat y\in\mathcal{C}^{r-1}([\hat t-\sigma,\hat t],\mathbb{R}^m)$ if $\sigma>0$ or $\hat y\in\mathbb{R}^{rm}$ if $\sigma=0$ such that $\mathcal{U}_T(M,\hat t,\hat y)\neq\emptyset$. Then, $\mathcal{U}_T(M,\hat t,\hat y)$ is equal to the set $\tilde{\mathcal{U}}_T(M,\hat t,\hat y)$ defined by*

$$\left\{ u\in L^\infty([\hat t,\hat t+T],\mathbb{R}^m) \;\middle|\; \begin{array}{l} x(t;\hat t,\hat y,u) \text{ satisfies (1) for all}\\ t\in[\hat t,\hat t+T],\ \|u\|_\infty\le M,\\ \int_{\hat t}^{\hat t+T}\ell_{\psi_r}(t,\zeta(t),u(t))\,\mathrm{d}t<\infty,\\ \zeta(t) := x(t;\hat t,\hat y,u)-\chi(y_{\mathrm{ref}})(t) \end{array} \right\}.$$

Proof. We adapt the proof of [2, Thm. 4.3] to the current setting. Given $u\in\mathcal{U}_T(M,\hat t,\hat y)$, it follows from the definition of $\mathcal{U}_T(M,\hat t,\hat y)$ that $\zeta(t) := x(t;\hat t,\hat y,u) - \chi(y_{\mathrm{ref}})(t) \in \mathcal{D}^r_t$ for all $t\in[\hat t,\hat t+T]$. Thus,

$$\forall\, t\in[\hat t,\hat t+T]:\ \|e_r(\zeta(t))\| < \psi_r(t).$$

We use the shorthand notation $e_r(t) := e_r(\zeta(t))$. Due to continuity of the involved functions, there exists $\varepsilon\in(0,1)$ with $\|e_r(t)\|^2 \le \psi_r(t)^2 - \varepsilon$ for all $t\in[\hat t,\hat t+T]$. Then, $\ell_{\psi_r}(t,\zeta(t),u(t)) \ge 0$ for all $t\in[\hat t,\hat t+T]$ and

$$\begin{aligned} &\int_{\hat t}^{\hat t+T} |\ell_{\psi_r}(t,\zeta(t),u(t))|\ \mathrm{d}t\\ &= \int_{\hat t}^{\hat t+T}\left|\frac{\|e_r(t)\|^2}{\psi_r(t)^2 - \|e_r(t)\|^2} + \lambda_u\|u(t)\|^2\right|\ \mathrm{d}t\\ &\le \int_{\hat t}^{\hat t+T}\frac{\|\psi_r\|_\infty}{\varepsilon} + \lambda_u\|u\|_\infty^2\,\mathrm{d}t \le \left(\frac{\|\psi_r\|_\infty}{\varepsilon} + \lambda_u M^2\right)T < \infty. \end{aligned}$$

Therefore, $\mathcal{U}_T(M,\hat t,\hat y)$ is contained in $\tilde{\mathcal{U}}_T(M,\hat t,\hat y)$.

Let $u\in\tilde{\mathcal{U}}_T(M,\hat t,\hat y)$. We show that $\zeta(t)\in\mathcal{D}^r_t$ for all $t\in[\hat t,\hat t+T]$. Since $\mathcal{U}_T(M,\hat t,\hat y)\neq\emptyset$, we have $\chi(\zeta)(\hat t) = \chi(\hat y - y_{\mathrm{ref}})(\hat t) \in \mathcal{D}^r_{\hat t}$ (where we identify $(\hat y)^{(i-1)}(\hat t) = \hat y_i$ for $i=1,\dots,r$ if $\sigma=0$). According to Lemma 3.2 it suffices to show that $\|e_r(t)\| < \psi_r(t)$ for all $t\in[\hat t,\hat t+T]$. Assume there exists $t\in[\hat t,\hat t+T]$ with $\|e_r(t)\| > \psi_r(t)$. By continuity of the involved functions, there exists

$$\tilde t := \min\left\{\ t\in[\hat t,\hat t+T]\ \middle|\ \|e_r(t)\| = \psi_r(t)\ \right\}.$$

Recalling the definition of the Lebesgue integral, see e.g. [22, Def 11.22], $\int_{\hat t}^{\hat t+T}\ell_{\psi_r}(t,\zeta(t),u(t))\,\mathrm{d}t < \infty$ implies $\int_{\hat t}^{\hat t+T}\left(\ell_{\psi_r}(t,\zeta(t),u(t))\right)^+\ \mathrm{d}t<\infty$. Thus,

$$\begin{aligned} \int_{\hat t}^{\tilde t}\frac{1}{1-\frac{\|e_r(t)\|^2}{\psi_r(t)^2}}\,\mathrm{d}t &= \int_{\hat t}^{\tilde t}\frac{\|e_r(t)\|^2}{\psi_r(t)^2 - \|e_r(t)\|^2} + 1\,\mathrm{d}t\\ &\le \int_{\hat t}^{\hat t+T}\left(\frac{\|e_r(t)\|^2}{\psi_r(t)^2 - \|e_r(t)\|^2}\right)^+\ \mathrm{d}t + T\\ &\le \int_{\hat t}^{\hat t+T}\left(\frac{\|e_r(t)\|^2}{\psi_r(t)^2 - \|e_r(t)\|^2}\right)^+ + \lambda_u\|u\|^2\ \mathrm{d}t + T\\ &= \int_{\hat t}^{\hat t+T}\left(\ell_{\psi_r}(t,\zeta(t),u(t))\right)^+\mathrm{d}t + T < \infty. \end{aligned}$$

We seek to apply [2, Lem. 4.1]. To this end, we show that $1-\frac{\|e_r(\cdot)\|^2}{\psi_r(\cdot)^2}$ is Lipschitz continuous on $[\hat t,\tilde t]$. If a function is bounded, with bounded derivative, then it is Lipschitz continuous. Hence, since $\psi_r\in\mathcal{G}$ it is Lipschitz continuous on $[\hat t,\tilde t]$. Clearly, e_r is bounded by ψ_r, and we show that $\dot{e}_r$ is bounded. First observe that, since $x(\cdot;\hat{t},\hat{y},u)$ is continuous, it is bounded on the compact interval $[\hat{t},\tilde{t}]$. Since f and g are continuous and by the BIBO property of the operator $\mathbf{T}$, $f(\mathbf{T}(x(\cdot;\hat{t},\hat{y},u)))$ and $g(\mathbf{T}(x(\cdot;\hat{t},\hat{y},u)))$ are bounded on the interval $[\hat{t},\tilde{t}]$. As in (18) in the proof proof of Lemma 3.3, for $e_i(t) := e_i(\zeta)(t)$, we may show that $e_i^{(j)}$ is bounded by μ_i^j as defined in (15) for $i = 1,\dots,r$ and $j = 0,\dots r-i-1$. Finally, it follows from (17) that

$$\dot{e}_r = f(\mathbf{T}(\chi(y))(\cdot)) + g(\mathbf{T}(\chi(y))(\cdot))u - y_{\mathrm{ref}}^{(r)} + \sum_{j=1}^{r-1} k_j e_j^{(r-j)},$$

which is bounded on $[\hat{t},\tilde{t}]$ by the above observations. Since ψ_r and e_r are Lipschitz continuous and products and sums of Lipschitz continuous functions on a compact interval are again Lipschitz continuous, we may infer that $1-\frac{\|e_r(\cdot)\|^2}{\psi_r(\cdot)^2}$ is Lipschitz continuous on $[\hat{t},\tilde{t}]$. Now [2, Lem. 4.1] yields that it is strictly positive, i.e., $\psi_r(t)^2 > \|e_r(t)\|^2$ for all $t \in [\hat{t},\tilde{t}]$, which contradicts the definition of $\tilde{t}$. Hence, $\tilde{\mathcal{U}}_T(M,\hat{t},\hat{y}) \subseteq \mathcal{U}_T(M,\hat{t},\hat{y})$. □

Lemma 3.5. *Under the assumptions of Theorem 2.3, consider the functions $\psi_2,\dots,\psi_r \in \mathcal{G}$ defined in (10). Further, let $T>0$, $M>0$, $\hat{t} \geq t_0$, and $\hat{y} \in \mathcal{C}^{r-1}([\hat{t}-\sigma,\hat{t}],\mathbb{R}^m)$ if $\sigma > 0$ or $\hat{y} \in \mathbb{R}^{rm}$ if $\sigma = 0$ such that $\mathcal{U}_T(M,\hat{t},\hat{y}) \neq \emptyset$. Then, there exists $u^\star \in \mathcal{U}_T(M,\hat{t},\hat{y})$ such that $u^\star$ solves the OCP (5) for $\theta = \psi_r$.*

Proof. As a consequence of Lemma 3.4, solving the OCP (5) is equivalent to minimizing the function

$$J : L^\infty([\hat{t},\hat{t}+T],\mathbb{R}^m) \to \mathbb{R}\cup\{\infty\},$$
$$u \mapsto \begin{cases} \int_{\hat{t}}^{\hat{t}+T} \ell_{\psi_r}(t,\zeta(t),u(t))\,\mathrm{d}t, & u \in \mathcal{U}_T(M,\hat{t},\hat{y}),\\ \infty, & \text{else,}\end{cases}$$

where $\zeta(t) := x(t;\hat{t},\hat{y},u) - \chi(y_{\mathrm{ref}})(t)$. For every $u \in \mathcal{U}_T(M,\hat{t},\hat{y})$ we have $\|e_r(\zeta(t))\| < \psi_r(t)$ for all $t \in [\hat{t},\hat{t}+T]$, thus $J(u) \geq 0$. Hence, the infimum $J^\star := \inf_{u\in\mathcal{U}_T(M,\hat{t},\hat{y})} J(u)$ exists. Let $(u_k) \in \left(\mathcal{U}_T(M,\hat{t},\hat{y})\right)^{\mathbb{N}}$ be a minimizing sequence, meaning $J(u_k) \to J^\star$. By definition of $\mathcal{U}_T(M,\hat{t},\hat{y})$, we have $\|u_k\| \leq M$ for all $k \in \mathbb{N}$. Since $L^\infty([\hat{t},\hat{t}+T],\mathbb{R}^m) \subset L^2([\hat{t},\hat{t}+T],\mathbb{R}^m)$, we conclude that (u_k) is a bounded sequence in the Hilbert space L^2, thus u_k converges weakly, up to a subsequence, to a function $u^\star \in L^2([\hat{t},\hat{t}+T],\mathbb{R}^m)$. Let $(x_k) := (x(\cdot;\hat{t};\hat{y};u_k)) \in \mathcal{C}([\hat{t}-\sigma,\hat{t}+T],\mathbb{R}^{rm})^{\mathbb{N}}$ be the sequence of associated responses. By $u_k \in \mathcal{U}_T(M,\hat{t},\hat{y})$ we have $x_k(t) \in \mathcal{D}_t^r$ for all t in $[\hat{t},\hat{t}+T]$. Since the set $\bigcup_{t\in[\hat{t},\hat{t}+T]} \mathcal{D}_t^r$ is compact and independent of $k \in \mathbb{N}$, the sequence (x_k) is uniformly bounded. A repetition of Steps 2–4 of the proof of [2, Thm. 4.6] yields that (x_k) has a subsequence (which we do not relabel) that converges uniformly to $x^\star = x(\cdot;\hat{t};\hat{y};u^\star)$ and that $\|u^\star\|_\infty \leq M$. Along the lines of Steps 5–7 of the proof of [2, Thm. 4.6] it follows that $u^\star \in \mathcal{U}_T(M,\hat{t},\hat{y})$ and $J(u^\star) = J^\star$. This completes the proof. □

3.1. Proof of Theorem 2.3

Choosing the bound $M>0$ from Lemma 3.3 and utilizing Lemma 3.5 this can be shown by a straightforward adaption of the proof of [2, Thm. 2.10]. □

4. Simulations

To demonstrate the application of the FMPC Algorithm 2.1 we consider the example of a mass-spring system mounted on a car from [23]. Consider a car with mass m_1, on which a ramp is mounted and inclined by the angle $\theta \in [0,\frac{\pi}{2})$. On this ramp a mass m_2, which is coupled to the car by spring-damper component with spring constant $k>0$ and damping coefficient $d>0$, moves frictionless, see Figure 2. A control force $F = u$ can be applied

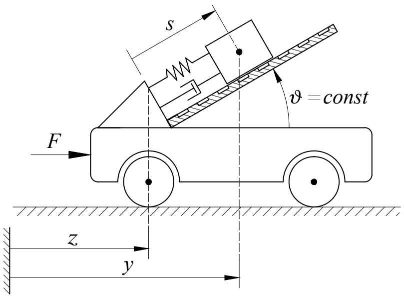


Figure 2: Mass-on-car system.

to the car. The dynamics of the system can be described by the equations

$$\begin{bmatrix} m_1+m_2 & m_2\cos(\vartheta)\\ m_2\cos(\vartheta) & m_2\end{bmatrix}\begin{pmatrix}\ddot{z}(t)\\ \ddot{s}(t)\end{pmatrix} + \begin{pmatrix}0\\ ks(t)+d\dot{s}(t)\end{pmatrix} = \begin{pmatrix}u(t)\\ 0\end{pmatrix}, \tag{19}$$

where $z(t)$ is the horizontal position of the car and $s(t)$ the relative position of the mass on the ramp at time t. The output y of the system is the horizontal position of the mass on the ramp, given by

$$y(t) = z(t) + s(t)\cos(\vartheta).$$

For the simulation we choose the parameters $m_1 = 4$, $m_2 = 1$, $k = 2$, $d = 1$, $\theta = \frac{\pi}{4}$, and initial values $z(0) = s(0) = \dot{z}(0) = \dot{s}(0) = 0$. As outlined in [4], for these parameters the system (19) belongs to the class $\mathcal{N}^{1,2}$. The objective is tracking of the reference signal $y_{\mathrm{ref}}(t) = \cos(t)$ within predefined boundaries described by a function $\psi \in \mathcal{G}$. This means that the tracking error $e(t) = y(t) - y_{\mathrm{ref}}(t)$ should satisfy $\|e(t)\| < \psi(t)$ for all $t \geq 0$.

We compare the FMPC Algorithm Algorithm 2.1 with the original FMPC scheme from [2], for which only feasibility for systems with relative degree one has been shown so far, and the FMPC scheme from [1], which uses feasibility constraints to ensure recursive feasibility for systems with higher relative degree. Since, in comparison to the set $\mathcal{G}$, the set of admissible funnel functions for control scheme from [1] is quite restrictive, the same funnel function $\psi(t) = 1/10 + 11\mathrm{e}^{-27x/20} - 7\mathrm{e}^{-3x/2}$ as in [1] was chosen. Straightforward calculations show that $\alpha = 1.5$, $\beta = \frac{3}{20}$, $\gamma = 0.5$, and $k_1 = 14$ satisfy the requirements of Theorem 2.3. With these parameters, the funnel function θ in (9) is given by

$$\theta(t) = 28e^{-3t/2} + \frac{1}{5}.$$

For the stage cost function ℓ_θ as in (4) the parameter $\lambda_u = \frac{1}{100}$ has been chosen. Further, the maximal input was limited to $\|u\|_\infty \leq 20$, i.e., $M = 20$ was chosen.

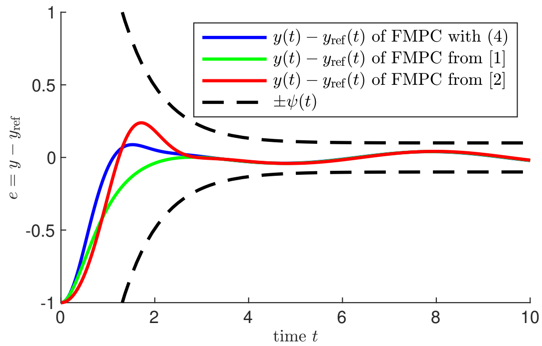
Inserting the definition of the function e_2 from (2) with parameter k_1, the stage cost thus reads

$$\ell_\theta(t, \xi_1, \xi_2, u) = \frac{\|\xi_2 + k_1\xi_1\|^2}{\theta(t)^2 - \|\xi_2 + k_1\xi_1\|^2} + \lambda_u \|u\|^2$$

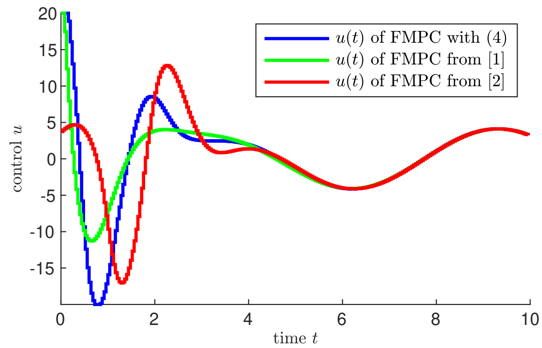
for $\|\xi_2 + k_1\xi_1\| \neq \theta(t)$. With this and $e(t) = y(t) - y_{\text{ref}}(t)$ the OCP (5) becomes

$$\underset{\substack{u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m), \\ \|u\|_\infty \leq M}}{\text{minimize}} \int_{\hat{t}}^{\hat{t}+T} \ell_\theta(t, e(t), \dot{e}(t), u(t))\, \mathrm{d}t.$$

As in [1] and [2] only step functions with constant step length 0.04 are considered in the OCP (5) due to discretisation. The prediction horizon and the time shift are chosen as $T = 0.6$ and $\delta = 0.04$.



(a) Tracking error e and funnel boundary ψ



(b) Control input

Figure 3: Simulation of system (19) under FMPC Algorithm 2.1 and FMPC from [1, 2]

All simulations are performed with MATLAB and the toolkit CASADI on the interval $[0, 10]$ and are depicted in Figure 3. The tracking errors resulting from the application of the different FMPC schemes from [1], [2] and Algorithm 2.1 to system (19) are shown in Figure 3a. The corresponding control signals are displayed in Figure 3b. It is evident that all three control schemes achieve the control objective, the evolution of the tracking error with in the performance boundaries given by ψ.

Overall, the performance of all three FMPC schemes is comparable. After $t = 4$ the computed control signals and the corresponding tracking errors of all three control schemes are almost identical. However, FMPC from [1] requires feasibility constraints in the OCP to achieve initial and recursive feasibility; together with the more complex stage cost, this severely increases the computational effort. Furthermore, the parameters involved in the feasibility constraints are very hard to determine and usually (as in the simulations performed here) conservative estimates must be used. But then again, initial and recursive feasibility cannot be guaranteed. Concerning the FMPC scheme from [2], it is still an open problem to show that it is initially and recursively feasible for systems with relative degree larger than one.

5. Conclusion

In the present paper we proposed a new model predictive control algorithm for a class of nonlinear systems with arbitrary relative degree, which achieves tracking of a reference signal with prescribed performance. The new FMPC scheme resolves the drawbacks of earlier approaches in [2] (no proof of initial and recursive feasibility for relative degree larger than one) and [1] (requirement of feasibility constraints, design parameters difficult to determine, high computational effort). All advantages of these approaches (no terminal costs or conditions, no requirements on the prediction horizon) are retained. Essentially, this solves the open problems formulated in the conclusions of [1, 2]. Compared to previous works on FMPC, the class of nonlinear systems considered here includes systems with nonlinear delays and infinite-dimensional internal dynamics. An interesting question which remains for future research is, whether the weighted sum of the tracking error derivatives $e, \dot{e}, \ldots, e^{(r-1)}$ used in the cost functional in (5) can be replaced by a sole error signal e, when instead the prediction horizon T is chosen sufficiently long.

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