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On stability of time-varying linear differential-algebraic equations

Masterarbeit zur Erlangung des akademischen Grades Master of Science

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Abstract

Differential-algebraic equations are becoming increasingly important in a lot of technical areas, such as electrical engineering. Since they are not explicitly solvable in most cases, or have hardly manageable solutions, and solutions even need not to be unique, one focuses on qualitative statements about the system behavior. Stability of linear time-varying differential-algebraic equations of the form $E(t)\dot{x} = A(t)x + f(t)$ is studied in this thesis. A detailed investigation of such systems without any restrictions seems not to be available. A main goal of this thesis is to develop a relationship between the stability behavior of the solutions of this system and the stability behavior of the trivial solution of its associated homogeneous system. Furthermore, we develop, via a Lyapunov-approach, conditions for a restricted form of exponential stability. Moreover, we give a detailed investigation of the solution and stability theory of systems which are transferable into standard canonical form. Regarding this we state a representation of the general solution and a condition under which it exists. We introduce consistent initial values and, for homogeneous systems, the generalized transition matrix and determine properties of it, which can be seen as direct generalizations of the properties of the transition matrix of an ordinary linear differential equation. Furthermore, we introduce the projected generalized time-varying Lyapunov-equation, and derive necessary and sufficient conditions for exponential stability utilizing this equation. In this context the solvability of the Lyapunov-equation as well as the uniqueness and representation of its solution is investigated.

Zusammenfassung

Differential-algebraische Gleichungen gewinnen in vielen technischen Gebieten, wie zum Beispiel der Elektrotechnik, immer mehr an Bedeutung. Da sie aber in den meisten Fällen nicht explizit lösbar sind, oder schwer handhabbare Lösungen besitzen, und die Lösungen auch nicht eindeutig sein müssen, konzentriert man sich auf qualitative Aussagen über das Systemverhalten. Die Stabilität linearer zeitvarianter differential-algebraischer Gleichungen der Form $E(t)\dot{x} = A(t)x + f(t)$ wird in dieser Arbeit studiert. Eine detailierte Untersuchung solcher Systeme ohne irgendwelche Einschränkungen scheint bisher nicht verfügbar zu sein. Ein zentrales Ziel dieser Arbeit ist es eine Verbindung zwischen dem Stabilitätsverhalten der Lösungen dieses Systems und dem Stabilitätsverhalten der trivialen Lösung des zugehörigen homogenen Systems herzustellen. Weiterhin entwickeln wir, mittels einer Lyapunov-Methode, Bedingungen für eine eingeschränkte Form von exponentieller Stabilität. Des Weiteren führen wir eine detailierte Untersuchung der Lösungs- und Stabilitätstheorie von Systemen, die sich in Standard-Normalform überführen lassen durch. Dies betreffend geben wir eine Darstellung der allgemeinen Lösung an und eine Bedingung unter der diese existiert. Wir führen konsistente Anfangswerte und, für homogene Systeme, die verallgemeinerte Übergangsmatrix ein und bestimmen ihre Eigenschaften, welche als direkte Verallgemeinerungen der Eigenschaften der Übergangsmatrix einer gewöhnlichen linearen Differentialgleichung angesehen werden können. Weiterhin führen wir die projizierte verallgemeinerte zeitvariante Lyapunov-Gleichung ein und leiten unter der Benutzung dieser notwendige und hinreichende Bedingungen für exponentielle Stabilität her. In diesem Zusammenhang untersuchen wir auch die Lösbarkeit der Lyapunov-Gleichung sowie die Eindeutigkeit und Darstellung der Lösung.

Nomenclature

$\mathbb{N}$	the set of the natural numbers $\{1, 2, 3, ...\}$
$\mathbb{R}$	the set of the real numbers
$\mathbb{R}^n$	the vector space of the real vectors of length n
$\mathbb{R}^{m\times n}$	the set of the real $(m \times n)$-matrices
$\|x\|$	$:= \sqrt{x^\top x}$, the Euclidean norm of the vector $x \in \mathbb{R}^n$
$\mathcal{B}_\delta(x^0)$	$:= \{\ x \in \mathbb{R}^n \ \mid \|x - x^0\| < \delta\ \}$, the open ball of radius $\delta > 0$ around $x^0 \in \mathbb{R}^n$
$\|A\|$	$:= \sup\{\ \|Ax\| \ \mid \|x\| = 1\ \}$, the spectral norm of the matrix $A \in \mathbb{R}^{m\times n}$
$A^\top$	the transpose of the matrix $A \in \mathbb{R}^{m\times n}$
$I = I_n$	$:= \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$, the identity matrix of dimension n
$\ker A$	the kernel of the matrix $A \in \mathbb{R}^{m\times n}$
$\operatorname{im} A$	the image of the matrix $A \in \mathbb{R}^{m\times n}$
$C^k(\mathcal{J} \to \mathbb{R}^n)$	the set of all k-times continuously differentiable functions mapping from $\mathcal{J}$ to $\mathbb{R}^n$, $k \in \mathbb{N}$
$\operatorname{dom} f$	the domain of the function f
$f\mid_{\mathcal{M}}$	the restriction of the function f on a set $\mathcal{M} \subseteq \operatorname{dom}(f)$

For matrices $A, B \in \mathbb{R}^{n\times n}$ we will shortly write $A \le B$, if for all $x \in \mathbb{R}^n$ the condition $x^\top Ax \le x^\top Bx$ holds. And for $C, D : (\tau, \infty) \to \mathbb{R}^{n\times n}$, $\tau \in [-\infty, \infty)$, $\mathcal{U} \subseteq (\tau, \infty) \times \mathbb{R}^n$ we will write $C(\cdot) \le_{\mathcal{U}} D(\cdot)$, if

$$\forall\, (t, x) \in \mathcal{U} : x^\top C(t)x \le x^\top D(t)x.$$

We introduce " $=_{\mathcal{U}}$ " in the same way. Furthermore, let

$$\mathcal{P}_{\mathcal{U}} := \left\{ M : (\tau, \infty) \to \mathbb{R}^{n\times n} \ \middle|\ \begin{array}{l} M \text{ is continuous and symmetric,} \\ \exists\, m_1, m_2 > 0 :\ m_1 I_n \le_{\mathcal{U}} M(\cdot) \le_{\mathcal{U}} m_2 I_n \end{array} \right\}$$

Contents

Chapter 1

Introduction

1.1 Motivation

In a lot of areas of physics or engineering stability theory plays a fundamental role. Differential equations are the central tool for describing time-varying processes. Unfortunately a lot of these equations can not be solved explicitly, or have hardly manageable solutions. With the help of numerical methods the solutions can be approximated very well on fixed time intervals, but more often one is interested in a qualitative behavior of the solutions. A primary topic of this qualitative theory of differential equations is the stability theory, as studied in this master thesis.
Since the stability theory of ordinary differential equations is well investigated, we consider differential-algebraic equations, which have come to some importance in technical areas like electrical engineering during the last decades. The most general form of a differential-algebraic equation is an implicit differential equation of the form

$$F(t, x, \dot{x}) = 0, \tag{1.1}$$

where the function $F : G \to \mathbb{R}^n$ is continuous on the open and connected set $G \subseteq \mathbb{R}^{1+n+n}$, $G \neq \emptyset$, $n \in \mathbb{N}$, see also [BCP89, Sec. 1.1]. $\frac{\partial F}{\partial y}(t, x, y)$ may be singular at some points $(t, x, y) \in G$. In many cases, some of the relations within (1.1) do not involve $\dot{x}$ at all, hence they are pure algebraic equations. This motivates to call (1.1) a differential-algebraic equation.

1.2 Implicit differential equations

Consider the implicit differential equation (1.1).

By a *solution* of (1.1) we mean a continuously differentiable function $x : \mathcal{J} \to \mathbb{R}^n$, which solves (1.1) for all $t \in \mathcal{J}$, and $\mathcal{J}$ is an open interval such that $\{ (t, x(t), \dot{x}(t)) \mid t \in \mathcal{J} \} \subseteq G$.

Further we need a concept of extendability of solutions. Hence we define right maximal and global solutions, similar to [Ama90, Sec. 5] where it was done for (explicit) ordinary differential equations, as follows.

1.2.1 Definition (Right maximal/global solutions). A solution $\tilde{x} : (a, \tilde{b}) \to \mathbb{R}^n$ of (1.1) is called a *(right) extension* of a solution $x : (a, b) \to \mathbb{R}^n$ of (1.1), if $\tilde{b} \geq b$ and $x = \tilde{x}\mid_{(a,b)}$.

A solution $x : (a, b) \to \mathbb{R}^n$ of (1.1) is called *right maximal*, if $b = \tilde{b}$ for every extension $\tilde{x} : (a, \tilde{b}) \to \mathbb{R}^n$ of it. It is called *right global*, if $b = \sup \mathcal{T}(G)$, where $\mathcal{T}(G) := \{ t \in \mathbb{R} \mid \exists\, x, \tilde{x} \in \mathbb{R}^n : (t, x, \tilde{x}) \in G \}$, and, if $(a, b) = \mathcal{T}(G)$, $x(\cdot)$ is called *global*. ⋄

For a right maximal solution $x : (a, b) \to \mathbb{R}^n$ of (1.1) which is not right global, i.e. $b < \sup \mathcal{T}(G)$, we further consider the following two cases:

(i) If $\limsup_{t \nearrow b} \|x(t)\| = \infty$, then $x(\cdot)$ is said to have a *finite escape time*.

(ii) If $x(\cdot)$ has no finite escape time, then $x(\cdot)$ is called *non-extendable*.

Note that the notion "non-extendable" is often used for solutions which are right maximal in our terms, see e.g. [Ama90, Har82].

Define

$$\mathcal{S}(t^0, x^0) := \left\{ x : \mathcal{J} \to \mathbb{R}^n \;\middle|\; \begin{array}{l} \mathcal{J} \text{ open interval, } t^0 \in \mathcal{J},\ x(t^0) = x^0, \\ x(\cdot) \text{ is a right maximal solution of (1.1)} \end{array} \right\}$$

to be the set of all right maximal solutions $x(\cdot)$ of (1.1) with $x(t^0) = x^0$. In the following we define the concepts of Lyapunov-stability studied in this work.

1.2.2 Definition (Stability)**.** Let $x : (\tau, \infty) \to \mathbb{R}^n$, $\tau \in [-\infty, \infty)$, be a right global solution of (1.1).

$x(\cdot)$ is *stable* $:\Longleftrightarrow$

$$\forall\, \varepsilon > 0\ \forall\, t^0 > \tau\ \exists\, \delta > 0\ \forall\, y^0 \in \mathcal{B}_\delta(x(t^0))\ \forall\, y(\cdot) \in \mathcal{S}(t^0, y^0) :$$

$$[t^0, \infty) \subseteq \operatorname{dom} y \quad \wedge \quad \forall\, t \geq t^0 : y(t) \in \mathcal{B}_\varepsilon(x(t)).$$

$x(\cdot)$ is *attractive* $:\Longleftrightarrow$

$$\forall\, t^0 > \tau\ \exists\, \eta > 0\ \forall\, y^0 \in \mathcal{B}_\eta(x(t^0))\ \forall\, y(\cdot) \in \mathcal{S}(t^0, y^0) :$$

$$[t^0, \infty) \subseteq \operatorname{dom} y \quad \wedge \quad \lim_{t\to\infty} (y(t) - x(t)) = 0.$$

$x(\cdot)$ is *asymptotically stable* $:\Longleftrightarrow$ $x(\cdot)$ is stable and attractive.

$x(\cdot)$ is *exponentially stable* $:\Longleftrightarrow$

$$\exists\, \alpha, \beta > 0\ \forall\, t^0 > \tau\ \exists\, \eta > 0\ \forall\, y^0 \in \mathcal{B}_\eta(x(t^0))\ \forall\, y(\cdot) \in \mathcal{S}(t^0, y^0) :$$

$$[t^0, \infty) \subseteq \operatorname{dom} y \quad \wedge \quad \forall\, t \geq t^0 : \|y(t) - x(t)\| \leq \alpha e^{-\beta(t-t^0)} \|y(t^0) - x(t^0)\|.$$

1.2.3 Remark. Note that stability does not imply that every initial value problem is solvable in the neighborhood of the considered solution. Furthermore, a possibly existing solution does not have to be unique. The only requirement is that every existing solution in a neighborhood of the considered one stays in an ε-neighborhood of it. $\diamond$

1.2.4 Remark (Exponential stability)**.** Our definition of exponential stability is often called *uniform exponential stability*, since the constants α and β neither depend on the initial values nor on the initial times: See, in the case of linear systems, for instance [Rug96, Def. 6.5].
Furthermore, in contrast to the definition of asymptotic stability, in the definition of exponential stability it is not explicitly required that the solution has to be stable. However it is implicitly contained. $\diamond$

1.2.5 Proposition (Exponential stability implies stability). *Every solution* $x : (\tau, \infty) \to \mathbb{R}^n$ *of* (1.1) *satisfies:*

$$x(\cdot) \text{ is exponentially stable } \implies x(\cdot) \text{ is stable.}$$

Proof: Let $x : (\tau, \infty) \to \mathbb{R}^n$ be an exponentially stable solution of (1.1) and the constants $\alpha, \beta > 0$ be given as in Definition 1.2.2. Let $\varepsilon > 0$ and $t^0 > \tau$, then

$$\begin{aligned}
&\exists\, \eta > 0\ \forall\, y^0 \in \mathcal{B}_\eta(x(t^0))\ \forall\, y(\cdot) \in \mathcal{S}(t^0, y^0) : \\
&[t^0, \infty) \subseteq \operatorname{dom} y \wedge\ \forall\, t \geq t^0 : \|y(t) - x(t)\| \leq \alpha e^{-\beta(t-t^0)} \|y(t^0) - x(t^0)\|.
\end{aligned}$$

Define

$$\delta := \min\left\{\frac{\varepsilon}{\alpha}, \eta\right\}.$$

and let $y^0 \in \mathcal{B}_\delta(x(t^0))$ and $y(\cdot) \in \mathcal{S}(t^0, y^0)$. Since $y^0 \in \mathcal{B}_\delta(x(t^0)) \subseteq \mathcal{B}_\eta(x(t^0))$ it follows $[t^0, \infty) \subseteq \operatorname{dom} y(\cdot)$ and

$$\begin{aligned}
\forall\, t \geq t^0 : \|y(t) - x(t)\| &\leq \alpha e^{-\beta(t-t^0)} \|y(t^0) - x(t^0)\| \\
&< \alpha e^{-\beta(t-t^0)} \delta \leq \alpha e^{-\beta(t-t^0)} \frac{\varepsilon}{\alpha} \leq \varepsilon.
\end{aligned}$$

□

1.3 Structure of the thesis

In this master thesis time-varying linear differential-algebraic equations

$$E(t)\dot{x} = A(t)x + f(t), \tag{1.2}$$

for $\tau \in [-\infty, \infty)$ and continuous $E, A : (\tau, \infty) \to \mathbb{R}^{n\times n}$, $f : (\tau, \infty) \to \mathbb{R}^n$, $n \in \mathbb{N}$, are studied. In particular, $E(\cdot)$ may be singular for some or all $t \in (\tau, \infty)$. Systems of the form (1.2) naturally occur when modelling linear electrical circuits or simple mechanical systems. For the case that $E(\cdot)$ and $A(\cdot)$ are constant and $f(\cdot) = 0$ the stability behavior of (1.2) has been investigated in [Ber08]. Actually this thesis is the continuation of the work in [Ber08].

At first sight it seems that the theory of differential-algebraic equations is well developed, see e.g. the textbooks [Cam80, Cam82, Dai89, KM06] and the references therein. However, regarding the stability theory of systems (1.2) there only exist some first results: for constant $E(\cdot)$ and time-varying $A(\cdot)$ [DVP07], constant $E(\cdot)$ and periodically $A(\cdot)$ [SLSZ06], and the ansatz of "regularizing operators" to obtain Lyapunov stability criteria [SC04]. Moreover, there are results for differential-algebraic equations with index 1 or 2 [Tis94, LMW96]. However, a detailed investigation of systems with arbitrary index and arbitrary $E(\cdot)$ (particularly of variable rank) seems not to be available. One aim of this thesis is to develop a connection between the stability behavior of the solutions of (1.2) and the stability behavior of the trivial solution of its associated homogeneous system, where only continuity of $E(\cdot), A(\cdot)$ and $f(\cdot)$ is required (see Theorem 3.2.3). Furthermore, we develop, via a Lyapunov-approach, conditions for a restricted form of exponential stability (see Theorem 3.6.2).

The preliminaries for our investigations are given in Section 1.2, where the stability concepts studied in this work are defined, in Section 2.1, where some well-known results on ordinary linear differential equations are stated, and in Section 2.2, where we state the most common results of the Lyapunov-theory of ordinary linear differential equations; with detailed proofs, so the generalization to the case of differential-algebraic equations becomes more obvious and is not surprising.

Chapter 3 is the main part of this thesis. In Section 3.1 we investigate the behavior of the solutions of differential-algebraic equations and determine all cases which might occur, when a solution is right maximal, but not right global. This is a crucial preparatory work for the results of Section 3.2, where the already mentioned Theorem 3.2.3 is developed, which seems to be new. In Section 3.3 we introduce the concept of pairs of consistent initial values and the standard canonical form (SCF) of homogeneous systems

$$E(t)\dot{x} = A(t)x \tag{1.3}$$

and will henceforth restrict the consideration to the case of systems which are transferable into SCF. We determine uniqueness of the SCF and develop a representation for the solutions of initial value problems (1.3), $x(t^0) = x^0$. This motivates the introduction of the generalized transition matrix, which seems to be a new approach to such systems. We derive some properties of the generalized transition matrix which can be seen as direct generalizations of the properties of the transition matrix of an ordinary linear differential equation. In Section 3.4 we develop a representation for the solutions of initial value problems (1.2), $x(t^0) = x^0$. We introduce the concept of analytic solvability and derive a connection to the concept of transferability into SCF. In Section 3.5 we derive some equivalent representations for the considered stability concepts, which are more handy, similar to the case of ordinary linear differential equations. In Section 3.6 we introduce the projected generalized time-varying Lyapunov-equation as a new tool for the investigation of exponential stability. Then, in Theorem 3.6.2, we derive, using this equation, necessary conditions for a restricted form of exponential stability of systems (1.2) (not necessarily transferable into SCF). Furthermore, we derive necessary and sufficient conditions for exponential stability of systems (1.2) which are transferable into SCF, and, moreover, a condition under which the projected generalized time-varying Lyapunov-equation has a unique solution as well as a representation for this solution. All results of Section 3.6 seem to be new.

Chapter 2

Time-varying linear differential equations

In this chapter we consider the linear system

$$\dot{x} = A(t)x + b(t), \tag{2.1}$$

for $\tau \in [-\infty, \infty)$ and continuous $A : (\tau, \infty) \to \mathbb{R}^{n\times n}$, $b : (\tau, \infty) \to \mathbb{R}^n$, $n \in \mathbb{N}$; the associated homogeneous system is

$$\dot{x} = A(t)x. \tag{2.2}$$

2.1 Preliminaries and stability theory

We denote by $x(\cdot\,; t^0, x^0)$ the unique global solution of the initial value problem

$$\dot{x} = A(t)x, \quad x(t^0) = x^0.$$

The space of all global solutions of (2.2) is an n-dimensional vector space. Let $x_1(\cdot), ..., x_n(\cdot)$ be a basis for this vector space. Then the matrix $X(\cdot) := [x_1(\cdot), ..., x_n(\cdot)]$ is called a *fundamental matrix* for (2.2). We define the *transition matrix* of (2.2) by

$$\Phi(\cdot, t^0) := X(\cdot)X(t^0)^{-1}, \quad t^0 > \tau.$$

The transition matrix $\Phi(\cdot, t^0)$ does not depend on the choice of the special basis vectors $x_i(\cdot)$; $\Phi(\cdot, t^0)$ is the unique global solution of the matrix differential equation

$$\dot{X} = A(t)X, \quad X(t^0) = I_n.$$

We get the following immediate properties of $\Phi(\cdot, \cdot)$, the proofs of which can be found in [Ama90, Sec. 11] for instance.

2.1.1 Lemma (Properties of $\Phi(\cdot, \cdot)$)**.** *Consider the system* (2.2)*. For arbitrary* $t, r, s \in (\tau, \infty)$ *the following statements hold:*

(i) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, s) = A(t)\Phi(t, s)$*,*

(ii) $\Phi(t, t) = I_n$*,*

(iii) $\Phi(t, r)\Phi(r, s) = \Phi(t, s)$*,*

(iv) $\Phi(t, s)^{-1} = \Phi(s, t)$*,*

(v) $\Phi(s, \cdot)^\top$ *is the transition matrix of* $\dot{x} = -A(t)^\top x$*, i.e.*

$$\dot{X} = -A(t)^\top X, \quad X(s) = I_n$$

has the unique solution $\Phi(s, \cdot)^\top$*.*

(vi) $\frac{d}{ds}\Phi(t, s) = -\Phi(t, s)A(s)$*.*

When investigating the stability behavior of the inhomogeneous system (2.1), especially the associated homogeneous system (2.2) is of importance, as the following result shows.

2.1.2 Proposition (Uniform stability behavior of all solutions)**.** *All global solutions of the inhomogeneous system* (2.1) *have one of the properties* $\{$*stable, attractive, asymptotically stable, exponentially stable*$\}$ *if, and only if, the trivial solution of the associated homogeneous system* (2.2) *has the respective property.*

Proof: The assertion regarding stable, attractive and asymptotically stable solutions follows from [Aul04, Satz 7.5.1]. An analogous proof shows the exponentially stable solutions. □

With respect to Proposition 2.1.2, we call a system of the form (2.1) *stable*, *attractive*, *asymptotically stable* or *exponentially stable*, if the global trivial solution of the associated homogeneous system (2.2) has the respective property. For asymptotic and exponential stability, in which we are especially interested, we get some further characterization.

2.1.3 Proposition (Attractivity implies stability [Aul04, Satz 7.5.3])**.** *Every attractive global solution of* (2.1) *is stable, and therefore asymptotically stable.*

2.1.4 Proposition (Transition matrix and stability [Aul04, Satz 7.5.4])**.** *The trivial solution of the homogeneous system* (2.2) *is*

(i) stable if, and only if,

$$\forall\, t^0 > \tau\ \exists\, \beta > 0\ \forall\, t \geq t^0 :\ \|\Phi(t, t^0)\| \leq \beta,$$

(ii) asymptotically stable if, and only if,

$$\forall\ t^0 > \tau :\ \lim_{t\to\infty} \Phi(t, t^0) = 0.$$

Actually in both cases it is sufficient to consider one single initial time $t^0 > \tau$*.*

The results so far now allow to state some corollaries, which are often used as definitions for the respective stability concepts.

2.1.5 Corollary. *The system* (2.1) *is asymptotically stable if, and only if, for every solution* $x : (\tau, \infty) \to \mathbb{R}^n$ *of the associated homogeneous system* (2.2) *holds*

$$\lim_{t\to\infty} x(t) = 0.$$

Proof: $(\Leftarrow)$: Follows immediately from Proposition 2.1.2 and Proposition 2.1.3.
$(\Rightarrow)$: Follows from Proposition 2.1.2, Proposition 2.1.4 and the fact that for all $(t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n$ we can write $x(\cdot\,; t^0, x^0) = \Phi(\cdot, t^0)x^0$. $\square$

2.1.6 Corollary. *The system* (2.1) *is exponentially stable if, and only if,*

$$\exists\, \alpha, \beta > 0\ \forall\, (t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n\ \forall\, t \geq t^0 :\ \|x(t; t^0, x^0)\| \leq \alpha e^{-\beta(t-t^0)}\|x^0\|.$$

Proof: $(\Leftarrow)$: Follows immediately from Proposition 2.1.2.
$(\Rightarrow)$: Let $(t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n$. Since (2.1) is exponentially stable, so it is the trivial solution of (2.2) and hence

$$\exists\, \alpha, \beta > 0\ \exists\, \delta = \delta(t^0)\ \forall\, y^0 \in \mathcal{B}_\delta(0)\ \forall\, t \geq t^0 :\ \|\Phi(t, t^0)y^0\| \leq \alpha e^{-\beta(t-t^0)}\|y^0\|.$$

If $x^0 = 0$ then $x(\,\cdot\,; t^0, x^0) = 0$, and if $x^0 \neq 0$ it holds true that

$$\begin{aligned} &\forall\, t \geq t^0 : \left\|\Phi(t, t^0)\frac{\delta x^0}{2\|x^0\|}\right\| \leq \alpha e^{-\beta(t-t^0)}\left\|\frac{\delta x^0}{2\|x^0\|}\right\| \\ \iff\ &\forall\, t \geq t^0 : \|\Phi(t, t^0)x^0\| \leq \alpha e^{-\beta(t-t^0)}\|x^0\|. \end{aligned}$$

□

2.2 Lyapunov theory

The aim of this section is to state necessary and sufficient conditions for exponential stability of (2.2). Furthermore, we want to state this without knowledge of the actual solutions of (2.2), and hence we use a Lyapunov-like approach. A. M. Lyapunov gave the foundations for this theory in his habilitation [Lya92].
Note that we do not deal with asymptotic stability. The reason for this is, that we will consider a Lyapunov-equation and can not guarantee existence of solutions to this equation, if the system (2.2) is not exponentially stable. Asymptotic stability is not sufficient, since there is a need for a certain convergence rate of the solutions of (2.2) to zero. Furthermore, the following example illustrates that asymptotic and exponential stability are not equivalent, as it is in the case, when $A(\cdot)$ is constant (see e.g. [Aul04, Sec. 7.5]).

2.2.1 Example. Consider the scalar equation

$$\dot{x} = -\frac{1}{t}x, \quad t \in \left(\frac{1}{2}, \infty\right). \tag{2.3}$$

Then for any $x^0 \in \mathbb{R}$, $t^0 > \frac{1}{2}$, the initial value problem (2.3), $x(t^0) = x^0$ has the unique global solution

$$x : \left(\frac{1}{2}, \infty\right) \to \mathbb{R},\ t \mapsto \frac{t^0 x^0}{t}$$

and hence (2.3) is asymptotically stable, but not exponentially stable. ◇

Our approach is mainly based on the Lyapunov-equation for time-varying linear systems (2.2). The following lemma states this equation and gives a solution under certain conditions.

2.2.2 Lemma (Solution of the Lyapunov-equation)**.** *Let* $A, Q : (\tau, \infty) \to \mathbb{R}^{n \times n}$ *be continuous and* $\Phi(\cdot,\cdot)$ *the transition matrix of the system* (2.2)*. If the integral*

$$P(t) := \int_t^\infty \Phi(s,t)^\top Q(s) \Phi(s,t)\, ds$$

exists for all $t > \tau$*, then the* time-varying Lyapunov-equation

$$\forall\, t > \tau : \ A(t)^\top P(t) + P(t)A(t) + \dot{P}(t) = -Q(t) \tag{2.4}$$

has the continuously differentiable solution $P : (\tau, \infty) \to \mathbb{R}^{n \times n}$*.*

Proof: For all $t > \tau$ we have

$$\begin{aligned}
& A(t)^\top P(t) + P(t)A(t) + \dot{P}(t) \\
= \quad & \int_t^\infty A(t)^\top \Phi(s,t)^\top Q(s)\Phi(s,t) + \Phi(s,t)^\top Q(s)\Phi(s,t)A(t)\, ds \\
& + \tfrac{\mathrm{d}}{\mathrm{d}t} \int_t^\infty \Phi(s,t)^\top Q(s)\Phi(s,t)\, ds \\
= \quad & \int_t^\infty A(t)^\top \Phi(s,t)^\top Q(s)\Phi(s,t) + \Phi(s,t)^\top Q(s)\Phi(s,t)A(t)\, ds \\
& + \int_t^\infty \tfrac{\mathrm{d}}{\mathrm{d}t} \big(\Phi(s,t)^\top Q(s)\Phi(s,t)\big)\, ds - \Phi(t,t)^\top Q(t)\Phi(t,t) \\
\overset{\text{Lemma 2.1.1}}{=} \quad & \int_t^\infty A(t)^\top \Phi(s,t)^\top Q(s)\Phi(s,t) + \Phi(s,t)^\top Q(s)\Phi(s,t)A(t)\, ds \\
& + \int_t^\infty (-\Phi(s,t)A(t))^\top Q(s)\Phi(s,t) + \Phi(s,t)^\top Q(s)(-\Phi(s,t)A(t))\, ds - Q(t) \\
= \quad & -Q(t).
\end{aligned}$$

Since $Q(\cdot)$ and $\Phi(\cdot,\cdot)$ are continuous, $P(\cdot)$ is continuously differentiable. □

We use the time-varying Lyapunov-equation (2.4) and the corresponding solution $P(\cdot)$ to prove exponential stability of the system (2.2). On the other hand, as already mentioned, we need exponential stability to prove that the time-varying Lyapunov-equation has a solution, that means the integral $P(t)$ exists for $t > \tau$. This is true, if

we have an exponential convergence rate of the solutions of (2.2) to zero.
For simplicity set

$$\mathcal{D} := (\tau, \infty) \times \mathbb{R}^n.$$

We obtain the following theorem.

2.2.3 Theorem (Necessary and sufficient conditions for exponential stability)**.** *Consider system* (2.2)*.*

(i) Let $A(\cdot)$ be bounded and $Q(\cdot) \in \mathcal{P}_{\mathcal{D}}$. If (2.2) *is exponentially stable, then there exists a continuously differentiable solution $P(\cdot) \in \mathcal{P}_{\mathcal{D}}$ to* (2.4)*.*

(ii) If there exist $P(\cdot), Q(\cdot) \in \mathcal{P}_{\mathcal{D}}$, such that $P(\cdot)$ is continuously differentiable and (2.4) *holds, then* (2.2) *is exponentially stable.*

Proof: The proof is essentially from: [Bro70, Chapt. 31, Thm. 5 & Thm. 6] and [Mar03, Thm. 4.6].
(i): $Q(\cdot) \in \mathcal{P}_{\mathcal{D}}$ means, in particular,

$$\exists\, q_1, q_2 > 0 \; \forall\, t > \tau : \; q_1 I_n \leq Q(t) \leq q_2 I_n. \tag{2.5}$$

Let $\Phi(\cdot, \cdot)$ be the transition matrix of system (2.2). Define

$$P : (\tau, \infty) \to \mathbb{R}^{n \times n}, \; t \mapsto \int_t^\infty \Phi(s,t)^\top Q(s) \Phi(s,t)\, ds.$$

First we show that $P(t)$ exists for all $t > \tau$.
It follows from the exponential stability of (2.2) and Corollary 2.1.6, that

$$\exists\, \mu, \nu > 0 \; \forall\, (t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n \; \forall\, t \geq t^0 : \; \|x(t; t^0, x^0)\| \leq \mu e^{-\nu(t - t^0)} \|x^0\|.$$

Let $(t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n$ be arbitrary and $t^1 > t^0$. For simplicity define $x(\cdot) :=$

$x(\cdot\,; t^0, x^0)$. Then

$$\begin{aligned}
\int_{t^0}^{t^1} (x^0)^\top \Phi(s,t^0)^\top Q(s) \Phi(s,t^0) x^0 \, ds &= \int_{t^0}^{t^1} x(s)^\top Q(s) x(s) \, ds \\
&\leq \int_{t^0}^{t^1} q_2 x(s)^\top x(s) \, ds \\
&= q_2 \int_{t^0}^{t^1} \|x(s)\|^2 \, ds \\
&\leq q_2 \int_{t^0}^{t^1} \mu^2 e^{-2\nu(s-t^0)} \|x^0\|^2 \, ds \\
&= -\frac{q_2 \mu^2}{2\nu} \|x^0\|^2 e^{-2\nu(s-t^0)} \Big|_{t^0}^{t^1} \\
&= \frac{q_2 \mu^2}{2\nu} (x^0)^\top x^0 \left(1 - e^{-2\nu(t^1 - t^0)}\right).
\end{aligned}$$

Since t^1 is arbitrary, $P(t^0)$ exists and we have

$$\forall\, t > \tau : \; P(t) \leq \frac{q_2 \mu^2}{2\nu} I_n.$$

Then Lemma 2.2.2 yields that $P(\cdot)$ is a solution of (2.4). Due to the continuity of $Q(\cdot)$ and $\Phi(\cdot,\cdot)$, $P(\cdot)$ is continuously differentiable and due to the symmetry of $Q(\cdot)$, $P(\cdot)$ is also symmetric. It remains to show, that $P(\cdot)$ is bounded from below.
Boundedness of $A(\cdot)$ means

$$\exists\, a > 0 \; \forall\, t > \tau : \; \|A(t)\| \leq a.$$

Furthermore, for all $y \in \mathbb{R}^n$ and all $t > \tau$ we have

$$|y^\top A(t) y| \leq \|y\| \|A(t) y\| \leq \|A(t)\| \|y\|^2 = \|A(t)\| y^\top y.$$

Hence

$$\begin{array}{rcl} (x^0)^\top P(t^0)x^0 & = & \displaystyle\int_{t^0}^{\infty} x(s)^\top Q(s)x(s)\,ds \\ & \geq & \displaystyle\int_{t^0}^{\infty} q_1 x(s)^\top x(s)\,ds \\ & \geq & q_1 \displaystyle\int_{t^0}^{\infty} \frac{\|A(s)\|}{a} x(s)^\top x(s)\,ds \\ & \geq & \displaystyle\frac{q_1}{a}\int_{t^0}^{\infty} |x(s)^\top A(s)x(s)|\,ds \\ & \geq & \displaystyle\frac{q_1}{a}\left|\int_{t^0}^{\infty} x(s)^\top \left(\frac{d}{ds}x(s)\right)\,ds\right| \\ & = & \displaystyle\frac{q_1}{a}\left|\int_{t^0}^{\infty} \frac{1}{2}\frac{d}{ds}\left(x(s)^\top x(s)\right)\,ds\right| \\ & = & \displaystyle\left|\frac{q_1}{2a}\,x(s)^\top x(s)\Big|_{t^0}^{\infty}\right| \\ & = & \displaystyle\frac{q_1}{2a} x(t^0)^\top x(t^0) \\ & = & \displaystyle\frac{q_1}{2a}(x^0)^\top x^0. \end{array}$$

This shows $P(\cdot) \in \mathcal{P}_\mathcal{D}$.

(ii): Define

$$V : \mathbb{R}^n \times (\tau, \infty) \to \mathbb{R},\ (x,t) \mapsto x^\top P(t)x.$$

Let $(t^0, x^0) \in (\tau,\infty) \times \mathbb{R}^n$ be arbitrarily given and again $x(\cdot) := x(\cdot\,; t^0, x^0)$. We show that there exists an estimate of the form

$$\exists\, c \in \mathbb{R}\ \forall\, t \geq t^0 :\ \tfrac{\mathrm{d}}{\mathrm{d}t} V(x(t),t) \leq cV(x(t),t).$$

We have, for all $t \geq t^0$,

$$\begin{array}{rcl} \tfrac{\mathrm{d}}{\mathrm{d}t} V(x(t),t) & = & \dot{x}(t)^\top P(t)x(t) + x(t)^\top \dot{P}(t)x(t) + x(t)^\top P(t)\dot{x}(t) \\ & = & x(t)^\top \big(A(t)^\top P(t) + \dot{P}(t) + P(t)A(t)\big)x(t) \\ & \stackrel{(2.4)}{=} & -x(t)^\top Q(t)x(t) \\ & \stackrel{(2.5)}{\leq} & -q_1 x(t)^\top x(t). \end{array}$$

From $P(\cdot) \in \mathcal{P}_\mathcal{D}$ we obtain

$$\exists\, p_1, p_2 > 0\ \forall\, t > \tau :\ p_1 I_n \leq P(t) \leq p_2 I_n, \tag{2.6}$$

hence

$$\forall\, t \geq t^0 : \; -I_n \leq -\frac{1}{p_2} P(t),$$

and thus

$$\forall\, t \geq t^0 : \; \tfrac{\mathrm{d}}{\mathrm{d}t} V(x(t), t) \leq -\frac{q_1}{p_2} x(t)^\top P(t) x(t) = -\frac{q_1}{p_2} V(x(t), t).$$

Therefore $y : (\tau, \infty) \to \mathbb{R},\; t \mapsto V(x(t), t)$ solves the differential inequality

$$\dot{y} \leq -\frac{q_1}{p_2} y, \quad y(t^0) = V(x^0, t^0).$$

and separation of variables gives

$$V(x(t), t) = y(t) \leq e^{-\frac{q_1}{p_2}(t - t^0)} V(x^0, t^0), \quad t \geq t^0.$$

We use this to get a suitable estimate for the norm of $x(\cdot)$.

It further holds by assumption that for all $t \geq t^0$

$$V(x(t), t) = x(t)^\top P(t) x(t) \geq p_1 x(t)^\top x(t),$$

and so

$$\begin{aligned} \|x(t)\|^2 = x(t)^\top x(t) \quad &\leq \quad \frac{1}{p_1} V(x(t), t) \\ &\leq \quad \frac{1}{p_1} e^{-\frac{q_1}{p_2}(t - t^0)} V(x^0, t^0). \end{aligned}$$

Finally we get

$$\begin{aligned} \|x(t)\| \quad &\leq \quad \left(\frac{1}{p_1} e^{-\frac{q_1}{p_2}(t - t^0)} V(x^0, t^0) \right)^{\frac{1}{2}} \\ &= \quad \sqrt{\frac{1}{p_1}} e^{-\frac{q_1}{2p_2}(t - t^0)} \sqrt{(x^0)^\top P(t^0) x^0} \\ &\leq \quad \sqrt{\frac{1}{p_1}} e^{-\frac{q_1}{2p_2}(t - t^0)} \sqrt{p_2 \|x^0\|^2} \\ &= \quad \sqrt{\frac{p_2}{p_1}} e^{-\frac{q_1}{2p_2}(t - t^0)} \|x^0\| \end{aligned}$$

for $t \geq t^0$, what is exactly the exponential stability of (2.2) due to Corollary 2.1.6, since the solution $x(\cdot)$ was arbitrary. $\square$

Now the question arises whether the solution $P(\cdot)$ of the time-varying Lyapunov-equation (2.4) is unique or not. Note that the following proposition states uniqueness of $P(\cdot)$ without the claim for symmetry of $P(\cdot)$ or $Q(\cdot)$, resp., and likewise asymptotic stability of (2.2) is sufficient. However asymptotic stability is not sufficient to deduce existence of a solution, so the second part of the proposition gives a representation for $P(\cdot)$ under this assumption.

2.2.4 Proposition (Unique solvability of the Lyapunov-equation). *Let* (2.2) *be asymptotically stable and* $Q : (\tau, \infty) \to \mathbb{R}^{n\times n}$ *be continuous. If* $P_1 : (\tau, \infty) \to \mathbb{R}^{n\times n}$ *and* $P_2 : (\tau, \infty) \to \mathbb{R}^{n\times n}$ *are continuously differentiable solutions to* (2.4) *and satisfy*

$$\forall\, i \in \{1, 2\}\ \exists\, \alpha_i, \beta_i > 0\ \forall\, t > \tau : \ \alpha_i I_n \le P_i(t) \le \beta_i I_n,$$

then

$$\forall\, t > \tau : \ P_1(t) = P_2(t).$$

If, furthermore, (2.2) *is exponentially stable,* $A(\cdot)$ *is bounded and* $Q(\cdot)$ *satisfies* (2.5), *then*

$$P : (\tau, \infty) \to \mathbb{R}^{n\times n}, t \mapsto \int_t^\infty \Phi(s, t)^\top Q(s)\Phi(s, t)\, ds$$

is the unique solution of (2.4), (2.6).

Proof: Let $t^0 > \tau$. Define

$$\mu(t) := \Phi(t, t^0)^\top \big(P_1(t) - P_2(t)\big)\Phi(t, t^0), \quad t \ge t^0.$$

Then, obviously,

$$\forall\, t \ge t^0 : \ \dot\mu(t) = 0,$$

since $P_1(\cdot)$ and $P_2(\cdot)$ both solve the time-varying Lyapunov-equation. Hence $\mu(\cdot)$ must be constant,

$$\forall\, t \ge t^0 : \ \mu(t) = \mu(t^0) = P_1(t^0) - P_2(t^0).$$

Further we deduce

$$\begin{aligned}
&\alpha_1\Phi(t, t^0)^\top\Phi(t, t^0) - \beta_2\Phi(t, t^0)^\top\Phi(t, t^0)\\
\le\ &\underbrace{\Phi(t, t^0)^\top P_1(t)\Phi(t, t^0) - \Phi(t, t^0)^\top P_2(t)\Phi(t, t^0)}_{=\mu(t)}\\
\le\ &\beta_1\Phi(t, t^0)^\top\Phi(t, t^0) - \alpha_2\Phi(t, t^0)^\top\Phi(t, t^0)
\end{aligned}$$

for all $t \geq t^0$. Since $\lim_{t\to\infty} \Phi(t,t^0) = 0$ due to the asymptotic stability of (2.2) and Corollary 2.1.5, it follows

$$\lim_{t\to\infty} \mu(t) = 0,$$

and hence

$$0 = \mu(t^0) = P_1(t^0) - P_2(t^0).$$

Since t^0 was arbitrary we may conclude that

$$\forall\, t > \tau : \; P_1(t) = P_2(t).$$

The second statement, for an exponentially stable system (2.2), follows from the first one and Lemma 2.2.2, since the integral $P(t)$ exists for all $t > \tau$ and $P(\cdot)$ is continuously differentiable and fulfills (2.6), as it has been shown in the proof of Theorem 2.2.3 (i). □

One may also wonder whether the assumption of the boundedness of $A(\cdot)$ in Theorem 2.2.3 (i) is necessary. The following example illustrates that (2.4) has possibly no solutions in the case of an exponentially stable system and unbounded $A(\cdot)$.

2.2.5 Example. Consider the scalar system

$$\dot{x} = -tx, \quad t \in \mathbb{R}. \tag{2.7}$$

We show that the time-varying Lyapunov-equation (2.4) has no continuously differentiable solution $p(\cdot) \in \mathcal{P}_{\mathcal{D}}$. For simplicity we choose $q(\cdot) \equiv 1$. (Note that $\mathcal{D} = \mathbb{R}^2$ in this example.)
The global solution $x(\cdot)$ of the initial value problem (2.7), $x(t^0) = x^0$ for $(t^0, x^0) \in \mathbb{R}^2$ is given by

$$x(t) = e^{-\frac{1}{2}(t^2-(t^0)^2)}x^0, \quad t \in \mathbb{R}.$$

Obviously, the system is exponentially stable. The Lyapunov-equation (2.4) has the shape

$$\dot{p} = 2tp - 1. \tag{2.8}$$

Define

$$\tilde{p} : \mathbb{R} \to \mathbb{R}, \; t \mapsto e^{t^2} \int_t^\infty e^{-s^2}\, ds.$$

Clearly the integral exists and $\tilde{p}(\cdot)$ solves (2.8). Hence the global solution $p(\cdot)$ of (2.8), $p(t^0) = p^0$ for $(t^0, p^0) \in \mathbb{R}^2$ is given by

$$p(t) = \frac{p^0 - \tilde{p}(t^0)}{e^{(t^0)^2}} e^{t^2} + \tilde{p}(t) = e^{t^2} \left(\frac{p^0}{e^{(t^0)^2}} + \int_t^\infty e^{-s^2}\, ds - \int_{t^0}^\infty e^{-s^2}\, ds \right).$$

Using integration by parts gives, for $t > 0$,

$$\begin{aligned} \int_t^\infty e^{-s^2}\, ds &= -\frac{1}{2} \int_t^\infty \frac{1}{s}\, d\left(e^{-s^2}\right) \\ &= -\frac{1}{2} \left[\frac{e^{-s^2}}{s} \right]_t^\infty - \frac{1}{2} \int_t^\infty \frac{e^{-s^2}}{s^2}\, ds \\ &= \frac{e^{-t^2}}{2t} + \frac{1}{4} \int_t^\infty \frac{1}{s^3}\, d\left(e^{-s^2}\right) \\ &= \frac{e^{-t^2}}{2t} - \frac{e^{-t^2}}{4t^3} - \ldots, \end{aligned} \tag{2.9}$$

thus having

$$p(t) = e^{t^2} \left(\frac{p^0}{e^{(t^0)^2}} - \int_{t^0}^\infty e^{-s^2}\, ds \right) + \frac{1}{2t} - \frac{1}{4t} - \ldots, \quad \text{for } t > 0.$$

Hence $p(\cdot)$ is bounded if, and only if, $p^0 = e^{(t^0)^2} \int_{t^0}^\infty e^{-s^2}\, ds = \tilde{p}(t^0)$. This shows that $p(\cdot) = \tilde{p}(\cdot)$ is the only bounded solution of (2.8). Invoking again (2.9) we find

$$\lim_{t \to \infty} e^{t^2} \int_t^\infty e^{-s^2}\, ds = 0,$$

and therefore

$$\lim_{t \to \infty} \tilde{p}(t) = 0,$$

which contradicts existence of $p_1 > 0$ such that $p(t) \geq p_1$ for all $t \in \mathbb{R}$. Hence no solution of (2.8) is an element of $\mathcal{P}_{\mathcal{D}}$. $\diamond$

Chapter 3

Time-varying linear differential-algebraic equations

Consider the system

$$E(t)\dot{x} = A(t)x + f(t), \tag{3.1}$$

for $\tau \in [-\infty, \infty)$ and continuous $E, A : (\tau, \infty) \to \mathbb{R}^{n\times n}$, $f : (\tau, \infty) \to \mathbb{R}^n$, $n \in \mathbb{N}$; the associated homogeneous system is

$$E(t)\dot{x} = A(t)x. \tag{3.2}$$

For $(t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n$ let

$$\mathcal{S}_f(t^0, x^0) := \left\{ \; x : \mathcal{J} \to \mathbb{R}^n \;\middle|\; \begin{array}{l} \mathcal{J} \text{ open interval, } t^0 \in \mathcal{J}, \ x(t^0) = x^0, \\ x(\cdot) \text{ is a right maximal solution of (3.1)} \end{array} \right\}$$

be the set of all right maximal solutions of the initial value problem

$$E(t)\dot{x} = A(t)x + f(t), \quad x(t^0) = x^0.$$

In particular, $\mathcal{S}_0(t^0, x^0)$ is the set of all right maximal solutions of the homogeneous problem

$$E(t)\dot{x} = A(t)x, \quad x(t^0) = x^0.$$

Furthermore, we introduce the following subsets of $\mathcal{S}_f(t^0,x^0)$ and $\mathcal{S}_0(t^0,x^0)$, resp.:

$$\mathcal{G}_f(t^0,x^0) :=\{x(\cdot) \in \mathcal{S}_f(t^0,x^0) \mid x(\cdot) \text{ is right global solution of (3.1)}\},$$
$$\mathcal{G}_0(t^0,x^0) :=\{x(\cdot) \in \mathcal{S}_0(t^0,x^0) \mid x(\cdot) \text{ is right global solution of (3.2)}\}.$$

We will use the latter sets in Section 3.6.

3.1 Singular behavior of the solutions

In this section we concentrate on the initial value problem

$$E(t)\dot{x} = A(t)x + f(t), \quad x(t^0) = x^0, \tag{3.3}$$

and determine the behavior which a right maximal, but not right global, solution may show at its right endpoint.
In the case of ordinary differential equations

$$\dot{x} = f(t,x), \quad f \in C(D \to \mathbb{R}^n),\ D \subseteq \mathbb{R}^{1+n} \text{ open},$$

there are only 2 possibilities for the behavior of a right maximal, but not right global, solution $x : (a,b) \to \mathbb{R}^n$ (see [Wal98, p. 68] for the case $n = 1$ and [Wal98, § 10, Thm. VI] for $n > 1$):

(a) $\limsup_{t \nearrow b} \|x(t)\| = \infty$,

(b) $\lim_{t \nearrow b} \operatorname{dist}((t,x(t)),\partial D) = 0$.

In the case of differential-algebraic equations there arise more possibilities for the behavior of the solutions. Since we concentrate on systems of the form (3.1) something like (b) does not occur (the domain has no bound), but besides solutions with finite escape time we may discern 3 cases for non-extendable solutions. We show this via the following illustrative example from [KM06, Ex. 3.1], appropriately extended for our purposes.

3.1.1 Example. Consider (3.3) with $\tau = -\infty$, $x^0 = 0$, $t^0 \in \mathbb{R}$ and

$$E(t) = \begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix}, \quad A(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.4}$$

for $t \in \mathbb{R}$. Then simple calculations show that $x : \mathcal{J} \to \mathbb{R}^n$ is a solution of the initial value problem if, and only if, $\mathcal{J} \subseteq \mathbb{R}$ is an open interval and $x(t) = c(t) \begin{bmatrix} t \\ 1 \end{bmatrix}$, $t \in \mathcal{J}$, for some $c(\cdot) \in C^1(\mathcal{J} \to \mathbb{R})$ with $c(t^0) = 0$. Obviously (3.3) has uncountable many solutions.

Now the following situations may occur:

(i) (3.3) *has a global solution.* For example the trivial solution is a global solution of (3.3).

(ii) (3.3) *has a right maximal solution with finite escape time.* Choose $\omega \in (t^0, \infty)$ and let $c(t) = -\frac{1}{t-\omega} + \frac{1}{t^0-\omega}$, $t < \omega$. Then we calculate that $x : (-\infty, \omega) \to \mathbb{R}^n, t \mapsto c(t)[t,1]^\top$ is a solution of (3.3) and $\limsup_{t \nearrow \omega} \|x(t)\| = \infty$.

(iii) (3.3) *has a right maximal solution which has no finite escape time at* $\omega \in (t^0, \infty)$ *and is not continuous at* ω. Choose $c(t) = \sin \frac{a}{t-\omega}$, $t < \omega$, $a = \pi(t^0 - \omega)$. Then $x : (-\infty, \omega) \to \mathbb{R}^n, t \mapsto c(t)[t,1]^\top$ is a solution of (3.3) and the limit $\lim_{t \nearrow \omega} x(t)$ does not exist.

(iv) (3.3) *has a right maximal solution which is continuous but not differentiable at a finite time* $\omega \in (t^0, \infty)$. Choose $c(t) = (t-\omega) \sin \frac{a}{t-\omega}$, $t < \omega$, $a = \pi(t^0 - \omega)$. Then $x : (-\infty, \omega) \to \mathbb{R}^n, t \mapsto c(t)[t,1]^\top$ is a solution of (3.3) and the limit of the difference quotient $\lim_{t \nearrow \omega} \frac{x(t)-\tilde{x}}{t-\omega}$, where $\tilde{x} = \lim_{t \nearrow \omega} x(t)$, does not exist.

(v) (3.3) *has a right maximal solution which is continuous and differentiable at a finite time* $\omega \in (t^0, \infty)$*, but its derivative is not continuous at* ω. Choose $c(t) = (t-\omega)^2 \sin \frac{a}{t-\omega}$, $t < \omega$, $a = \pi(t^0 - \omega)$. Then $x : (-\infty, \omega) \to \mathbb{R}^n, t \mapsto c(t)[t,1]^\top$ is a solution of (3.3) and the limit $\lim_{t \nearrow \omega} \dot{x}(t)$ does not exist.

We stress that example (3.4) is analytic and t^0, x^0 are the same for all cases (i)-(v). Furthermore the matrix pencil $A(t) - \lambda E(t)$ is regular[1] for each $t \in \mathbb{R}$.
In (iii)-(v) there exists no extension of the solution over ω. In particular, (iii)-(v) represent all (distinct) possibilities for the behavior of any non-extendable solution at its right endpoint. $\diamond$

Finalizing this section, we treat the question whether a system (3.1) always has the property

$$\begin{aligned} & x_1(\cdot) \in \mathcal{S}_f(t^0, x^1),\ x_2(\cdot) \in \mathcal{S}_f(t^0, x^2) \\ \Longrightarrow\ & \Big((x_1 - x_2) : \operatorname{dom} x_1 \cap \operatorname{dom} x_2 \to \mathbb{R}^n\Big) \in \mathcal{S}_0(t^0, x^1 - x^2), \end{aligned} \tag{3.5}$$

i.e. that the difference of two right maximal solutions of (3.1), defined on the intersection of their domains, is a right maximal solution of (3.2). The following example shows that (3.5) does *not* hold in general.

3.1.2 Example. Consider the scalar equation

$$t\dot{x} = -tx + 1, \quad t \in \mathbb{R} \tag{3.6}$$

and the associated homogeneous equation

$$t\dot{x} = -tx, \quad t \in \mathbb{R}. \tag{3.7}$$

In the regions $(-\infty, 0)$ and $(0, \infty)$ the system (3.7) is equivalent to $\dot{x} = -x$ and has unique global solutions, resp. Obviously, the solutions in both regions can be uniquely extended over $t = 0$, and hence we obtain unique global solutions of (3.7), defined on all of $\mathbb{R}$. We see

$$\mathcal{S}_0(t^0, x^0) = \left\{ x : (t^-, \infty) \to \mathbb{R} \;\middle|\; t^- \in [-\infty, t^0),\ x(t) = e^{-(t-t^0)} x^0 \right\}, \tag{3.8}$$

$(t^0, x^0) \in \mathbb{R}^2$.
To show that (3.5) does not hold for this example, first observe that (3.6) reads $0 = 1$

[1] i.e. $\Big(\lambda \mapsto \det\big(A(t) - \lambda E(t)\big)\Big) \neq 0$; for the theory of (regular) matrix pencils see the textbook [KM06] for instance

for $t = 0$, and hence no solution of (3.6) can be defined for $t = 0$. In the regions $(-\infty, 0)$ and $(0, \infty)$ the system (3.6) becomes

$$\dot{x} = -x + \frac{1}{t}, \tag{3.9}$$

and for $x^0 \in \mathbb{R}$ we find

$$\mathcal{S}_1(t^0, x^0) = \left\{ \, x : (t^-, 0) \to \mathbb{R} \;\middle|\; t^- \in [-\infty, t^0),\ x(t) = e^{-(t-t^0)} x^0 + \int_{t^0}^{t} \frac{e^{-(t-s)}}{s}\, ds \, \right\}$$

for $t^0 < 0$,

$$\mathcal{S}_1(t^0, x^0) = \left\{ \, x : (t^-, \infty) \to \mathbb{R} \;\middle|\; t^- \in [0, t^0),\ x(t) = e^{-(t-t^0)} x^0 + \int_{t^0}^{t} \frac{e^{-(t-s)}}{s}\, ds \, \right\} \tag{3.10}$$

for $t^0 > 0$. Let

$$\begin{aligned} x_1 &: (-\infty, 0) \to \mathbb{R}, t \mapsto e^{-(t-t^0)} x^1 + \int_{t^0}^{t} \frac{e^{-(t-s)}}{s}\, ds \quad \text{for } t^0 < 0, x^1 \in \mathbb{R}, \\ x_2 &: (-\infty, 0) \to \mathbb{R}, t \mapsto e^{-(t-t^0)} x^2 + \int_{t^0}^{t} \frac{e^{-(t-s)}}{s}\, ds \quad \text{for } x^2 \in \mathbb{R} \setminus \{x^1\}. \end{aligned}$$

Then $x(\cdot) := x_1(\cdot) - x_2(\cdot)$ with $\operatorname{dom} x(\cdot) = (-\infty, 0) = \operatorname{dom} x_1(\cdot) \cap \operatorname{dom} x_2(\cdot)$ is not a right maximal solution of (3.7), since we could extend it to all of $\mathbb{R}$, however $x_1(\cdot)$ and $x_2(\cdot)$ are right maximal solutions of (3.6). $\diamond$

3.2 Stability behavior of the solutions

Now the question arises whether every solution of the inhomogeneous system (3.1) is stable if, and only if, the trivial solution of the homogeneous system (3.2) is stable, as in the case of an ordinary linear differential equation. In general, the answer is non-affirmative.

3.2.1 Example. Revisit Example 3.1.2. It is clear (see (3.8)) that the trivial solution of (3.7) is exponentially stable. Since

$$\lim_{t \nearrow 0} \int_{-1}^{t} \frac{e^{-(t-s)}}{s}\, ds = -\infty,$$

$\left(x : (-1,0) \to \mathbb{R}^n, t \mapsto e^{-(t-1)} + \int_{-1}^{t} \frac{e^{-(t-s)}}{s}\, ds\right) \in \mathcal{S}_1(-1,1)$ and has a finite escape time. Therefore $x(\cdot)$ can not be exponentially stable. So the inhomogeneity $f(\cdot)$ in (3.1) can lead to finite escape times, and therefore change the stability behavior, even though the trivial solution of the homogeneous equation (3.2) is exponentially stable. But nevertheless from (3.10) it is easy to deduce that every right global solution of (3.6) is exponentially stable.
Note that the solutions of any ordinary linear differential equation can not have a finite escape time for any continuous inhomogeneity. If we consider (3.6) in the region $(-\infty, 0)$ we obtain the ordinary differential equation (3.9) and we find that any global solution $x(\cdot)$ of (3.9) (on $(-\infty,0)$) fulfills $\lim_{t\nearrow 0} |x(t)| = \infty$. Hence the system (3.9) has a singular point at $t = 0$, where the inhomogeneity is not continuous. When we multiply the system by t and therefore obtain the differential-algebraic equation (3.6) we keep the singular point, but the inhomogeneity becomes continuous. $\diamond$

The following lemma is crucial. By Example 3.1.1, property (3.5) is incorrect in general, and hence system (3.1) can have a singular behavior. But by a slight modification of the assumptions, i.e. either $x_1(\cdot)$ or $x_2(\cdot)$ has to be right global, we may deduce that the difference is right maximal.

3.2.2 Lemma (Right maximal solutions). *Consider systems* (3.1) *and* (3.2), *and let* $x^0, y^0 \in \mathbb{R}^n, t^0 > \tau$.

(i) If $x(\cdot) \in \mathcal{S}_f(t^0, x^0)$ *is right global and* $y(\cdot) \in \mathcal{S}_f(t^0, y^0)$, *then* $(x - y : \operatorname{dom} x \cap \operatorname{dom} y \to \mathbb{R}^n) \in \mathcal{S}_0(t^0, x^0 - y^0)$.

(ii) If $x(\cdot) \in \mathcal{S}_f(t^0, x^0)$ *is right global and* $y(\cdot) \in \mathcal{S}_0(t^0, y^0)$, *then* $(x + y : \operatorname{dom} x \cap \operatorname{dom} y \to \mathbb{R}^n) \in \mathcal{S}_f(t^0, x^0 + y^0)$.

Proof: *(i)*: Note that $z = x - y : \operatorname{dom} x \cap \operatorname{dom} y \to \mathbb{R}^n$ is a solution of the initial value problem

$$E(t)\dot z = A(t)z, \quad z(t^0) = x^0 - y^0.$$

Now let $(\alpha, \omega) := \operatorname{dom} z(\cdot)$. If $\omega = \infty$, then the claim holds. Let $\omega < \infty$. Since $y(\cdot)$ is right maximal and $\omega = \sup \operatorname{dom} y(\cdot)$ there are 4 distinct possibilities for the behavior of $y(\cdot)$ at ω (see also Example 3.1.1):

(a) $y(\cdot)$ has a finite escape time, i.e. $\limsup_{t\nearrow\omega}\|y(t)\| = \infty$,

(b) $y(\cdot)$ has no finite escape time and the limit $\lim_{t\nearrow\omega} y(t)$ does not exist,

(c) $y(\cdot)$ is continuous at ω ($\lim_{t\nearrow\omega} y(t)$ exists), but $\lim_{t\nearrow\omega}\frac{y(t)-\tilde{y}}{t-\omega}$, where $\tilde{y} = \lim_{t\nearrow\omega} y(t)$, does not exist,

(d) $y(\cdot)$ is continuous and differentiable at ω ($\tilde{y} = \lim_{t\nearrow\omega} y(t)$ and $\lim_{t\nearrow\omega}\frac{y(t)-\tilde{y}}{t-\omega}$ exist), but $\lim_{t\nearrow\omega}\dot{y}(t)$ does not exist.

Since $x(\cdot)$ is right global and has therefore no such singular behavior at ω, the difference $z(\cdot)$ inherits the behavior from $y(\cdot)$. Since the cases (a)-(d) are distinct it is easy to see that if $y(\cdot)$ fulfills one of them, then $z(\cdot)$ fulfills the same.
We show that $z(\cdot)$ is right maximal. Let $\mu : (\alpha,\tilde{\omega}) \to \mathbb{R}^n$ be an extension of $z(\cdot)$, i.e.

$$\omega \le \tilde{\omega} \quad \text{and} \quad z = \mu\,|_{(\alpha,\omega)}\,.$$

Then $\mu(\cdot)$ has the same singular behavior as $z(\cdot)$ at ω and since $\mu(\cdot)$ is a solution of (3.2) it follows that $\tilde{\omega} \le \omega$ and hence $\omega = \tilde{\omega}$.
(ii): The proof is analogous and omitted. □

3.2.3 Theorem (Uniform stability behavior of all right global solutions)**.** *Consider the inhomogeneous system* (3.1) *and the associated homogeneous system* (3.2)*.*

(i) If the trivial solution of (3.2)*, restricted to* (α,∞) *for some* $\alpha \ge \tau$*, has one of the properties {stable, attractive, asymptotically stable, exponentially stable}, then every right global solution* $x : (\beta,\infty) \to \mathbb{R}^n$ *of* (3.1) *with* $\beta \ge \alpha$ *has the respective property.*

(ii) If there exists a right global solution $x(\cdot)$ *of* (3.1) *with one of the properties {stable, attractive, asymptotically stable, exponentially stable}, then the trivial solution of* (3.2)*, restricted to* $\operatorname{dom} x(\cdot)$*, has the respective property.*

Proof: The idea for this proof is due to [Aul04, Satz 7.4.10]. We prove the claim for stability, the other concepts are proved similarly.

(i): Let the trivial solution of (3.2), restricted to (α, ∞) for some $\alpha \geq \tau$, be stable and let $\mu : (\beta, \infty) \to \mathbb{R}^n$ be a right global solution of (3.1), $\beta \geq \alpha$. We show that $\mu(\cdot)$ is stable.

Let $\varepsilon > 0$ and $t^0 > \beta$. Since the trivial solution of (3.2), restricted to (α, ∞), is stable, Definition 1.2.2 yields

$$\begin{aligned} &\exists\, \delta > 0\ \forall\, y^0 \in \mathcal{B}_\delta(0)\ \forall\, y(\cdot) \in \mathcal{S}_0(t^0, y^0) : \\ &\qquad\qquad y(\cdot) \text{ is right global } \wedge\ \left[\forall\, t \geq t^0 :\ y(t) \in \mathcal{B}_\varepsilon(0)\right]. \end{aligned} \tag{3.11}$$

Let $\eta \in \mathcal{B}_\delta(\mu(t^0))$. If $\mathcal{S}_f(t^0, \eta) = \emptyset$, then the claim holds. Let $\lambda(\cdot) \in \mathcal{S}_f(t^0, \eta)$. By Lemma 3.2.2 (i) and since $t^0 \in \operatorname{dom}\lambda \cap \operatorname{dom}\mu$ we have $(\mu - \lambda : \operatorname{dom}\lambda \cap \operatorname{dom}\mu \to \mathbb{R}^n) \in \mathcal{S}_0(t^0, \mu(t^0) - \eta)$. Then $\mu(t^0) - \eta \in \mathcal{B}_\delta(0)$ and (3.11) yield that $(\mu - \lambda)(\cdot)$ is right global, and hence $\lambda(\cdot)$ must be right global, and

$$\left[\forall\, t \geq t^0 :\ \lambda(t) - \mu(t) \in \mathcal{B}_\varepsilon(0)\right] \quad \Longrightarrow \quad \left[\forall\, t \geq t^0 :\ \lambda(t) \in \mathcal{B}_\varepsilon(\mu(t))\right]$$

and therefore $\mu(\cdot)$ is stable.

(ii): Let $\mu : \mathcal{J} \to \mathbb{R}^n$ be a right global and stable solution of (3.1). We show that the trivial solution of (3.2), restricted to $\mathcal{J}$, is stable.

Let $\varepsilon > 0$ and $t^0 \in \mathcal{J}$. Since $\mu(\cdot)$ is stable, Definition 1.2.2 yields

$$\begin{aligned} &\exists\, \delta > 0\ \forall\, y^0 \in \mathcal{B}_\delta(\mu(t^0))\ \forall\, y(\cdot) \in \mathcal{S}_f(t^0, y^0) : \\ &\qquad\qquad y(\cdot) \text{ is right global } \wedge\ \forall\, t \geq t^0 :\ y(t) \in \mathcal{B}_\varepsilon(\mu(t)). \end{aligned} \tag{3.12}$$

Let $\eta \in \mathcal{B}_\delta(0)$. If $\mathcal{S}_0(t^0, \eta) = \emptyset$, then the claim holds. Let $\lambda(\cdot) \in \mathcal{S}_0(t^0, \eta)$. By Lemma 3.2.2 (ii) and since $t^0 \in \operatorname{dom}\lambda \cap \operatorname{dom}\mu$ we have $(\mu + \lambda : \operatorname{dom}\lambda \cap \operatorname{dom}\mu \to \mathbb{R}^n) \in \mathcal{S}_f(t^0, \mu(t^0) + \eta)$. Then $\mu(t^0) + \eta \in \mathcal{B}_\delta(\mu(t^0))$ and (3.12) yield that $(\mu + \lambda)(\cdot)$ is right global, and hence $\lambda(\cdot)$ must be right global, and

$$\left[\forall\, t \geq t^0 :\ \mu(t) + \lambda(t) \in \mathcal{B}_\varepsilon(\mu(t))\right] \quad \Longrightarrow \quad \left[\forall\, t \geq t^0 :\ \lambda(t) \in \mathcal{B}_\varepsilon(0)\right]$$

and therefore the trivial solution of (3.2), restricted to $\mathcal{J}$, is stable. □

We emphasize the generality of Theorem 3.2.3. There are no assumptions on the matrices $E(\cdot)$ and $A(\cdot)$ or on the solutions of the systems (3.1) and (3.2), resp. It holds for every differential-algebraic equation.

Theorem 3.2.3 justifies (similar to ordinary linear differential equations) the following definition.

3.2.4 Definition. The system (3.1) is called *stable*, *attractive*, *asymptotically stable* or *exponentially stable* if, and only if, the global trivial solution of (3.2) has the respective property. $\diamond$

We will use these notions in Sections 3.5 and 3.6.

3.3 Standard canonical form and homogeneous DAEs

In this section we investigate the homogeneous system (3.2). We introduce the concept of consistent initial values and an equivalence relation on the set of all pairs (E, A), $E, A \in C((\tau, \infty) \to \mathbb{R}^{n \times n})$. Then we define and consider a special class of pairs, which are equivalent to some canonical form. Finally, via this canonical form, we will be able to define a generalized transition matrix for the special class of differential-algebraic equations.

3.3.1 Definition (Pairs of consistent initial values)**.** A pair $(t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n$ is called a *pair of consistent initial values* of the system (3.2) if, and only if, there exists a solution $x(\cdot)$ of (3.2) with $t^0 \in \operatorname{dom} x(\cdot)$ and $x(t^0) = x^0$. We denote by $\mathcal{V}$ the *set of all pairs of consistent initial values* of (3.2). Furthermore, for $t^0 > \tau$,

$$\mathcal{V}(t^0) := \left\{ \, x^0 \in \mathbb{R}^n \;\middle|\; (t^0, x^0) \in \mathcal{V} \, \right\}$$

is called the *set of the initial values, which are consistent with* t^0. $\diamond$

This definition can be found in [KM06, Def. 1.1] for instance. For the case that $E(\cdot)$ and $A(\cdot)$ are constant, consistent initial values and the subspace of all consistent initial values has been investigated in [OD85].

3.3.2 Remark. The space $\mathcal{V}(t^0)$ has the following properties:

(i) $\forall\, t^0 > \tau :\ \mathcal{V}(t^0)$ is a linear subspace of $\mathbb{R}^n$,

(ii) If $x : \mathcal{J} \to \mathbb{R}^n$ is a solution of (3.2), then $x(t) \in \mathcal{V}(t)$ for all $t \in \mathcal{J}$. $\diamond$

3.3.3 Definition (Equivalence of pairs (E, A) [KM06, Def. 3.3])**.** Two pairs (E_i, A_i), $E_i, A_i \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$, $i = 1, 2$, of matrix functions are called *equivalent w.r.t.* $S \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$, $T \in C^1((\tau,\infty) \to \mathbb{R}^{n\times n})$ if, and only if, $S(\cdot)$ and $T(\cdot)$ are pointwise nonsingular and

$$E_2 = SE_1T, \quad A_2 = SA_1T - SE_1\dot{T} \tag{3.13}$$

as equality of functions. We write $(E_1, A_1) \sim (E_2, A_2)$ if, and only if, (E_1, A_1) and (E_2, A_2) are *equivalent*, i.e. there exist pointwise nonsingular $S \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$, $T \in C^1((\tau,\infty) \to \mathbb{R}^{n\times n})$ such that (3.13) holds. $\diamond$

Any $T \in C^1((\tau,\infty) \to \mathbb{R}^{n\times n})$ s.t. $\det T(t) \neq 0$ for all $t > \tau$ satisfies

$$\begin{aligned} & 0 = \tfrac{\mathrm{d}}{\mathrm{d}t} I_n = \tfrac{\mathrm{d}}{\mathrm{d}t}(T(t)T(t)^{-1}) = \dot{T}(t)T(t)^{-1} + T(t)\tfrac{\mathrm{d}}{\mathrm{d}t}(T(t)^{-1}) \\ \Rightarrow \quad & \tfrac{\mathrm{d}}{\mathrm{d}t}(T(t)^{-1}) = -T(t)^{-1}\dot{T}(t)T(t)^{-1}. \end{aligned} \tag{3.14}$$

3.3.4 Remark (Equivalence relation)**.** By [KM06, Lem. 3.4] the relation introduced in Definition 3.3.3 is an equivalence relation. In particular, using (3.14), we obtain

$$E_1 = S^{-1}E_2T^{-1}, \quad A_1 = S^{-1}A_2T^{-1} - S^{-1}E_2\tfrac{\mathrm{d}}{\mathrm{d}t}T^{-1}. \tag{3.15}$$

3.3.5 Remark (Transformation of solutions)**.** Let $E_i, A_i \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$, $i = 1, 2$, be equivalent w.r.t. $S \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$, $T \in C^1((\tau,\infty) \to \mathbb{R}^{n\times n})$. Then $x(\cdot)$ is a solution of

$$E_1(t)\dot{x} = A_1(t)x$$

if, and only if, $y(\cdot) := T(\cdot)^{-1}x(\cdot)$ is a solution of

$$E_2(t)\dot{y} = A_2(t)y.$$

3.3.6 Definition (Transferability into SCF). A system (3.2) is called *transferable into standard canonical form (SCF)* if, and only if, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$(E, A) \sim \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right), \tag{3.16}$$

where $J : (\tau, \infty) \to \mathbb{R}^{n_1 \times n_1}$ and $N : (\tau, \infty) \to \mathbb{R}^{n_2 \times n_2}$, such that, for all $t > \tau$, $N(t)$ is strictly lower triangular[1]. $\diamond$

The definition of the SCF as we use it here can be found in [CP83]; it had its origin in [Cam83]. The SCF is a generalization of the well-known Weierstraß form for time-invariant differential-algebraic equations.

Uniqueness of the SCF is dealt with in the following theorem. It is shown that n_1, n_2 are unique and J, N are unique up to $(I_{n_1}, J) \sim (I_{n_1}, \tilde{J}), \quad (N, I_{n_2}) \sim (\tilde{N}, I_{n_2})$.

3.3.7 Theorem (Uniqueness of the SCF). *Let $n_1, n_2, \tilde{n}_1, \tilde{n}_2 \in \mathbb{N}$ and $J_1 \in C((\tau,\infty) \to \mathbb{R}^{n_1 \times n_1}), J_2 \in C((\tau,\infty) \to \mathbb{R}^{\tilde{n}_1 \times \tilde{n}_1}), N_1 \in C((\tau,\infty) \to \mathbb{R}^{n_2 \times n_2}), N_2 \in C((\tau,\infty) \to \mathbb{R}^{\tilde{n}_2 \times \tilde{n}_2})$, such that $N_1(t)$ and $N_2(t)$ are strictly lower triangular for all $t > \tau$. If*

$$\left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1 \end{bmatrix}, \begin{bmatrix} J_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) \quad \text{and} \quad \left(\begin{bmatrix} I_{\tilde{n}_1} & 0 \\ 0 & N_2 \end{bmatrix}, \begin{bmatrix} J_2 & 0 \\ 0 & I_{\tilde{n}_2} \end{bmatrix} \right)$$

are equivalent w.r.t. some $S \in C((\tau,\infty) \to \mathbb{R}^{n \times n})$, $T \in C^1((\tau,\infty) \to \mathbb{R}^{n \times n})$, then

(i) $n_1 = \tilde{n}_1$, $n_2 = \tilde{n}_2$,

(ii) $S = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}, \quad T_{11} = S_{11}^{-1}$,

(iii) $(I_{n_1}, J_1) \sim (I_{n_1}, J_2), \quad (N_1, I_{n_2}) \sim (N_2, I_{n_2})$. $\diamond$

The following lemma is crucial for the proof of Theorem 3.3.7. It deals with the subsystem $N(t)\dot{z} = z$, which is a so called *pure differential-algebraic equation*. Such systems only have the trivial solution.

[1] A matrix is called *strictly lower triangular* if it has only zeros on its main diagonal and above.

3.3.8 Lemma (Solution of the pure DAE part). *Let $N(\cdot) \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$ such that $N(t)$ is strictly lower triangular for all $t > \tau$. Then $x(\cdot) = 0$ is the unique global solution of the pure differential-algebraic equation*

$$N(t)\dot{x} = x. \tag{3.17}$$

Furthermore, every (local) solution $z : \mathcal{J} \to \mathbb{R}^n$ of (3.17) fulfills $z(t) = 0$, $t \in \mathcal{J}$.

Proof: *Step 1*: Clearly $x(\cdot) = 0$ solves (3.17) for all $t > \tau$.
Step 2: We show that any solution $z : \mathcal{J} \to \mathbb{R}^n$ of (3.17) fulfills $z = 0$. Consider (3.17) row-wise. Let $N(t) = (n_{ij}(t))_{i,j=1,\dots,n}$ for $t > \tau$, then

$$z_i(t) = \sum_{j=1}^{i-1} n_{ij}(t)\dot{z}_j(t) \tag{3.18}$$

for $t \in \mathcal{J}$ and $i \in \{1, ..., n\}$. We prove

$$\forall\, i \in \{1,\dots,n\}\ \forall\, t \in \mathcal{J} :\ z_i(t) = 0$$

by induction over i. The assertion clearly holds true for $i = 1$. Suppose it holds for some $i \in \{1,\dots,n-1\}$. Then $\dot{z}_j(t) = 0$ for all $t \in \mathcal{J}$ and all $j \in \{1,\dots,i\}$, hence

$$\forall\, t \in \mathcal{J} :\ z_{i+1}(t) \overset{(3.18)}{=} \sum_{j=1}^{i} n_{ij}(t)\dot{z}_j(t) = 0.$$

This shows $z = 0$ and completes the proof of the lemma. □

3.3.9 Corollary. *The initial value problem*

$$N(t)\dot{x} = x, \quad x(t^0) = x^0,$$

where $N(\cdot) \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$ such that $N(t)$ is strictly lower triangular for all $t > \tau$, $(t^0, x^0) \in (\tau,\infty) \times \mathbb{R}^n$, has a unique global solution if, and only if, $x^0 = 0$.

Proof of Theorem 3.3.7: *Step 1*: Without loss of generality assume that $n_1 \geq \tilde{n}_1$. Write $T^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ for $T_{11} \in C^1((\tau,\infty) \to \mathbb{R}^{n_1\times n_1})$, $T_{22} \in C^1((\tau,\infty) \to \mathbb{R}^{n_2\times n_2})$ and T_{12}, T_{21} appropriate. We show that, for all $t > \tau$,

$$T_{21}(t) = 0, \quad \det T_{11}(t) \neq 0, \quad \det T_{22}(t) \neq 0.$$

Let $(t^0, x^1) \in (\tau, \infty) \times \mathbb{R}^{n_1}$. Then

$$x : (\tau, \infty) \to \mathbb{R}^n,\ t \mapsto \begin{bmatrix} \Phi_{J_1}(t, t^0)x^1 \\ 0 \end{bmatrix},$$

where $\Phi_{J_1}(\cdot, \cdot)$ denotes the transition matrix of $\dot{z} = J_1(t)z$, solves

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1(t) \end{bmatrix} \dot{x} = \begin{bmatrix} J_1(t) & 0 \\ 0 & I_{n_2} \end{bmatrix} x.$$

By Remark 3.3.5 $y(\cdot) := T(\cdot)^{-1}x(\cdot)$ solves

$$\begin{bmatrix} I_{\tilde{n}_1} & 0 \\ 0 & N_2(t) \end{bmatrix} \dot{y} = \begin{bmatrix} J_2(t) & 0 \\ 0 & I_{\tilde{n}_2} \end{bmatrix} y,$$

and it follows from Lemma 3.3.8 that $y(\cdot) = \begin{bmatrix} y_1(\cdot) \\ 0 \end{bmatrix}$ for some $y_1 \in C^1((\tau, \infty) \to \mathbb{R}^{\tilde{n}_1})$. Hence

$$\begin{bmatrix} T_{11}(t^0)x^1 \\ T_{21}(t^0)x^1 \end{bmatrix} = T(t^0)^{-1}x(t^0) = y(t^0) = \begin{bmatrix} y_1(t^0) \\ 0 \end{bmatrix}. \tag{3.19}$$

Since $n_2 \leq \tilde{n}_2$ it follows $T_{21}(t^0)x^1 = 0$, and since $x^1 \in \mathbb{R}^{n_1}$ was arbitrary it follows $T_{21}(t^0) = 0$. Thus $\det T_{11}(t^0) \det T_{22}(t^0) = \det T(t^0)^{-1}$, and invertibility of $T(t^0)$ yields invertibility of $T_{11}(t^0)$.

Step 2: We prove (i). Assume that $n_1 > \tilde{n}_1$. Let α be the last row of $T_{11}(t^0)$, $\alpha^\top \in \mathbb{R}^{n_1}$. Then (3.19) and $n_1 > \tilde{n}_1$ yield $\alpha x^1 = 0$, and, since x^1 was arbitrary, it follows $\alpha = 0$, which contradicts $\det T_{11}(t^0) \neq 0$.

Step 3: We prove (iii). Write $S^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ for $S_{11} \in C((\tau, \infty) \to \mathbb{R}^{n_1 \times n_1})$, $S_{22} \in C((\tau, \infty) \to \mathbb{R}^{n_2 \times n_2})$ and S_{12}, S_{21} appropriate. Then

$$\begin{aligned} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1 \end{bmatrix} &= S^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N_2 \end{bmatrix} T^{-1} = \begin{bmatrix} S_{11}T_{11} + S_{12}N_2T_{21} & S_{11}T_{12} + S_{12}N_2T_{22} \\ S_{21}T_{11} + S_{22}N_2T_{21} & S_{21}T_{12} + S_{22}N_2T_{22} \end{bmatrix}, \\ \begin{bmatrix} J_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} &\overset{(3.15)}{=} S^{-1} \begin{bmatrix} J_2 & 0 \\ 0 & I_{n_2} \end{bmatrix} T^{-1} - S^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N_2 \end{bmatrix} \tfrac{\mathrm{d}}{\mathrm{d}t}\big(T^{-1}\big) \end{aligned} \tag{3.20}$$

$$= \begin{bmatrix} S_{11}J_2T_{11} + S_{12}T_{21} - S_{11}\dot{T}_{11} - S_{12}N_2\dot{T}_{21} & S_{11}J_2T_{12} + S_{12}T_{22} - S_{11}\dot{T}_{12} - S_{12}N_2\dot{T}_{22} \\ S_{21}J_2T_{11} + S_{22}T_{21} - S_{21}\dot{T}_{11} - S_{22}N_2\dot{T}_{21} & S_{21}J_2T_{12} + S_{22}T_{22} - S_{21}\dot{T}_{12} - S_{22}N_2\dot{T}_{22} \end{bmatrix}. \tag{3.21}$$

Step 1 and the equations in the first n_1 columns in (3.20) yield

$$\forall\, t > \tau : \; S_{11}(t)^{-1} = T_{11}(t) \;\wedge\; S_{21}(t) = 0 \;\wedge\; \det S_{22}(t) \neq 0,$$

and therefore, by (3.20),

$$N_1 = S_{22}N_2T_{22} \tag{3.22}$$

and, by the lower right block in (3.21),

$$I_{n_2} = S_{22}T_{22} - S_{22}N_2\dot{T}_{22}. \tag{3.23}$$

Both (3.22) and (3.23) are equivalent to $(N_1, I_{n_2}) \sim (N_2, I_{n_2})$. On the other hand the upper left block in (3.21) yields $J_1 = S_{11}J_2T_{11} - S_{11}\dot{T}_{11}$, and invoking $S_{11} = T_{11}^{-1}$, we find $J_1 = T_{11}^{-1}J_2T_{11} - T_{11}^{-1}\dot{T}_{11}$ or, equivalently, $(I_{n_1}, J_1) \sim (I_{n_1}, J_2)$.

Step 4: It remains to prove $T_{12} = S_{12} = 0$. From (3.23) it follows

$$S_{22}^{-1} = T_{22} - N_2\dot{T}_{22}.$$

Observe that the upper right block in (3.21) yields

$$0 = S_{11}(J_2T_{12}\dot{T}_{12}) + S_{12}(T_{22} - N_2\dot{T}_{22}),$$

hence

$$S_{12} = -S_{11}(J_2T_{12} - \dot{T}_{12})S_{22}. \tag{3.24}$$

Then, from the upper right block in (3.20) we may deduce

$$T_{12} = -S_{11}^{-1}S_{12}N_2T_{22} \overset{(3.24)}{=} (J_2T_{12} - \dot{T}_{12})S_{22}N_2T_{22} \overset{(3.22)}{=} (J_2T_{12} - \dot{T}_{12})N_1. \tag{3.25}$$

Therefore we find

$$T_{12}e_{n_2} \overset{(3.25)}{=} (J_2T_{12} - \dot{T}_{12})N_1e_{n_2} = (J_2T_{12} - \dot{T}_{12}) \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = 0, \tag{3.26}$$

and, furthermore

$$T_{12}e_{n_2-1} \overset{(3.25)}{=} (J_2T_{12} - \dot{T}_{12})N_1e_{n_2-1} = (J_2T_{12} - \dot{T}_{12}) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix} \overset{(3.26)}{=} 0.$$

Proceeding in this way gives $T_{12} = 0$ and, invoking (3.24), we find $S_{12} = 0$. □

Next we derive a representation of the solutions of a system (3.2), which is transferable into SCF, and, furthermore, a representation for $\mathcal{V}$.

3.3.10 Theorem (Solution of the homogeneous DAE)**.** *Let* (3.2) *be transferable into SCF and use the notation from Definition 3.3.6.*

(i)

$$(t^0, x^0) \in \mathcal{V} \iff x^0 \in \operatorname{im} T(t^0) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}. \tag{3.27}$$

(ii) Any solution of the initial value problem (3.2)*,* $x(t^0) = x^0$*, where* $(t^0, x^0) \in \mathcal{V}$*, can be uniquely extended to a global solution* $x(\cdot)$*, and this solution satisfies*

$$x(t) = U(t,t^0)x^0, \quad U(t,t^0) := T(t) \begin{bmatrix} \Phi_J(t,t^0) & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1}, \ t > \tau, \tag{3.28}$$

where $\Phi_J(\cdot,\cdot)$ *denotes the transition matrix of* $\dot{z} = J(t)z$*.*

Proof: *Step 1*: We show that $x(\cdot)$ as in (3.28) solves (3.2) for all $t > \tau$:

$$\begin{aligned}
& E(t)\dot{x}(t) \\
= \quad & E(t)\left(\dot{T}(t)\begin{bmatrix}\Phi_J(t,t^0) & 0\\ 0 & 0\end{bmatrix} + E(t)T(t)\begin{bmatrix}J(t)\Phi_J(t,t^0) & 0\\ 0 & 0\end{bmatrix}\right)T(t^0)^{-1}x^0 \\
\overset{(3.16)}{=} \quad & \left(E(t)\dot{T}(t)\begin{bmatrix}\Phi_J(t,t^0) & 0\\ 0 & 0\end{bmatrix} + S(t)^{-1}\begin{bmatrix}I_{n_1} & 0\\ 0 & N(t)\end{bmatrix}\begin{bmatrix}J(t)\Phi_J(t,t^0) & 0\\ 0 & 0\end{bmatrix}\right)T(t^0)^{-1}x^0 \\
= \quad & \left(E(t)\dot{T}(t) + S(t)^{-1}\begin{bmatrix}J(t) & 0\\ 0 & I_{n_2}\end{bmatrix}\right)\begin{bmatrix}\Phi_J(t,t^0) & 0\\ 0 & 0\end{bmatrix}T(t^0)^{-1}x^0 \\
\overset{(3.16)}{=} \quad & (E(t)\dot{T}(t) + A(t)T(t) - E(t)\dot{T}(t))\begin{bmatrix}\Phi_J(t,t^0) & 0\\ 0 & 0\end{bmatrix}T(t^0)^{-1}x^0 \\
= \quad & A(t)T(t)\begin{bmatrix}\Phi_J(t,t^0) & 0\\ 0 & 0\end{bmatrix}T(t^0)^{-1}x^0 = A(t)x(t).
\end{aligned}$$

Step 2: We show that $x(t^0) = x^0$ for $x(\cdot)$ as in (3.28) if, and only, $x^0 \in \operatorname{im} T(t^0)\begin{bmatrix}I_{n_1}\\ 0\end{bmatrix}$. Set

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} := T(t^0)^{-1}x^0,$$

where $\alpha \in \mathbb{R}^{n_1}, \beta \in \mathbb{R}^{n_2}$. Then

$$x(t^0) = T(t^0)\begin{bmatrix}I_{n_1} & 0\\ 0 & 0\end{bmatrix}T(t^0)^{-1}x^0 = T(t^0)\begin{bmatrix}I_{n_1} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = x^0 - T(t^0)\begin{bmatrix}0\\ \beta\end{bmatrix},$$

and hence $x(t^0) = x^0$ if, and only if, $\beta = 0$ or, equivalently, $x^0 \in \operatorname{im} T(t^0)\begin{bmatrix}I_{n_1}\\ 0\end{bmatrix}$.

Step 3: We show that every solution $z : \mathcal{J} \to \mathbb{R}^n$ of (3.2), $z(t^0) = x^0$, $(t^0, x^0) \in \mathcal{V}$ fulfills $z = x\,|_{\mathcal{J}}$ for $x(\cdot)$ as in (3.28). Clearly $(z-x) : \mathcal{J} \to \mathbb{R}^n$ solves $E(t)\frac{\mathrm{d}}{\mathrm{d}t}(z-x)(t) = A(t)(z-x)(t)$ for all $t \in \mathcal{J}$. Then $[y_1(\cdot)^\top, y_2(\cdot)^\top]^\top = y(\cdot) := T(\cdot)^{-1}(z-x)(\cdot)$ solves

$$\begin{aligned}
\dot{y}_1 &= J(t)y_1,\\
N(t)\dot{y}_2 &= y_2,
\end{aligned}$$

and by Lemma 3.3.8 it follows $y_2(t) = 0$ for all $t \in \mathcal{J}$. Then, invoking $y(t^0) = T(t^0)^{-1}(x^0 - x(t^0))$,

$$\begin{aligned} 0 = \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} y(t^0) &= \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} T(t^0)^{-1} \left(x^0 - T(t^0) \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1} x^0 \right) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} T(t^0)^{-1} x^0 = T(t^0)^{-1}(x^0 - x(t^0)) = y(t^0). \end{aligned}$$

Hence $y_1(t) = 0$ for all $t \in \mathcal{J}$ and therefore $z = x\,|_{\mathcal{J}}$. This concludes the proof. □

Now we are in a position to define the main tool for the further investigations (especially stability theory), the generalized transition matrix.

3.3.11 Definition (Generalized transition matrix)**.** Let (3.2) be transferable into SCF, i.e. (3.16) holds for some pointwise nonsingular $S \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$, $T \in C^1((\tau,\infty) \to \mathbb{R}^{n\times n})$. The *generalized transition matrix* $U(\cdot\,,\cdot)$ of the system (3.2) is defined by

$$U(t,s) := T(t) \begin{bmatrix} \Phi_J(t,s) & 0 \\ 0 & 0 \end{bmatrix} T(s)^{-1}, \quad t,s > \tau,$$

and does not depend on the special choice of S, T. ◇

The following proposition shows uniqueness of $U(\cdot\,,\cdot)$ and generalizes Lemma 2.1.1.

3.3.12 Proposition (Uniqueness and properties of $U(\cdot\,,\cdot)$)**.** *Let J_1, N_1, J_2, N_2 as in Theorem 3.3.7. If*

$$\left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1 \end{bmatrix}, \begin{bmatrix} J_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) \quad \textit{and} \quad \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_2 \end{bmatrix}, \begin{bmatrix} J_2 & 0 \\ 0 & I_{n_2} \end{bmatrix} \right)$$

are equivalent w.r.t. some $S \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$, $T \in C^1((\tau,\infty) \to \mathbb{R}^{n\times n})$, then

$$\begin{bmatrix} \Phi_{J_2}(t,s) & 0 \\ 0 & 0 \end{bmatrix} = T(t) \begin{bmatrix} \Phi_{J_1}(t,s) & 0 \\ 0 & 0 \end{bmatrix} T(s)^{-1}, \quad t,s > \tau.$$

If (3.2) *is transferable into SCF, then, for arbitrary $t, r, s \in (\tau,\infty)$,*

(i) $E(t)\frac{\mathrm{d}}{\mathrm{d}t}U(t,s) = A(t)U(t,s)$, *(ii)* $\operatorname{im} U(t,s) = \mathcal{V}(t)$,

(iii) $U(t,r)U(r,s) = U(t,s)$,

(iv) $U(t,t)^2 = U(t,t)$,

(v) $\forall\, x \in \mathcal{V}(t) :\ U(t,t)x = x$.

Proof: *Step 1*: We prove uniqueness of $U(\cdot\,,\cdot)$. Choose arbitrary $(s,\alpha) \in (\tau,\infty)\times\mathbb{R}^{n_1}$. Since, by Theorem 3.3.7,

$$T = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}, \quad S = \begin{bmatrix} T_{11}^{-1} & 0 \\ 0 & S_{22} \end{bmatrix},$$

we find $x^0 := T(s)\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11}(s)\alpha \\ 0 \end{bmatrix}$, and hence

$$x : (\tau,\infty) \to \mathbb{R}^n,\ t \mapsto \begin{bmatrix} \Phi_{J_2}(t,s) & 0 \\ 0 & 0 \end{bmatrix} x^0$$

solves

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_2(t) \end{bmatrix} \dot{x} = \begin{bmatrix} J_2(t) & 0 \\ 0 & I_{n_2} \end{bmatrix} x, \quad x(s) = x^0.$$

Furthermore, $y(\cdot) := T(\cdot)^{-1}x(\cdot)$ solves

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1(t) \end{bmatrix} \dot{y} = \begin{bmatrix} J_1(t) & 0 \\ 0 & I_{n_2} \end{bmatrix} y, \quad y(s) = T(s)^{-1}x^0 = \begin{bmatrix} \alpha \\ 0 \end{bmatrix},$$

and since the global solution of this initial value problem is unique by Theorem 3.3.10, we find

$$y(t) = \begin{bmatrix} \Phi_{J_1}(t,s)\alpha \\ 0 \end{bmatrix}, \quad t > \tau.$$

This gives

$$T(t)\begin{bmatrix} \Phi_{J_1}(t,s) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = T(t)y(t) = x(t) = \begin{bmatrix} \Phi_{J_2}(t,s) & 0 \\ 0 & 0 \end{bmatrix} T(s)\begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \quad t > \tau,$$

and since $\alpha \in \mathbb{R}^{n_1}$ was arbitrary it follows that

$$\Phi_{J_1}(t,s) = T_{11}(t)\Phi_{J_2}(t,s)T_{11}(s).$$

Finally we may deduce

$$T(t)\begin{bmatrix}\Phi_{J_1}(t,s) & 0\\ 0 & 0\end{bmatrix}T(s)^{-1} = \begin{bmatrix}T_{11}(t)\Phi_{J_1}(t,s)T_{11}(s)^{-1} & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}\Phi_{J_2}(t,s) & 0\\ 0 & 0\end{bmatrix}, \quad t > \tau,$$

and since $s > \tau$ was arbitrary the assertion follows.

Step 2: We prove (i)-(v). Recall the representation of $E(\cdot)$ and $A(\cdot)$ given by (3.16).

(i) This is a straightforward calculation as in Step 1 of the proof of Theorem 3.3.10.

(ii) The invertibility of $\Phi_J(t,s)$ and (3.27) yield

$$\operatorname{im} U(t,s) = \operatorname{im}\left(T(t)\begin{bmatrix}\Phi_J(t,s) & 0\\ 0 & 0\end{bmatrix}T(s)^{-1}\right) = \operatorname{im}\left(T(t)\begin{bmatrix}I_{n_1} & 0\\ 0 & 0\end{bmatrix}\right) = \mathcal{V}(t).$$

(iii) It immediately follows from Lemma 2.1.1 (iii) that

$$\begin{aligned} U(t,r)U(r,s) &= T(t)\begin{bmatrix}\Phi_J(t,r) & 0\\ 0 & 0\end{bmatrix}T(r)^{-1}T(r)\begin{bmatrix}\Phi_J(r,s) & 0\\ 0 & 0\end{bmatrix}T(s)^{-1}\\ &= T(t)\begin{bmatrix}\Phi_J(t,r)\Phi_J(r,s) & 0\\ 0 & 0\end{bmatrix}T(s)^{-1}\\ &= T(t)\begin{bmatrix}\Phi_J(t,s) & 0\\ 0 & 0\end{bmatrix}T(s)^{-1} = U(t,s). \end{aligned}$$

(iv) Follows from (iii).

(v) Let $x \in \mathcal{V}(t)$. By (ii) we get $x \in \operatorname{im} U(t,t)$ and hence there exists $y \in \mathbb{R}^n$ such that $U(t,t)y = x$. Then

$$U(t,t)x = U(t,t)^2 y \overset{\text{(iv)}}{=} U(t,t)y = x. \qquad \square$$

Having Theorem 3.3.10 and the generalized transition matrix $U(\cdot,\cdot)$ we may immediately derive a vector space isomorphism between $\mathcal{V}(t^0)$ and the set of all global solutions of (3.2), for all $t^0 > \tau$.

3.3.13 Corollary (Vector space isomorphism)**.** *Let* (3.2) *be transferable into SCF and* $t^0 > \tau$*. Set*

$$\mathcal{B}(E,A) := \left\{ \ x : (\tau,\infty) \to \mathbb{R}^{n\times n} \ \middle| \ x(\cdot) \text{ is a global solution of (3.2)} \ \right\}.$$

Then the linear map

$$\varphi : \mathcal{V}(t^0) \to \mathcal{B}(E,A), \ x^0 \mapsto \big((\tau,\infty) \ni t \mapsto U(t,t^0)x^0\big)$$

is a vector space isomorphism.

Proof: *Step 1*: By Theorem 3.3.10 we get

$$\forall\, x^0 \in \mathcal{V}(t^0): \ \big((\tau,\infty) \ni t \mapsto U(t,t^0)x^0\big) \in \mathcal{B}(E,A),$$

and since, furthermore, $U(\cdot\,,\cdot)$ is well-defined we obtain that $\varphi(\cdot)$ is well-defined.
Step 2: We show that $\varphi(\cdot)$ is surjective. Let $x(\cdot) \in \mathcal{B}(E,A)$. Then, clearly, $x(t^0) \in \mathcal{V}(t^0)$ and from Theorem 3.3.10 (ii) it follows

$$\forall\, t > \tau: \ x(t) = U(t,t^0)x(t^0),$$

thus having $\varphi(x(t^0))(\cdot) = x(\cdot)$.
Step 3: We show that $\varphi(\cdot)$ is injective. Let $x^1, x^2 \in \mathcal{V}(t^0)$ such that $\varphi(x^1)(\cdot) = \varphi(x^2)(\cdot)$. Then

$$x^1 \overset{3.3.12\,(v)}{=} U(t^0,t^0)x^1 = \varphi(x^1)(t^0) = \varphi(x^2)(t^0) = U(t^0,t^0)x^2 \overset{3.3.12\,(v)}{=} x^2.$$

This completes the proof. □

3.3.14 Remark. If (3.2) is transferable into SCF we find, by Proposition 3.3.12 (ii), that $\mathcal{V}(t) = \operatorname{im} U(t,t)$ for all $t > \tau$. Since $t \mapsto U(t,t)$ is continuously differentiable, the representation of $\mathcal{V}(t)$ depends continuously differentiable on the time t and the dimension of $\mathcal{V}(t)$ does not change: $\dim \mathcal{V}(t) = n_1$ for all $t > \tau$.
These properties do not hold true for systems which are not transferable into SCF, as shown by the following example. ◇

3.3.15 Example. Consider the system

$$t\dot{x} = (1-t)x, \quad t \in \mathbb{R}. \tag{3.29}$$

For $t^0 \neq 0$, $x^0 \in \mathbb{R}$, the unique global solution $x(\cdot)$ of (3.29), $x(t^0) = x^0$ is

$$x : \mathbb{R} \to \mathbb{R},\ t \mapsto \frac{te^{-t}}{t^0 e^{-t^0}} x^0.$$

Hence every (local) solution $x : \mathcal{J} \to \mathbb{R}$ of (3.29), $x(0) = 0$ can be uniquely extended to

$$x_c : \mathbb{R} \to \mathbb{R},\ t \mapsto cte^{-t},$$

where $c = \frac{e^{\tilde{t}}}{\tilde{t}} x(\tilde{t})$ for some $\tilde{t} \in \mathcal{J} \setminus \{0\}$. The solutions $x_c(\cdot)$, $c \in \mathbb{R}$, are the only global solutions of (3.29), $x(0) = 0$. Furthermore, any initial value problem (3.29), $x(0) = x^0$ with $x^0 \neq 0$ has no solution. Hence we find $\mathcal{V}(t) = \mathbb{R}$ for $t \neq 0$, but $\mathcal{V}(0) = \{0\}$. We stress that the coefficients of (3.29) are analytic. ◇

3.4 Analytic solvability and inhomogeneous DAEs

In this section we investigate the inhomogeneous system (3.1). For (3.1) transferable into SCF we derive a representation for the unique global solutions of (3.1), $x(t^0) = x^0$ and conditions under which this solution exists. Furthermore, we introduce the concept of analytic solvability and determine its connection to transferability into SCF.

3.4.1 Proposition (Solution of the inhomogeneous pure DAE)**.** *Let $N(\cdot) \in \mathcal{C}^n((\tau,\infty) \to \mathbb{R}^{n\times n})$ such that $N(t)$ is strictly lower triangular for all $t > \tau$, $f(\cdot) \in \mathcal{C}^n((\tau,\infty) \to \mathbb{R}^n)$ and $(t^0, x^0) \in (\tau,\infty) \times \mathbb{R}^n$. Then the initial value problem*

$$N(t)\dot{x} = x + f(t), \quad x(t^0) = x^0, \tag{3.30}$$

has a solution if, and only if,

$$-\sum_{k=0}^{n-1} \left(N(\cdot)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k f(\cdot)\Big|_{t=t^0} = x^0. \tag{3.31}$$

Any solution of (3.30) *can be uniquely extended to a global solution* $x(\cdot)$*, and this solution satisfies*

$$x(t) = -\sum_{k=0}^{n-1} \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k f(t), \quad t > \tau. \tag{3.32}$$

Proof: *Step 1*: We show, for any $g(\cdot) \in C^n(\mathcal{J} \to \mathbb{R}^n)$, that

$$\forall\, t \in \mathcal{J}: \ \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^n g(t) = 0. \tag{3.33}$$

Let

$$g_0(\cdot) := \dot g(\cdot), \quad g_{k+1}(\cdot) := N(\cdot)\dot g_k(\cdot) + \dot N(\cdot) g_k(\cdot),\ k = 0, \ldots, n-2.$$

Then $\left(N(\cdot)\frac{\mathrm{d}}{\mathrm{d}t}\right)^n g(\cdot) = N(\cdot) g_{n-1}(\cdot)$, and hence we find, for all $t \in \mathcal{J}$,

$$\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^n g(t) = \sum_{j_0=0}^{n-1} \cdots \sum_{j_n=0}^{n-1} \alpha_{j_0,j_1,\ldots,j_n} N^{(j_0)}(t) \cdots N^{(j_{n-1})}(t) g^{(j_n+1)}(t) \tag{3.34}$$

for some $\alpha_{j_0,j_1,\ldots,j_n} \in \mathbb{R}$ for $(j_0, ..., j_n) \in \{0, ..., n-1\}^{n+1}$. Since $N(\cdot)$ is strictly lower triangular, the derivatives of $N(\cdot)$ are also strictly lower triangular. Obviously the product of n strictly lower triangular matrices must be zero and so (3.33) follows from (3.34).

Step 2: We show that $x(\cdot)$ as in (3.32) solves $N(t)\dot x(t) = x(t) + f(t)$ for all $t > \tau$:

$$\begin{aligned} N(t)\dot x(t) &= -(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}) \sum_{k=0}^{n-1} \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k f(t) \\ &= -\sum_{k=0}^{n-1} \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k f(t) + f(t) - \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t) \\ &\overset{(3.33)}{=} x(t) + f(t). \end{aligned}$$

Step 3: Clearly $x(t^0) = x^0$ for $x(\cdot)$ as in (3.32) if, and only if, (3.31) holds.

Step 4: We show that any solution $z : \mathcal{J} \to \mathbb{R}^n$ of (3.30) fulfills $z = x\,|_{\mathcal{J}}$ for $x(\cdot)$ as in (3.32). Obviously $(z - x) : \mathcal{J} \to \mathbb{R}^n$ solves $N(t)\frac{\mathrm{d}}{\mathrm{d}t}(z - x)(t) = (z - x)(t)$ for all $t \in \mathcal{J}$ and by Lemma 3.3.8 it follows immediately $(z - x)(\cdot) = 0$. □

3.4.2 Theorem (Solution of the inhomogeneous DAE)**.** *Let* $E(\cdot), A(\cdot) \in C^n((\tau,\infty) \to \mathbb{R}^{n\times n})$ *and* (3.2) *be transferable into SCF and use the notation from Definition 3.3.6. Furthermore, let* $S(\cdot), T(\cdot) \in C^n((\tau,\infty) \to \mathbb{R}^{n\times n})$*. Then, for* $f(\cdot) \in C^n((\tau,\infty) \to \mathbb{R}^n)$*, the following statements hold:*

(i) The initial value problem (3.1)*,* $x(t^0) = x^0$ *has a solution if, and only if,*

$$x^0 + T(t^0)\begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \left(\sum_{k=0}^{n_1-1} \left(N(\cdot)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]S(\cdot)f(\cdot) \right)\Bigg|_{t=t^0} \in \operatorname{im} T(t^0)\begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}. \tag{3.35}$$

(ii) Any solution of (3.1)*,* $x(t^0) = x^0$*, s.t.* (3.35) *holds, can be uniquely extended to a global solution* $x(\cdot)$*, and this solution satisfies*

$$\begin{aligned} x(t) \;=\;& U(t,t^0)x^0 + \int_{t^0}^{t} U(t,s)T(s)S(s)f(s)\,ds \\ & - T(t)\begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \sum_{k=0}^{n_1-1} \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]S(t)f(t), \quad t > \tau. \end{aligned} \tag{3.36}$$

where $U(\cdot\,,\cdot)$ *is the generalized transition matrix of* (3.2)*, see Definition 3.3.11.*

Proof: *Step 1*: We show that $x(\cdot)$ as in (3.36) solves (3.1) for all $t > \tau$:

$$\begin{aligned} & E(t)\dot{x}(t) \\ =\;& E(t)\tfrac{\mathrm{d}}{\mathrm{d}t}U(t,t^0)x^0 + E(t)\int_{t^0}^{t} \tfrac{\mathrm{d}}{\mathrm{d}t}U(t,s)T(s)S(s)f(s)\,ds \\ & + E(t)U(t,t)T(t)S(t)f(t) \\ & - E(t)\dot{T}(t)\begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \sum_{k=0}^{n_1-1} \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]S(t)f(t) \\ & - E(t)T(t)\begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \sum_{k=0}^{n_1-1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]S(t)f(t) \\ \overset{3.3.12\,(i)}{=}\;& A(t)U(t,t^0)x^0 + A(t)\int_{t^0}^{t} U(t,s)T(s)S(s)f(s)\,ds + E(t)U(t,t)T(t)S(t)f(t) \\ & - E(t)\dot{T}(t)\begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \sum_{k=0}^{n_1-1} \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]S(t)f(t) \\ & - E(t)T(t)\begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \sum_{k=0}^{n_1-1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]S(t)f(t) \end{aligned} \tag{3.37}$$

Since, for $t > \tau$,

$$
\begin{aligned}
& E(t)T(t)\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\sum_{k=0}^{n_1-1}(\tfrac{\mathrm{d}}{\mathrm{d}t})\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k[0,I_{n_2}]S(t)f(t)\\
\overset{(3.16)}{=}\quad & S(t)^{-1}\begin{bmatrix}I_{n_1} & 0\\ 0 & N(t)\end{bmatrix}\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\sum_{k=0}^{n_1-1}(\tfrac{\mathrm{d}}{\mathrm{d}t})\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k[0,I_{n_2}]S(t)f(t)\\
=\quad & S(t)^{-1}\sum_{k=0}^{n_1-1}\begin{bmatrix}0 & 0\\ 0 & \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{k+1}\end{bmatrix}S(t)f(t)\\
=\quad & S(t)^{-1}\left(\sum_{k=0}^{n_1-1}\begin{bmatrix}0 & 0\\ 0 & \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{k}\end{bmatrix}-\begin{bmatrix}0 & 0\\ 0 & I_{n_2}\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{n_1}\end{bmatrix}\right)S(t)f(t)\\
\overset{(3.33)}{=}\quad & S(t)^{-1}\sum_{k=0}^{n_1-1}\begin{bmatrix}0 & 0\\ 0 & \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{k}\end{bmatrix}S(t)f(t)-S(t)^{-1}\begin{bmatrix}0 & 0\\ 0 & I_{n_2}\end{bmatrix}S(t)f(t),
\end{aligned}
$$

and

$$
E(t)U(t,t)T(t)S(t)f(t)\overset{(3.16)}{=}S(t)^{-1}\begin{bmatrix}I_{n_1} & 0\\ 0 & 0\end{bmatrix}S(t)f(t),
$$

we find that the last 3 terms in (3.37) equal

$$
\begin{aligned}
& S(t)^{-1}\begin{bmatrix}I_{n_1} & 0\\ 0 & 0\end{bmatrix}S(t)f(t)+S(t)^{-1}\begin{bmatrix}0 & 0\\ 0 & I_{n_2}\end{bmatrix}S(t)f(t)\\
& -E(t)\dot T(t)\sum_{k=0}^{n_1-1}\begin{bmatrix}0 & 0\\ 0 & \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{k}\end{bmatrix}S(t)f(t)-S(t)^{-1}\sum_{k=0}^{n_1-1}\begin{bmatrix}0 & 0\\ 0 & \left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{k}\end{bmatrix}S(t)f(t)\\
=\quad & f(t)-\left(E(t)\dot T(t)+S(t)^{-1}\right)\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\sum_{k=0}^{n_1-1}\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k[0,I_{n_2}]S(t)f(t)\\
\overset{(3.16)}{=}\quad & f(t)-\left(A(t)T(t)-S(t)^{-1}\begin{bmatrix}J(t) & 0\\ 0 & I_{n_2}\end{bmatrix}+S(t)^{-1}\right)\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\\
& \cdot\sum_{k=0}^{n_1-1}\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k[0,I_{n_2}]S(t)f(t)\\
=\quad & f(t)-A(t)T(t)\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\sum_{k=0}^{n_1-1}\left(N(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k[0,I_{n_2}]S(t)f(t),
\end{aligned}
$$

and arrive at $E(t)\dot{x}(t) = A(t)x(t) + f(t)$, $t > \tau$.

Step 2: We show that $x(t^0) = x^0$ for $x(\cdot)$ as in (3.36) if, and only, (3.35) holds. Set

$$\eta := T(t^0) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \left(\sum_{k=0}^{n_1-1} \left(N(\cdot)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]S(\cdot)f(\cdot) \right)\Bigg|_{t=t^0}, \qquad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} := T(t^0)^{-1}\left(x^0 + \eta\right),$$

where $\alpha \in \mathbb{R}^{n_1}, \beta \in \mathbb{R}^{n_2}$. Then

$$\begin{aligned} x(t^0) &= T(t^0) \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1}x^0 - \eta = T(t^0) \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &\quad - T(t^0) \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \left(\sum_{k=0}^{n_1-1} \left(N(\cdot)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}]S(\cdot)f(\cdot) \right)\Bigg|_{t=t^0} - \eta \\ &= T(t^0) \begin{bmatrix} \alpha \\ 0 \end{bmatrix} - \eta = x^0 - T(t^0) \begin{bmatrix} 0 \\ \beta \end{bmatrix}, \end{aligned}$$

and hence $x(t^0) = x^0$ if, and only if, $\beta = 0$ or, equivalently, (3.35) holds.

Step 3: Let $(t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n$ such that (3.1), $z(t^0) = x^0$ has a solution. We show that every solution $z : \mathcal{J} \to \mathbb{R}^n$ of (3.1), $z(t^0) = x^0$ fulfills $z = x\,|_{\mathcal{J}}$ for $x(\cdot)$ as in (3.36). Clearly $(z - x) : \mathcal{J} \to \mathbb{R}^n$ solves $E(t)\frac{\mathrm{d}}{\mathrm{d}t}(z - x)(t) = A(t)(z - x)(t)$ for all $t \in \mathcal{J}$. Then Theorem 3.3.10 gives $(z - x)(t^0) \in \operatorname{im} T(t^0) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}$, and since, by Step 2, $x^0 - x(t^0) = T(t^0) \begin{bmatrix} 0 \\ \beta \end{bmatrix} \in \operatorname{im} T(t^0) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix}$, we conclude

$$z(t^0) - x(t^0) \in \operatorname{im} T(t^0) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} \cap \operatorname{im} T(t^0) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} = \{0\}.$$

Therefore, a repeated application of Theorem 3.3.10 yields $z = x\,|_{\mathcal{J}}$. This concludes the proof. □

3.4.3 Remark (Consistent initial values)**.** Let η be as in the proof of Theorem 3.4.2. Then (3.35) reads $x^0 + \eta \in \mathcal{V}(t^0)$, where $\mathcal{V}(t^0)$ is the set of initial values consistent with t^0 of the associated homogeneous system (3.2). Hence the *set of initial values* *consistent with* t^0 *of* (3.1) is the affine subspace

$$-\eta + \mathcal{V}(t^0) = -T(t^0) \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \left(\sum_{k=0}^{n_1-1} \left(N(\cdot)\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k [0, I_{n_2}] S(\cdot) f(\cdot) \right)\Bigg|_{t=t^0} + \mathcal{V}(t^0).$$

◇

We introduce the concept of analytic solvability.

3.4.4 Definition (Analytic solvability [CP83]). A system (3.1) is called *analytically solvable* if, and only if, we have, for all $f(\cdot) \in C^n((\tau,\infty) \to \mathbb{R}^n)$,

(i) $\exists$ solution to (3.1),

(ii) $\forall$ solutions $y : \mathcal{J} \to \mathbb{R}^n$ of (3.1) :

$$\exists \text{ global solution } x(\cdot) \text{ of (3.1) with } x\,|_{\mathcal{J}} = y,$$

(iii) $\forall$ global solutions $x_1(\cdot),\ x_2(\cdot)$ of (3.1) :

$$\left[\exists\, t^0 > \tau :\ x_1(t^0) \neq x_2(t^0)\right] \ \Rightarrow\ \left[\forall\, t > \tau :\ x_1(t) \neq x_2(t)\right].$$

3.4.5 Remark. Roughly speaking system (3.1) is *analytically solvable* if, and only if, for any inhomogeneity $f(\cdot) \in C^n((\tau,\infty) \to \mathbb{R}^n)$ there exist solutions to (3.1), and solutions when they exist, can be extended to all of (τ,∞) and are uniquely determined by their value at any $t^0 \in (\tau,\infty)$. ◇

3.4.6 Remark. One may also wonder whether in Definition 3.4.4 the conditions (i) and (ii) already imply (iii), or, in other words, if the following holds: Any initial value problem (3.1), $x(t^0) = x^0$, which has a solution, either has a unique (right) global solution, or there exists a (right) maximal solution which is not (right) global. Revisiting Example 3.3.15, one observes that it is possible for an initial value problem to have more than one global solution, and every local solution can be uniquely extended to one of the global solutions. ◇

We derive a relationship between transferability into SCF and analytic solvability.

3.4.7 Theorem (Relationship between SCF and analytic solvability)**.** *Consider system* (3.1)*. Then*

$$(3.1) \textit{ is analytically solvable} \quad \Longleftrightarrow \quad (3.2) \textit{ is transferable into SCF.}$$

For the implication ($\Leftarrow$), *it suffices to assume* $E, A \in C^n((\tau,\infty) \to \mathbb{R}^n)$ *and, using the notation from Definition 3.3.6,* $S, T \in C^n((\tau,\infty) \to \mathbb{R}^n)$.
For the implication ($\Rightarrow$), *real analyticity of* E *and* A *can, in general, not be dispensed.*

Proof: ($\Leftarrow$): Follows immediately from Theorem 3.4.2.
($\Rightarrow$): See [CP83, Thm. 2] and Example 3.4.8 for the latter statement. Note that the "SCF" constructed in the proof of [CP83, Thm. 2] is a pair $\left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) =: (\tilde{E}, \tilde{A})$, where $N(t)$ is strictly *upper* triangular for all $t > \tau$. Choosing $S \equiv T \equiv \begin{bmatrix} I_{n_1} & 0 \\ 0 & \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \end{bmatrix}$ we get that $(\tilde{E}, \tilde{A})$ and $\left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & \tilde{N} \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right)$ are equivalent w.r.t. S, T and $\tilde{N}(t)$ is strictly lower triangular for all $t > \tau$. $\square$

3.4.8 Example. [CP83, Ex. 2] states that the system

$$E(t)\dot{x} = -x + f(t), \tag{3.38}$$

where

$$E(t) = t^3 \begin{bmatrix} \sin(t^{-1}) \\ \cos(t^{-1}) \end{bmatrix} [\cos(t^{-1}), -\sin(t^{-1})], \quad E(0) = 0, \tag{3.39}$$

is an analytically solvable system, but any $S(\cdot), T(\cdot)$ putting (3.38),(3.39) into SCF must be discontinuous at zero. Note that for (3.38) it holds $A(\cdot) = -I$ and therefore $A(\cdot)$ is real analytic and $A(t) - \lambda E(t)$ is a regular matrix pencil for all $t \in \mathbb{R}$. $\diamond$

3.5 Stability

In the following, with respect to Theorem 3.2.3, we concentrate on the homogeneous system (3.2). Furthermore, we restrict ourselves to systems which are transferable into SCF. In this section we give some useful characterizations of the various concepts of stability introduced in Definition 1.2.2. Thereto we generalize Proposition 2.1.3 - Corollary 2.1.6, which hold for ordinary linear differential equations. The main tool for deriving stability results is the generalized transition matrix introduced in Definition 3.3.11.

3.5.1 Remark (Non-existence of the $J(\cdot)$-block in the SCF)**.** Let (3.2) be transferable into SCF and suppose that $n_1 = 0$. Then

$$\forall\, t^0 > \tau :\ U(\,\cdot\,, t^0) \equiv 0,$$

and hence (3.2) is exponentially stable. ◇

3.5.2 Proposition (Attractivity implies stability)**.** *Let* (3.2) *be transferable into SCF. If system* (3.2) *is attractive, then it is stable, and hence asymptotically stable.*

Proof: We use an argument, which is similar to the proof of Satz 7.5.3 in [Aul04]. Let $\varepsilon > 0$ and $t^0 > \tau$. Since (3.2) is attractive we have

$$\begin{gathered}\exists\, \delta = \delta(t^0) > 0\ \forall\, x^0 \in \mathcal{B}_\delta(0) \cap \mathcal{V}(t^0)\ \forall\, x(\cdot) \in \mathcal{S}_0(t^0, x^0) :\\ 0 = \lim_{t\to\infty} x(t) = \lim_{t\to\infty} U(t, t^0)x^0.\end{gathered}$$

If $n_1 = 0$, then exponential stability of (3.2) follows from Remark 3.5.1. Let $n_1 > 0$. Consider the matrix

$$X^0 := [X_1^0, \ldots, X_{n_1}^0] := \frac{\delta}{2\|T(t^0)\|} T(t^0) \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}.$$

From (3.27) we deduce $X_i^0 \in \mathcal{V}(t^0)$, $i \in \{1, ..., n_1\}$, and since

$$\forall\, i \in \{1, ..., n_1\} :\ \|X_i^0\| = \frac{\delta}{2\|T(t^0)\|} \left\| T(t^0) \begin{bmatrix} e_i \\ 0 \end{bmatrix} \right\| \le \frac{\delta}{2} < \delta$$

we get $X_i^0 \in \mathcal{B}_\delta(0) \cap \mathcal{V}(t^0)$, $i \in \{1, ..., n_1\}$. Hence

$$0 = \lim_{t\to\infty} U(t, t^0)X^0 = \frac{\delta}{2\|T(t^0)\|} \lim_{t\to\infty} T(t) \begin{bmatrix} \Phi_J(t, t^0) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}.$$

From this it follows

$$\lim_{t\to\infty} T(t)\begin{bmatrix}\Phi_J(t,t^0) & 0\\ 0 & 0\end{bmatrix} = 0, \quad \text{thus} \quad \lim_{t\to\infty} U(t,t^0) = 0,$$

and hence there exists $\lambda = \lambda(t^0) > 0$, such that

$$\forall\, t \geq t^0 : \ \|U(t,t^0)\| \leq \lambda.$$

Define $\eta = \eta(\varepsilon, t^0) := \frac{\varepsilon}{\lambda}$. Then

$$\begin{aligned}&\forall\, x^0 \in \mathcal{B}_\eta(0) \cap \mathcal{V}(t^0)\ \forall\, x(\cdot) \in \mathcal{S}_0(t^0,x^0)\ \forall\, t \geq t^0 :\\ &\|x(t)\| = \|U(t,t^0)x^0\| \leq \|U(t,t^0)\|\|x^0\| < \lambda\frac{\varepsilon}{\lambda} = \varepsilon.\end{aligned}$$

Therefore (3.2) is stable. □

3.5.3 Remark. Proposition 3.5.2 does not hold true for systems which are not transferable into SCF: Revisiting Example 3.3.15 one observes that (3.29) is attractive, but not stable. ◇

3.5.4 Corollary. *Let* (3.2) *be transferable into SCF.* (3.2) *is asymptotically stable if, and only if, every global solution* $x : (\tau,\infty) \to \mathbb{R}^n$ *of* (3.2) *satisfies*

$$\lim_{t\to\infty} x(t) = 0.$$

Proof: ($\Leftarrow$): By Proposition 3.3.10 every local solution of (3.2) can be uniquely extended to a global solution, thus every right maximal solution is right global. Then attractivity of (3.2) follows immediately, and by Proposition 3.5.2 the asymptotic stability.

($\Rightarrow$): Let $(t^0,x^0) \in \mathcal{V}$ and $x(\cdot)$ be the global solution of (3.2), $x(t^0) = x^0$. Since (3.2) is asymptotically stable it is attractive. Then, as in the proof of Proposition 3.5.2, it follows that

$$\lim_{t\to\infty} U(t,t^0) = 0, \quad \text{thus having} \quad \lim_{t\to\infty} x(t) = \lim_{t\to\infty} U(t,t^0)x^0 = 0.$$ □

3.5.5 Corollary. *Let* (3.2) *be transferable into SCF. Then* (3.2) *is exponentially stable if, and only if,*

$$\exists\, \alpha,\beta > 0\ \forall\, (t^0,x^0) \in \mathcal{V}\ \forall\, t \geq t^0 : \ \|U(t,t^0)x^0\| \leq \alpha e^{-\beta(t-t^0)}\|x^0\|.$$

Proof: If $n_1 = 0$, then there is nothing to prove, due to Remark 3.5.1. Let $n_1 > 0$.
($\Leftarrow$): Since any solution $x : \mathcal{J} \to \mathbb{R}^n$ of (3.2) has the representation

$$\forall\, t, t^0 \in \mathcal{J} : \; x(t) = U(t,t^0)x(t^0)$$

by Proposition 3.3.10, exponential stability of (3.2) follows immediately.
($\Rightarrow$): Let $(t^0, x^0) \in \mathcal{V}$. Since (3.2) is exponentially stable we have

$$\begin{gathered}\exists\, \alpha, \beta > 0 \;\exists\, \delta = \delta(t^0) > 0 \;\forall\, y^0 \in \mathcal{B}_\delta(0) \cap \mathcal{V}(t^0) \;\forall\, t \geq t^0 : \\ \|U(t,t^0)y^0\| \leq \alpha e^{-\beta(t-t^0)} \|y^0\|.\end{gathered}$$

If $x^0 = 0$ then $U(t,t^0)x^0 = 0$ for all $t \geq t^0$, and if $x^0 \neq 0$ it holds true that

$$\begin{aligned} & \forall\, t \geq t^0 : \; \left\| U(t,t^0) \frac{\delta x^0}{2\|x^0\|} \right\| \leq \alpha e^{-\beta(t-t^0)} \left\| \frac{\delta x^0}{2\|x^0\|} \right\| \\ \iff & \forall\, t \geq t^0 : \|U(t,t^0)x^0\| \leq \alpha e^{-\beta(t-t^0)} \|x^0\|. \end{aligned}$$ □

Finalizing this section we derive a relationship between the stability behavior of system (3.2) and the stability behavior of its SCF. We show that the transformation (3.13) preserves exponential stability under some condition on T.

3.5.6 Proposition (Exponential stability is preserved by uniformly bounded transformations). *Let $E_i, A_i \in C((\tau,\infty) \to \mathbb{R}^n)$, $i = 1, 2$, and (E_1, A_1) and (E_2, A_2) be equivalent w.r.t. some $S \in C((\tau,\infty) \to \mathbb{R}^n)$, $T \in C^1((\tau,\infty) \to \mathbb{R}^n)$. Furthermore, let*

$$E_1(t)\dot{x} = A_1(t)x \tag{3.40}$$

be transferable into SCF, $\mathcal{V}_1$ be the set of all pairs of consistent initial values of (3.40) *and $T(\cdot)^{-\top}T(\cdot)^{-1} \in \mathcal{P}_{\mathcal{V}_1}$. Then* (3.40) *is exponentially stable if, and only if,*

$$E_2(t)\dot{x} = A_2(t)x \tag{3.41}$$

is exponentially stable.

Proof: Since (3.40) is transferable into SCF and $\sim$ is an equivalence relation, also (3.41) is transferable into SCF. Let $U_1(\cdot,\cdot)$ or $U_2(\cdot,\cdot)$ denote the generalized transition matrix of (3.40) or (3.41), resp. As a consequence of Remark 3.3.5 and the uniqueness of the generalized transition matrix (see Proposition 3.3.12) we find

$$\forall\, t,s > \tau : \; U_1(t,s) = T(t)U_2(t,s)T(s)^{-1}.$$

Set $\mathcal{V}_1(t) := \{\, x \in \mathbb{R}^n \mid (t,x) \in \mathcal{V}_1 \,\}$ and $\mathcal{V}_2(t) := T(t)^{-1}\mathcal{V}_1(t)$ for $t > \tau$; $\mathcal{V}_2 := \{\, (t,x) \in (\tau,\infty) \times \mathbb{R}^n \mid x \in \mathcal{V}_2(t) \,\}$. Clearly $\mathcal{V}_2$ is the set of all pairs of consistent initial values of system (3.41). $T(\cdot)^{-\top}T(\cdot)^{-1} \in \mathcal{P}_{\mathcal{V}_1}$ means

$$\exists\, \alpha,\beta > 0 : \; \alpha I_n \le_{\mathcal{V}_1} T(\cdot)^{-\top}T(\cdot)^{-1} \le_{\mathcal{V}_1} \beta I_n. \tag{3.42}$$

Let $(t,x) \in \mathcal{V}_2$. Then $(t,T(t)x) \in \mathcal{V}_1$ and hence

$$\frac{1}{\beta}\|x\|^2 = \frac{1}{\beta}\|T(t)^{-1}(T(t)x)\|^2 \overset{(3.42)}{\le} \|T(t)x\|^2 \overset{(3.42)}{\le} \frac{1}{\alpha}\|T(t)^{-1}(T(t)x)\|^2 = \frac{1}{\alpha}\|x\|^2, \tag{3.43}$$

which gives $T(\cdot)^\top T(\cdot) \in \mathcal{P}_{\mathcal{V}_2}$. Now we are in the position to prove the assertion of the proposition.

(3.40) is exponentially stable if, and only if, (cf. Corollary 3.5.5) there exist $\mu,\nu > 0$ such that

$$\forall\, (t^0,x^1) \in \mathcal{V}_1 \;\forall\, t \ge t^0 : \; \|U_1(t,t^0)x^1\| \le \mu e^{-\nu(t-t^0)}\|x^1\|. \tag{3.44}$$

Due to the preparations above we may deduce

$$\begin{aligned}
(3.44) \quad &\iff \quad \forall\, (t^0,x^2) \in \mathcal{V}_2 \;\forall\, t \ge t^0 : \; \|T(t)U_2(t,t^0)x^2\| \le \mu e^{-\nu(t-t^0)}\|T(t^0)x^2\| \\
&\overset{3.3.12\,(ii)}{\underset{(3.43)}{\Longrightarrow}} \quad \forall\, (t^0,x^2) \in \mathcal{V}_2 \;\forall\, t \ge t^0 : \; \frac{1}{\beta}\|U_2(t,t^0)x^2\|^2 \le \frac{\mu^2}{\alpha}e^{-2\nu(t-t^0)}\|x^2\|^2 \\
&\iff \quad \forall\, (t^0,x^2) \in \mathcal{V}_2 \;\forall\, t \ge t^0 : \; \|U_2(t,t^0)x^2\| \le \frac{\mu\sqrt{\beta}}{\sqrt{\alpha}}e^{-\nu(t-t^0)}\|x^2\|,
\end{aligned}$$

and

$$\begin{aligned}
&\forall\, (t^0,x^2) \in \mathcal{V}_2 \;\forall\, t \ge t^0 : \; \frac{1}{\beta}\|U_2(t,t^0)x^2\|^2 \le \frac{\mu^2}{\alpha}e^{-2\nu(t-t^0)}\|x^2\|^2 \\
\overset{3.3.12\,(ii)}{\underset{(3.43)}{\Longrightarrow}} \quad &\forall\, (t^0,x^2) \in \mathcal{V}_2 \;\forall\, t \ge t^0 : \; \|T(t)U_2(t,t^0)x^2\| \le \frac{\mu\beta}{\alpha}e^{-\nu(t-t^0)}\|T(t^0)x^2\|.
\end{aligned}$$

This yields the assertion of the proposition. □

As an immediate consequence of Proposition 3.5.6 we obtain the following result, which yields a suitable relationship between the system (3.2) and the subsystem $\dot{z} = J(t)z$ of its SCF.

3.5.7 Corollary (Exponential stability is inherited from subsystem)**.** *Let* (3.2) *be transferable into SCF and use the notation from Definition 3.3.6. Suppose* $T(\cdot)^{-\top}T(\cdot)^{-1} \in \mathcal{P}_{\mathcal{V}}$*. Then* (3.2) *is exponentially stable if, and only if, either* $n_1 = 0$ *or the system* $\dot{z} = J(t)z$ *is exponentially stable.*

3.6 Exponential stability and projected generalized time-varying Lyapunov-equation

In this section we investigate exponential stability of (3.2) via a Lyapunov-like approach. Thereto we generalize Theorem 2.2.3. In the case, where $E(\cdot)$ and $A(\cdot)$ are constant, the (generalized) Lyapunov-equation is

$$A^\top PE + E^\top PA = -Q, \tag{3.45}$$

and one uses

$$V : \mathcal{V}^* \to \mathbb{R},\ x \mapsto (Ex)^\top P(Ex),$$

where $\mathcal{V}^* = \mathcal{V}(t)$ for all $t \in \mathbb{R}$, as a Lyapunov-function (see e.g. [OD85, Thm. 2.2]). Hence it is obvious to use

$$V : \mathcal{V} \to \mathbb{R},\ (t, x) \mapsto (E(t)x)^\top P(t)(E(t)x)$$

as a Lyapunov-function in the time-varying case. Together with (3.45) and (2.4) this motivates the investigation of the *generalized time-varying Lyapunov-equation*

$$\forall\, t > \tau : A(t)^\top P(t)E(t) + E(t)^\top P(t)A(t) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(t)^\top P(t)E(t)\big) = -Q(t). \tag{3.46}$$

However, performing a generalization of Theorem 2.2.3 (ii), one realizes that it is not possible to ensure that all right maximal solutions in a neighborhood of the trivial solution tend exponentially to zero (we can not assure that they are right global at all). But if we only consider right global solutions of (3.2) it is possible. Hence we introduce the following notations.

$$\begin{aligned}
&\mathcal{G} := \{\ (t,x) \in (\tau,\infty) \times \mathbb{R}^n \ \mid \mathcal{G}_0(t,x) \neq \emptyset\ \}\,,\\
&\forall\, t > \tau :\ \mathcal{G}(t) := \{\ x \in \mathbb{R}^n \ \mid (t,x) \in \mathcal{G}\ \}\,,\\
&\mathcal{E}\mathcal{G} := \{\ (t,x) \in (\tau,\infty) \times \mathbb{R}^n \ \mid x \in E(t)\mathcal{G}(t)\ \}\,.
\end{aligned}$$

3.6.1 Remark (Properties of $\mathcal{G}(t)$)**.** The space $\mathcal{G}(t)$ has the following properties:

(i) $\forall\, t > \tau :\ \mathcal{G}(t)$ is a linear subspace of $\mathbb{R}^n$,

(ii) If $x : (a,\infty) \to \mathbb{R}^n$ is a right global solution of (3.2), then $x(t) \in \mathcal{G}(t)$ for all $t > a$. $\diamond$

Now one experiences that it is sufficient to consider the matrix-valued functions $Q(\cdot)$ and $E(\cdot)^\top P(\cdot)E(\cdot)$ as well as the generalized time-varying Lyapunov-equation (3.46) on the set $\mathcal{G}$, since the graphs of right global solutions are always located in this set due to Remark 3.6.1. Therefore one comes up with the investigation of the *projected generalized time-varying Lyapunov-equation*

$$\boxed{A(\cdot)^\top P(\cdot)E(\cdot) + E(\cdot)^\top P(\cdot)A(\cdot) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(\cdot)^\top P(\cdot)E(\cdot)\big) =_{\mathcal{G}} -Q(\cdot).} \tag{3.47}$$

The next theorem states necessary conditions for a restricted form of exponential stability of the trivial solution of (3.2). We use the notion "restricted", since it is not quite exponential stability. As already mentioned above we can only guarantee that all right global solutions in a neighborhood of the trivial solution tend exponentially to zero.

3.6.2 Theorem (Necessary conditions for restricted exponential stability)**.** *Consider system* (3.2). *If there exist* $Q(\cdot) \in \mathcal{P}_{\mathcal{G}}$ *and* $P \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$*, such that* $E(\cdot)^\top P(\cdot)E(\cdot) \in \mathcal{P}_{\mathcal{G}} \cap\ C^1((\tau,\infty) \to \mathbb{R}^{n\times n})$ *and* (3.47) *holds, then*

$$\exists\, \alpha,\beta > 0\ \forall\, (t^0,x^0) \in (\tau,\infty)\times\mathbb{R}^n\ \forall\, x(\cdot) \in \mathcal{G}_0(t^0,x^0)\ \forall\, t \geq t^0 :\ \|x(t)\| \leq \alpha e^{-\beta(t-t^0)}\|x^0\|.$$

Proof: $Q(\cdot), E(\cdot)^\top P(\cdot)E(\cdot) \in \mathcal{P}_{\mathcal{G}}$ mean

$$\exists\, q_1,q_2 > 0 :\ q_1 I_n \leq_{\mathcal{G}} Q(\cdot) \leq_{\mathcal{G}} q_2 I_n, \tag{3.48}$$

$$\exists\, p_1, p_2 > 0 : \; p_1 I_n \leq_{\mathcal{G}} E(\cdot)^\top P(\cdot) E(\cdot) \leq_{\mathcal{G}} p_2 I_n. \tag{3.49}$$

Define

$$V : \mathcal{G} \to \mathbb{R}, \; (t,x) \mapsto (E(t)x)^\top P(t)(E(t)x).$$

Let $(t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n$ be arbitrary. If $\mathcal{G}_0(t^0, x^0) = \emptyset$ there is nothing to show. Hence let $x^0 \in \mathcal{G}(t^0)$ and $x(\cdot) \in \mathcal{G}_0(t^0, x^0)$. We show that there exists an estimate of the form

$$\exists\, c \in \mathbb{R}\; \forall\, t \geq t^0 : \; \tfrac{\mathrm{d}}{\mathrm{d}t} V(t, x(t)) \leq c V(t, x(t)).$$

Since, by Remark 3.6.1, $(t, x(t)) \in \mathcal{G}$ for all $t \geq t^0$, we may deduce, for all $t \geq t^0$,

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} V(t,x(t)) \quad &= \quad \dot{x}(t)^\top E(t)^\top P(t) E(t) x(t) + x(t)^\top \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(t)^\top P(t) E(t)\big) x(t) \\
&\quad\quad + x(t)^\top E(t)^\top P(t) E(t) \dot{x}(t) \\
&= \quad x(t)^\top \left(A(t)^\top P(t) E(t) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(t)^\top P(t) E(t)\big) + E(t)^\top P(t) A(t)\right) x(t) \\
&\overset{(3.47)}{=} \quad -x(t)^\top Q(t) x(t) \\
&\overset{(3.48)}{\leq} \quad -q_1 x(t)^\top x(t) \\
&\overset{(3.49)}{\leq} \quad -\frac{q_1}{p_2} (E(t)x(t))^\top P(t)(E(t)x(t)) \\
&= \quad -\frac{q_1}{p_2} V(x(t), t).
\end{aligned}$$

Then separation of variables yields (cf. p. 15)

$$\forall\, t \geq t^0 : \; V(x(t), t) \leq e^{-\frac{q_1}{p_2}(t - t^0)} V(x^0, t^0). \tag{3.50}$$

Now we are in a position to derive an estimate for the norm of $x(\cdot)$:

$$\begin{aligned}
\forall\, t \geq t^0 : \; \|x(t)\|^2 \quad &\overset{(3.49)}{\leq} \quad \frac{1}{p_1} (E(t)x(t))^\top P(t)(E(t)x(t)) \\
&\overset{(3.50)}{\leq} \quad \frac{1}{p_1} e^{-\frac{q_1}{p_2}(t-t^0)} V(x^0, t^0).
\end{aligned}$$

Finally we get for $t \geq t^0$:

$$\begin{aligned}
\|x(t)\| \leq \quad &\left(\frac{1}{p_1} e^{-\frac{q_1}{p_2}(t-t^0)} V(x^0, t^0)\right)^{\frac{1}{2}} \\
= \quad &\sqrt{\frac{1}{p_1}} e^{-\frac{q_1}{2p_2}(t-t^0)} \sqrt{(x^0)^\top E(t^0)^\top P(t^0) E(t^0) x^0} \\
\overset{(3.49)}{\leq} \quad &\sqrt{\frac{p_2}{p_1}} e^{-\frac{q_1}{2p_2}(t-t^0)} \|x^0\|
\end{aligned}$$

□

Theorem 3.6.2 shows that not the solution $P(\cdot)$ of (3.47), but $E(\cdot)^\top P(\cdot)E(\cdot)$, is the object of interest. Symmetry, differentiability and the boundary conditions are not claimed for $P(\cdot)$, but for $E(\cdot)^\top P(\cdot)E(\cdot)$. Under an additional condition on $E(\cdot)$ one might get a result, where all requirements are made for $P(\cdot)$.

3.6.3 Lemma (Relationship between $P(\cdot)$ and $E(\cdot)^\top P(\cdot)E(\cdot)$)**.** *Consider system* (3.2) *and let* $E(\cdot)^\top E(\cdot) \in \mathcal{P}_\mathcal{G}$ *and* $P \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$ *be symmetric. Then*

$$P(\cdot) \in \mathcal{P}_{\mathcal{E}\mathcal{G}} \quad \Longleftrightarrow \quad E(\cdot)^\top P(\cdot)E(\cdot) \in \mathcal{P}_\mathcal{G}.$$

Proof: $E(\cdot)^\top E(\cdot) \in \mathcal{P}_\mathcal{G}$ means

$$\exists\, \alpha, \beta > 0 : \ \alpha I_n \le_\mathcal{G} E(\cdot)^\top E(\cdot) \le_\mathcal{G} \beta I_n. \tag{3.51}$$

($\Rightarrow$): By $P(\cdot) \in \mathcal{P}_{\mathcal{E}\mathcal{G}}$ we find

$$\exists\, p_1, p_2 > 0 : \ p_1 I_n \le_{\mathcal{E}\mathcal{G}} P(\cdot) \le_{\mathcal{E}\mathcal{G}} p_2 I_n. \tag{3.52}$$

Let $(t,x) \in \mathcal{G}$. Then $E(t)x \in E(t)\mathcal{G}(t)$ and

$$\begin{aligned} p_1\alpha x^\top x &\overset{(3.51)}{\le} p_1 x^\top E(t)^\top E(t)x \overset{(3.52)}{\le} x^\top E(t)^\top P(t)E(t)x \\ &\overset{(3.52)}{\le} p_2 x^\top E(t)^\top E(t)x \overset{(3.51)}{\le} p_2\beta x^\top x. \end{aligned}$$

($\Leftarrow$): Since $E(\cdot)^\top P(\cdot)E(\cdot) \in \mathcal{P}_\mathcal{G}$ we get

$$\exists\, r_1, r_2 > 0 : \ r_1 I_n \le_\mathcal{G} E(\cdot)^\top P(\cdot)E(\cdot) \le_\mathcal{G} r_2 I_n. \tag{3.53}$$

Let $(t,x) \in \mathcal{E}\mathcal{G}$. Then $x \in E(t)\mathcal{G}(t)$ and therefore there exists $y \in \mathcal{G}(t)$ such that $x = E(t)y$ and

$$\begin{aligned} \frac{r_1}{\beta} x^\top x &= \frac{r_1}{\beta}(E(t)y)^\top (E(t)y) \overset{(3.51)}{\le} r_1 y^\top y \overset{(3.53)}{\le} \underbrace{y^\top E(t)^\top P(t)E(t)y}_{=x^\top P(t)x} \\ &\overset{(3.53)}{\le} r_2 y^\top y \overset{(3.51)}{\le} \frac{r_2}{\alpha}(E(t)y)^\top (E(t)y) = \frac{r_2}{\alpha} x^\top x. \end{aligned}$$

$\square$

3.6.4 Corollary (Alternative of Theorem 3.6.2)**.** *Consider system* (3.2)*. Let* $E(\cdot)$ *be continuously differentiable and satisfy* $E(\cdot)^\top E(\cdot) \in \mathcal{P}_\mathcal{G}$*. If there exist* $Q(\cdot) \in \mathcal{P}_\mathcal{G}$ *and a continuously differentiable* $P(\cdot) \in \mathcal{P}_{\mathcal{E}\mathcal{G}}$*, such that* (3.47) *holds, then*

$$\exists\, \alpha, \beta > 0\ \forall\, (t^0, x^0) \in (\tau,\infty)\times\mathbb{R}^n\ \forall\, x(\cdot) \in \mathcal{G}_0(t^0,x^0)\ \forall\, t \ge t^0 : \ \|x(t)\| \le \alpha e^{-\beta(t-t^0)}\|x^0\|.$$

Proof: From Lemma 3.6.3 it follows $E(\cdot)^\top P(\cdot)E(\cdot) \in \mathcal{P}_\mathcal{G} \cap C^1((\tau,\infty) \to \mathbb{R}^{n\times n})$. Then Theorem 3.6.2 yields the assertion. $\square$

3.6.5 Remark. Consider the case, where $E(\cdot)$ and $A(\cdot)$ are constant. Then Theorem 3.6.2 and Corollary 3.6.4 improve the well-known results concerning generalized Lyapunov equations and exponential stability of (3.2) (see e.g. [Sty02, OD85] for these results), since regularity of the matrix pencil $\lambda E - A$ is not required, and also not implied by any of the assumptions. $\diamond$

In the remainder of this section we derive a version of Theorem 3.6.2 (and Corollary 3.6.4) for systems (3.2) which are transferable into SCF. Furthermore, we state a converse of this result. First we introduce the notation

$$\mathcal{E}\mathcal{V} := \{\ (t,x) \in (\tau,\infty) \times \mathbb{R}^n \ \mid x \in E(t)\mathcal{V}(t)\ \}.$$

Then note that, due to Theorem 3.3.10, we find

$$\mathcal{V} = \mathcal{G},\ \mathcal{E}\mathcal{V} = \mathcal{E}\mathcal{G},\ \forall\, t > \tau:\ \mathcal{V}(t) = \mathcal{G}(t),$$

for systems (3.2) transferable into SCF.

We derive a result concerning the set $\mathcal{V}(t)$ of initial values consistent with t of (3.2) and $\ker E(t)$.

3.6.6 Proposition (The kernel of $E(t)$ does not lie in $\mathcal{V}(t)$)**.** *Let* (3.2) *be transferable into SCF. Then*

$$\forall\, t > \tau:\ \mathcal{V}(t) \cap \ker E(t) = \{0\},$$

and, in particular, for all $(t^0, x^0) \in \mathcal{V}$ *and for all* $t > \tau$,

$$E(t)U(t,t^0)x^0 = 0 \quad \Longleftrightarrow \quad U(t,t^0)x^0 = 0.$$

Proof: Let $x^0 \in \mathcal{V}(t^0) \cap \ker E(t^0)$ and use the notation from Definition 3.3.6. Then

$$\begin{aligned}
0 &= T(t^0)S(t^0)^{-1}E(t^0)x^0 \\
&\overset{3.3.12(\mathrm{v})}{=} T(t^0)S(t^0)^{-1}E(t^0)U(t^0,t^0)x^0 \\
&\overset{(3.16)}{=} T(t^0)\begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t^0) \end{bmatrix} T(t^0)^{-1}T(t^0)\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1}x^0 \\
&= T(t^0)^{-1}\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T(t^0)^{-1}x^0 \\
&= U(t^0,t^0)x^0 \overset{3.3.12(\mathrm{v})}{=} x^0.
\end{aligned}$$

The second assertion then follows by Proposition 3.3.12 (ii). □

One may wonder whether $E(\cdot)^\top E(\cdot) \in \mathcal{P}_\mathcal{V}$ holds true for every system (3.2) which is transferable into SCF. Clearly

$$\exists\, \beta > 0 : \; E(\cdot)^\top E(\cdot) \le_\mathcal{V} \beta I_n$$

does not necessarily hold. But since, due to Proposition 3.6.6, $\ker E(t) \cap \mathcal{V}(t) = \{0\}$ for all $t > \tau$ it could be expected, that

$$\exists\, \alpha > 0 : \; \alpha I_n \le_\mathcal{V} E(\cdot)^\top E(\cdot) \tag{3.54}$$

holds true. The following example shows that this is not true.

3.6.7 Example. Consider

$$E(t) = \begin{bmatrix} \frac{1}{t^2} & 0 \\ 0 & 0 \end{bmatrix}, \quad A(t) = \begin{bmatrix} \frac{1}{t^2} + \frac{1}{t^3} & 0 \\ 0 & 1 \end{bmatrix}, \quad t > \tau := 0.$$

Defining $S(t) := T(t) := \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$, $t > 0$, we obtain that

$$(E, A) \quad \text{and} \quad \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

are equivalent w.r.t. S, T. Now let $t^0 > \tau$ and $x^0 \in \mathcal{V}(t^0) = \operatorname{im} \begin{bmatrix} t^0 \\ 0 \end{bmatrix}$. This means $x^0 = \begin{bmatrix} \alpha t^0 \\ 0 \end{bmatrix}$ for some $\alpha \in \mathbb{R}$. Then

$$\|E(t^0)x^0\| = \left\| \begin{bmatrix} \frac{\alpha}{t^0} \\ 0 \end{bmatrix} \right\| = \frac{|\alpha|}{t^0} \xrightarrow[t^0 \to \infty]{} 0,$$

and hence (3.54) can not hold true. ◇

The projected generalized time-varying Lyapunov-equation has the following shape for systems (3.2) which are transferable into SCF:

$$A(\cdot)^\top P(\cdot)E(\cdot) + E(\cdot)^\top P(\cdot)A(\cdot) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(\cdot)^\top P(\cdot)E(\cdot)\big) =_{\mathcal{V}} -Q(\cdot). \tag{3.55}$$

3.6.8 Remark (Projected generalized time-varying Lyapunov-equation)**.** One may wonder why we call (3.47) and (3.55) "projected". As a matter of fact there is no motivation to call (3.47) "projected", but since it is the more general version of (3.55) we kept the notation. For the motivation to call (3.55) "projected" consider (3.46) and any $V : (\tau, \infty) \to \mathbb{R}^{n\times n}$ with $\operatorname{im} V(t) = \mathcal{V}(t)$ for all $t > \tau$. We could choose $V(t) = U(t,t), t > \tau$, for instance. Then, multiplying (3.46) with $V(t)^\top$ from the left and with $V(t)$ from the right, we obtain

$$\begin{aligned}(V(t)x)^\top \Big(A(t)^\top P(t)E(t) + E(t)^\top P(t)A(t) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(t)^\top P(t)E(t)\big)\Big)(V(t)x)\\ = -(V(t)x)^\top Q(t)(V(t)x)\end{aligned}$$

for all $t > \tau$ and $x \in \mathbb{R}^n$. Since $(t, V(t)x) \in \mathcal{V}$ for all $t > \tau$ and $x \in \mathbb{R}^n$ it follows that (3.55) holds. Hence we obtain (3.55) through a projection via $V(\cdot)$ from (3.46).
This shows also that (3.55) is a more general equation, since if (3.46) holds then (3.55) holds, too. The converse is not true in general, as the following example shows.

3.6.9 Example. Let

$$E(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A(t) = \begin{bmatrix} -t & 0 \\ 0 & 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Then (E, A) is already in SCF and we may choose $S(\cdot) = T(\cdot) = I$. Hence $n_1 = n_2 = 1$ and $\mathcal{V} = \mathbb{R} \times \operatorname{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Let $p : \mathbb{R} \to \mathbb{R},\ t \mapsto e^{t^2} \int_t^\infty e^{-s^2}\, ds$, and

$$P(t) = \begin{bmatrix} p(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad Q(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Then $P(\cdot)$ and $Q(\cdot)$ solve (3.55), but not (3.46). $\diamond$

Now we are in the position to state the main theorem of this section.

3.6.10 Theorem (Necessary and sufficient conditions for exponential stability of systems in SCF)**.** *Let* (3.2) *be transferable into SCF and use the notation from Definition 3.3.6.*

(i) If there exist $Q(\cdot) \in \mathcal{P}_\mathcal{V}$ and $P \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$, such that $E(\cdot)^\top P(\cdot) E(\cdot) \in \mathcal{P}_\mathcal{V} \cap C^1((\tau,\infty) \to \mathbb{R}^{n\times n})$ and (3.55) *holds, then* (3.2) *is exponentially stable.*

(ii) Let $E(\cdot)$ be continuously differentiable and satisfy $E(\cdot)^\top E(\cdot) \in \mathcal{P}_\mathcal{V}$. If there exists $Q(\cdot) \in \mathcal{P}_\mathcal{V}$ and a continuously differentiable $P(\cdot) \in \mathcal{P}_{\mathcal{E}\mathcal{V}}$, such that (3.55) *holds, then* (3.2) *is exponentially stable.*

(iii) Let $E(\cdot)$ and $N(\cdot)$ be continuously differentiable. Furthermore, let $E(\cdot)^\top E(\cdot), Q(\cdot) \in \mathcal{P}_\mathcal{V}$, and $E(\cdot)$ and $\big(\dot{E}(\cdot) + A(\cdot)\big)$ be bounded. If (3.2) *is exponentially stable, then there exists a solution $P : (\tau, \infty) \to \mathbb{R}^{n\times n}$ to* (3.55) *with $E(\cdot)^\top P(\cdot) E(\cdot) \in \mathcal{P}_\mathcal{V} \cap C^1((\tau,\infty) \to \mathbb{R}^{n\times n})$.*

(iv) Let $E(\cdot)$ and $S(\cdot)$ be continuously differentiable. Furthermore, let $E(\cdot)^\top E(\cdot), Q(\cdot) \in \mathcal{P}_\mathcal{V}$, and $E(\cdot)$ and $\big(\dot{E}(\cdot)+A(\cdot)\big)$ be bounded. If (3.2) *is exponentially stable, then there exists a continuously differentiable solution $P(\cdot) \in \mathcal{P}_{\mathcal{E}\mathcal{V}}$ to* (3.55)*.*

Proof: *(i)*: Follows from Theorem 3.6.2, $\mathcal{V} = \mathcal{G}$ and Corollary 3.5.5.
(ii): Follows from Corollary 3.6.4, $\mathcal{V} = \mathcal{G}$ and Corollary 3.5.5.
(iii): $Q(\cdot), E(\cdot)^\top E(\cdot) \in \mathcal{P}_\mathcal{V}$ mean

$$\exists\, \alpha_1, \beta_1 > 0 : \ \alpha_1 I_n \leq_\mathcal{V} Q(\cdot) \leq_\mathcal{V} \beta_1 I_n, \tag{3.56}$$

$$\exists\, \alpha_2, \beta_2 > 0 : \; \alpha_2 I_n \leq_{\mathcal{V}} E(\cdot)^\top E(\cdot) \leq_{\mathcal{V}} \beta_2 I_n. \tag{3.57}$$

Define

$$P : (\tau, \infty) \to \mathbb{R}^{n\times n}, \; t \mapsto S(t)^\top T(t)^\top \int_t^\infty U(s,t)^\top Q(s) U(s,t)\, ds\; T(t)S(t).$$

Step 1: We show that $P(t)$ exists for all $t > \tau$. It follows from the exponential stability of (3.2) and Corollary 3.5.5, that

$$\exists\, \mu, \nu > 0 \; \forall\, (t^0, x^0) \in \mathcal{V} \; \forall\, t \geq t^0 : \; \|U(t,t^0)x^0\| \leq \mu e^{-\nu(t-t^0)} \|x^0\|. \tag{3.58}$$

Let $(t^0, x^0) \in (\tau, \infty) \times \mathbb{R}^n$ be arbitrary and $t^1 > t^0$. Set $\begin{bmatrix} v \\ w \end{bmatrix} := S(t^0)x^0$, $v \in \mathbb{R}^{n_1}, w \in \mathbb{R}^{n_2}$. Then $T(t^0)S(t^0)x^0 = T(t^0)\begin{bmatrix} v \\ w \end{bmatrix} = T(t^0)\begin{bmatrix} v \\ 0 \end{bmatrix} + T(t^0)\begin{bmatrix} 0 \\ w \end{bmatrix}$ and

$$\forall\, s > \tau : \; U(s,t^0)T(t^0)\begin{bmatrix} 0 \\ w \end{bmatrix} = T(t^0)\begin{bmatrix} \Phi_J(s,t^0) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ w \end{bmatrix} = 0. \tag{3.59}$$

Define $y^0 := T(t^0)\begin{bmatrix} v \\ 0 \end{bmatrix} \in \mathcal{V}(t^0)$. Then $U(s,t^0)y^0 \in \mathcal{V}(s)$ by Proposition 3.3.12 (ii) and

$$\begin{aligned}
& \int_{t^0}^{t^1} (x^0)^\top S(t^0)^\top T(t^0)^\top U(s,t^0)^\top Q(s) U(s,t^0) T(t^0) S(t^0) x^0 \, ds \\
\overset{(3.59)}{=} \; & \int_{t^0}^{t^1} (U(s,t^0)y^0)^\top Q(s) (U(s,t^0)y^0)\, ds \\
\overset{(3.56)}{\leq} \; & \int_{t^0}^{t^1} \beta_1 (U(s,t^0)y^0)^\top (U(s,t^0)y^0)\, ds \\
\overset{(3.58)}{\leq} \; & \beta_1 \int_{t^0}^{t^1} \mu^2 e^{-2\nu(s-t^0)} \|y^0\|^2 \, ds \\
= \; & -\frac{\beta_1\mu^2}{2\nu} \|y^0\|^2 e^{-2\nu(s-t^0)} \Big|_{t^0}^{t^1} \\
= \; & \frac{\beta_1\mu^2}{2\nu} \|y^0\|^2 \left(1 - e^{-2\nu(t^1-t^0)}\right).
\end{aligned} \tag{3.60}$$

Since $t^1 > t^0$ and $x^0 \in \mathbb{R}^n$ were arbitrary, $P(t^0)$ exists.

Step 2: We show that $E(\cdot)^\top P(\cdot)E(\cdot) \leq_{\mathcal{V}} cI_n$ for some $c > 0$. Let $(t,x) \in \mathcal{V}$. Then $x = T(t)\begin{bmatrix} v \\ 0 \end{bmatrix}$ for some $v \in \mathbb{R}^{n_1}$. Hence

$$\begin{aligned}
& x^\top E(t)^\top P(t)E(t)x \\
\overset{(3.16)}{=} \quad & [v^\top, 0]T(t)^\top T(t)^{-\top} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t)^\top \end{bmatrix} S(t)^{-\top} P(t) S(t)^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t) \end{bmatrix} T(t)^{-1} T(t) \begin{bmatrix} v \\ 0 \end{bmatrix} \\
= \quad & [v^\top, 0]T(t)^\top \int_t^\infty U(s,t)^\top Q(s) U(s,t)\, ds\; T(t) \begin{bmatrix} v \\ 0 \end{bmatrix} \\
= \quad & \int_t^\infty (U(s,t)x)^\top Q(s)(U(s,t)x)\, ds,
\end{aligned} \tag{3.61}$$

and therefore, as in (3.60),

$$x^\top E(t)^\top P(t)E(t)x \leq \frac{\beta_1 \mu^2}{2\nu}\|x\|^2,$$

thus having, since $(t,x) \in \mathcal{V}$ was arbitrary,

$$E(\cdot)^\top P(\cdot)E(\cdot) \leq_{\mathcal{V}} \frac{\beta_1 \mu^2}{2\nu} I_n.$$

Step 3: We may write, for all $t > \tau$,

$$\begin{aligned}
& E(t)^\top P(t) E(t) \\
\overset{(3.16)}{=} \; & T(t)^{-\top} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t)^\top \end{bmatrix} T(t)^\top \textstyle\int_t^\infty U(s,t)^\top Q(s) U(s,t)\, ds\; T(t) \begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t) \end{bmatrix} T(t)^{-1},
\end{aligned} \tag{3.62}$$

and since $Q(\cdot)$ and $U(\cdot,\cdot)$ are continuous, and $T(\cdot)$ and $N(\cdot)$ are continuously differentiable, $E(\cdot)^\top P(\cdot)E(\cdot)$ is continuously differentiable.

Furthermore, clearly $P(\cdot)$ is symmetric, due to the symmetry of $Q(\cdot)$, and therefore $E(\cdot)^\top P(\cdot)E(\cdot)$ is symmetric.

Step 4: We show that $cI_n \leq_{\mathcal{V}} E(\cdot)^\top P(\cdot)E(\cdot)$ for some $c > 0$. Boundedness of $E(\cdot)$ and $\big(\dot{E}(\cdot) + A(\cdot)\big)$ means

$$\exists\, e, a > 0\; \forall\, t > \tau: \; \|E(t)\| \leq e \;\wedge\; \Big\|\big(\dot{E}(\cdot) + A(\cdot)\big)\Big\| \leq a.$$

For arbitrary $(t,x) \in \mathcal{V}$ and $x(\cdot) := U(\,\cdot\,,t)x$, we find

$$\forall\, s > \tau:\ \tfrac{\mathrm{d}}{\mathrm{d}s}\left(E(s)x(s)\right) = \dot{E}(s)x(s) + E(s)\dot{x}(s) = \big(\dot{E}(s) + A(s)\big)x(s), \tag{3.63}$$

and

$$0 \le \|E(s)x(s)\| \le e\|U(s,t)x\| \xrightarrow[s\to\infty]{(3.58)} 0, \tag{3.64}$$

thus having

$$\begin{aligned}
x^\top E(t)^\top P(t)E(t)x \quad &\overset{(3.61)}{=} \quad \int_t^\infty x(s)^\top Q(s)x(s)\,ds\\
&\ge \quad \int_t^\infty \alpha_1 x(s)^\top x(s)\,ds\\
&\ge \quad \alpha_1\int_t^\infty \frac{\|E(s)\|\ \left\|\big(\dot{E}(s)+A(s)\big)\right\|}{ae} x(s)^\top x(s)\,ds\\
&\ge \quad \frac{\alpha_1}{ae}\int_t^\infty \left|(E(s)x(s))^\top(\dot{E}(s)+A(s))x(s)\right|\,ds\\
&\overset{(3.63)}{\ge} \quad \frac{\alpha_1}{ae}\left|\int_t^\infty (E(s)x(s))^\top\left(\tfrac{\mathrm{d}}{\mathrm{d}s}(E(s)x(s))\right)\,ds\right|\\
&= \quad \frac{\alpha_1}{ae}\left|\int_t^\infty \frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}s}\left((E(s)x(s))^\top(E(s)x(s))\right)\,ds\right|\\
&= \quad \left|\frac{\alpha_1}{2ae}\ \|E(s)x(s)\|^2\Big|_t^\infty\right|\\
&\overset{(3.64)}{=} \quad \frac{\alpha_1}{2ae}\|E(t)U(t,t)x\|^2\\
&\overset{3.3.12\,(v)}{=} \quad \frac{\alpha_1}{2ae}\|E(t)x\|^2\\
&\overset{(3.57)}{\ge} \quad \frac{\alpha_1\alpha_2}{2ae}\|x\|^2.
\end{aligned}$$

This shows

$$\frac{\alpha_1\alpha_2}{2ae}I_n \le_{\mathcal{V}} E(\cdot)^\top P(\cdot)E(\cdot),$$

and hence $E(\cdot)^\top P(\cdot)E(\cdot) \in \mathcal{P}_{\mathcal{V}}$.

Step 5: It remains to prove that (3.55) holds. Let $(t,x) \in \mathcal{V}$. Set

$$\begin{aligned}
\hat{P}(t) := \int_t^\infty U(s,t)^\top Q(s)\Bigg(&-T(s)\begin{bmatrix}\Phi_J(s,t)J(t) & 0\\ 0 & 0\end{bmatrix}T(t)^{-1}\\
&+T(s)\begin{bmatrix}\Phi_J(s,t) & 0\\ 0 & 0\end{bmatrix}\tfrac{\mathrm{d}}{\mathrm{d}t}\big(T(t)^{-1}\big)\Bigg)\ ds.
\end{aligned}$$

First we show that

$$x^\top \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(t)^\top P(t)E(t)\big)x = x^\top \hat{P}(t)x + x^\top \hat{P}(t)^\top x - x^\top Q(t)x. \tag{3.65}$$

Define

$$\hat{U}(t) := \int_t^\infty U(s,t)^\top Q(s)U(s,t)\,ds, \quad \hat{T}(t) := T(t)\begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t)\end{bmatrix} T(t)^{-1}.$$

Since $x = T(t)\begin{bmatrix} v \\ 0\end{bmatrix}$ for some $v \in \mathbb{R}^{n_1}$ we get

$$\begin{aligned}
& x^\top \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(t)^\top P(t)E(t)\big)x \\
\overset{(3.62)}{=} \; & x^\top \left(\tfrac{\mathrm{d}}{\mathrm{d}t}(\hat{T}(t))^\top \hat{U}(t)\hat{T}(t) + \hat{T}(t)^\top \tfrac{\mathrm{d}}{\mathrm{d}t}(\hat{U}(t))\hat{T}(t) + \hat{T}(t)^\top \hat{U}(t)\tfrac{\mathrm{d}}{\mathrm{d}t}(\hat{T}(t))\right) x.
\end{aligned}$$

Now we derive

$$\hat{T}(t)x = T(t)\begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t)\end{bmatrix} T(t)^{-1}T(t)\begin{bmatrix} v \\ 0\end{bmatrix} = T(t)\begin{bmatrix} v \\ 0\end{bmatrix} = x,$$

and

$$\begin{aligned}
& \tfrac{\mathrm{d}}{\mathrm{d}t}(\hat{T}(t))x \\
= \; & \left(\dot{T}(t)\begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t)\end{bmatrix} T(t)^{-1} + T(t)\begin{bmatrix} 0 & 0 \\ 0 & \dot{N}(t)\end{bmatrix} T(t)^{-1}\right. \\
& \left. + T(t)\begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t)\end{bmatrix} \tfrac{\mathrm{d}}{\mathrm{d}t}\big(T(t)^{-1}\big)\right) x \\
\overset{(3.14)}{=} \; & \dot{T}(t)\begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t)\end{bmatrix} T(t)^{-1}T(t)\begin{bmatrix} v \\ 0\end{bmatrix} + T(t)\begin{bmatrix} 0 & 0 \\ 0 & \dot{N}(t)\end{bmatrix} T(t)^{-1}T(t)\begin{bmatrix} v \\ 0\end{bmatrix} \\
& - T(t)\begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t)\end{bmatrix} T(t)^{-1}\dot{T}(t)T(t)^{-1}T(t)\begin{bmatrix} v \\ 0\end{bmatrix} \\
= \; & \left(I - T(t)\begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t)\end{bmatrix} T(t)^{-1}\right)\dot{T}(t)\begin{bmatrix} v \\ 0\end{bmatrix} \\
= \; & T(t)\begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} - N(t)\end{bmatrix} T(t)^{-1}\dot{T}(t)\begin{bmatrix} v \\ 0\end{bmatrix}.
\end{aligned}$$

Defining $\begin{bmatrix} 0 \\ w \end{bmatrix} := \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} - N(t) \end{bmatrix} T(t)^{-1}\dot{T}(t) \begin{bmatrix} v \\ 0 \end{bmatrix}$, $w \in \mathbb{R}^{n_2}$, we obtain

$$\begin{aligned} & x^\top \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(t)^\top P(t) E(t)\big) x \\ = \; & \left(T(t) \begin{bmatrix} 0 \\ w \end{bmatrix} \right)^\top \hat{U}(t) x + x^\top \tfrac{\mathrm{d}}{\mathrm{d}t}(\hat{U}(t)) x + x^\top \hat{U}(t) \left(T(t) \begin{bmatrix} 0 \\ w \end{bmatrix} \right) \\ \overset{(3.59)}{=} \; & x^\top \tfrac{\mathrm{d}}{\mathrm{d}t}(\hat{U}(t)) x \\ = \; & x^\top \left(\int_t^\infty \big(\tfrac{\mathrm{d}}{\mathrm{d}t} U(s,t)\big)^\top Q(s) U(s,t) + U(s,t)^\top Q(s) \big(\tfrac{\mathrm{d}}{\mathrm{d}t} U(s,t)\big)\, ds \right) x \\ & -(U(t,t)x)^\top Q(t) (U(t,t)x). \end{aligned}$$

Observe that, for all $s, t > \tau$,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} U(s,t) \;&=\; T(s) \begin{bmatrix} \tfrac{\mathrm{d}}{\mathrm{d}t}\Phi_J(s,t) & 0 \\ 0 & 0 \end{bmatrix} T(t)^{-1} + T(s) \begin{bmatrix} \Phi_J(s,t) & 0 \\ 0 & 0 \end{bmatrix} \tfrac{\mathrm{d}}{\mathrm{d}t}\big(T(t)^{-1}\big) \\ &\overset{2.1.1\,(vi)}{=} T(s) \begin{bmatrix} -\Phi_J(s,t) J(t) & 0 \\ 0 & 0 \end{bmatrix} T(t)^{-1} + T(s) \begin{bmatrix} \Phi_J(s,t) & 0 \\ 0 & 0 \end{bmatrix} \tfrac{\mathrm{d}}{\mathrm{d}t}\big(T(t)^{-1}\big), \end{aligned}$$

and therefore

$$\begin{aligned} & x^\top \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(t)^\top P(t) E(t)\big) x \\ = \; & x^\top \hat{P}(t)^\top x + x^\top \hat{P}(t) x - (U(t,t)x)^\top Q(t)(U(t,t)x) \\ \overset{3.3.12\,(v)}{=} \; & x^\top \hat{P}(t)^\top x + x^\top \hat{P}(t) x - x^\top Q(t) x. \end{aligned}$$

Now we prove that

$$x^\top E(t)^\top P(t) A(t) x = -x^\top \hat{P}(t) x. \tag{3.66}$$

Recall that by (3.16) and (3.15)

$$A(t) = S(t)^{-1} \begin{bmatrix} J(t) & 0 \\ 0 & I_{n_2} \end{bmatrix} T(t)^{-1} - S(t)^{-1} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t) \end{bmatrix} \tfrac{\mathrm{d}}{\mathrm{d}t}\big(T(t)^{-1}\big).$$

Then, similar to (3.61), we find

$$\begin{aligned}
& x^\top E(t)^\top P(t) A(t) x \\
= \; & \int_t^\infty (U(s,t)x)^\top Q(s) U(s,t)\, ds\; T(t) S(t) A(t) x \\
= \; & \int_t^\infty (U(s,t)x)^\top Q(s) T(s) \begin{bmatrix} \Phi_J(s,t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} J(t) & 0 \\ 0 & I_{n_2} \end{bmatrix} T(t)^{-1} x\, ds \\
& - \int_t^\infty (U(s,t)x)^\top Q(s) T(s) \begin{bmatrix} \Phi_J(s,t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t) \end{bmatrix} \tfrac{\mathrm{d}}{\mathrm{d}t}\big(T(t)^{-1}\big) x\, ds \\
= \; & -x^\top \hat{P}(t) x.
\end{aligned}$$

Due to symmetry it follows in the same way that

$$x^\top A(t)^\top P(t) E(t) x = -x^\top \hat{P}(t)^\top x,$$

and together with (3.65) and (3.66) this shows that (3.55) holds.

(iv): Let $P(\cdot)$ be the same matrix-valued function as in (iii). Since now $S(\cdot)$ is continuously differentiable by assumption it follows that $P(\cdot)$ is continuously differentiable. Symmetry of $P(\cdot)$ is obvious. As shown in (iii) it holds $E(\cdot)^\top P(\cdot) E(\cdot) \in \mathcal{P}_\mathcal{V}$ and therefore Lemma 3.6.3 yields $P(\cdot) \in \mathcal{P}_{\mathcal{E}\mathcal{V}}$. That (3.55) is satisfied has also been proved in (iii). $\square$

Analyzing the proof of Theorem 3.6.10 we obtain the following corollary.

3.6.11 Corollary. *Let* (3.2) *be transferable into SCF and use the notation from Definition 3.3.6. Furthermore, let* $Q \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$ *such that* $Q \leq_\mathcal{V} \alpha I_n$ *for some* $\alpha > 0$. *Suppose* (3.2) *is exponentially stable and* $E(\cdot)$ *and* $N(\cdot)$ *are continuously differentiable. Then the following statements hold:*

(i) There exists a solution $P : (\tau,\infty) \to \mathbb{R}^{n\times n}$ *to* (3.55)*, such that* $E(\cdot)^\top P(\cdot) E(\cdot)$ *is continuously differentiable and* $E(\cdot)^\top P(\cdot) E(\cdot) \leq_\mathcal{V} \beta I_n$ *for some* $\beta > 0$.

(ii) If $Q(\cdot)$ *is symmetric, then* $P(\cdot)$ *is symmetric.*

(iii) If $S(\cdot)$ *is continuously differentiable, then* $P(\cdot)$ *is continuously differentiable.*

(iv) If $E(\cdot)$ and $\big(\dot E(\cdot) + A(\cdot)\big)$ are bounded and there exist $\gamma, \delta > 0$ such that $E(\cdot)^\top E(\cdot) \geq_{\mathcal{V}} \gamma I_n$ and $Q(\cdot) \geq_{\mathcal{V}} \delta I_n$, then $E(\cdot)^\top P(\cdot) E(\cdot) \geq_{\mathcal{V}} \lambda I_n$ for some $\lambda > 0$.

3.6.12 Remark (Time-invariant case)**.** Consider the time-invariant case, i.e. $E(\cdot)$ and $A(\cdot)$ are constant. Then transferability of (3.2) into SCF means that $\lambda E - A$ is regular. Pick any $t^0 > \tau$ and set $\mathcal{V}^* := \mathcal{V}(t^0)$. Hence $\mathcal{V}(t) = \mathcal{V}^*$ for all $t > \tau$ and $\mathcal{V} = (\tau, \infty) \times \mathbb{R}^n$. $E^\top E \in \mathcal{P}_{\mathcal{V}}$ is equivalent to

$$\exists\, \alpha, \beta > 0\ \forall\, x \in \mathcal{V}^* :\ \alpha \|x\| \leq \|Ex\| \leq \beta \|x\|. \tag{3.67}$$

Choose $\beta := \|E\|$ and $\alpha := \min\{\ \|Ex\| \ |\ x \in \mathcal{V}^*, \|x\| = 1\ \}$. If $E \neq 0$ (otherwise (3.67) would be trivially satisfied by any α, β) and $\mathcal{V}^* \neq \{0\}$ (otherwise (3.67) would clearly hold true), then $\beta > 0$ and $\alpha > 0$ since $\ker E \cap \mathcal{V}^* = \{0\}$ by Proposition 3.6.6. Hence the assumption $E^\top E \in \mathcal{P}_{\mathcal{V}}$ is always fulfilled in the time-invariant case. Therefore it follows immediately from Lemma 3.6.3 that

$$P \in \mathcal{P}_{\mathcal{E}\mathcal{V}} \quad \Longleftrightarrow \quad E^\top P E \in \mathcal{P}_{\mathcal{V}}.$$

Hence in the time-invariant case there is no need to consider $E^\top P E$ and Theorem 3.6.10 (i) and (ii) are equivalent in this case (note that they are not equivalent in general). Nevertheless Theorem 3.6.10 (ii) (considered time-invariant) is still an improvement of the ubiquitous result [Sty02, Thm. 4.6], since Stykel does not consider the restriction of the generalized Lyapunov equation to the set $\mathcal{V}^*$.
Furthermore, Theorem 3.6.10 (iii) and (iv) are also equivalent in this case and we find that Corollary 3.6.11 is a generalization of [Sty02, Thm. 4.15 & Rem. 4.16], except for the uniqueness property, since, using the notation from [Sty02], the matrix P_r is just a projector onto $\mathcal{V}^*$, and hence G positive definite means $P_r^\top G P_r \in \mathcal{P}_{\mathcal{V}}$. The uniqueness condition for the solution of the generalized Lyapunov equation given in [Sty02, Thm. 4.15] can be generalized to a uniqueness condition in the time-varying case (see Corollary 3.6.16). ◇

Finalizing this section we prove that the solution $P(\cdot)$ of the projected generalized time-varying Lyapunov-equation (3.55) is unique on $\mathcal{E}\mathcal{V}$. Note that symmetry of $P(\cdot)$ or $Q(\cdot)$, resp., is not required, and likewise asymptotic stability of (3.2) is sufficient.

However, to ensure existence of a solution, exponential stability is necessary (see Corollary 3.6.16).

3.6.13 Proposition (Unique solvability of the Lyapunov-equation). *Let* (3.2) *be transferable into SCF and* $Q \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$. *Suppose* (3.2) *is asymptotically stable,* $P_1, P_2 : (\tau,\infty) \to \mathbb{R}^{n\times n}$ *are solutions to* (3.55)*, such that* $E(\cdot)^\top P_i(\cdot)E(\cdot)$ *is continuously differentiable,* $i = 1, 2$*, and*

$$\forall\, i \in \{1,2\}\ \exists\, \alpha, \beta > 0 :\ \alpha_i I_n \leq_{\mathcal{V}} E(\cdot)^\top P_i(\cdot)E(\cdot) \leq_{\mathcal{V}} \beta_i I_n. \tag{3.68}$$

Then $E(\cdot)^\top P_1(\cdot)E(\cdot) =_{\mathcal{V}} E(\cdot)^\top P_2(\cdot)E(\cdot)$ *or, equivalently,* $P_1(\cdot) =_{\mathcal{E}\mathcal{V}} P_2(\cdot)$.

Proof: Let $s > \tau$ and define

$$M(t) := U(t,s)^\top E(t)^\top (P_1(t) - P_2(t))E(t)U(t,s), \quad t \geq s.$$

Then

$$\begin{aligned}
\dot{M}(t) \quad &= \quad (E(t)\tfrac{\mathrm{d}}{\mathrm{d}t}U(t,s))^\top (P_1(t) - P_2(t))E(t)U(t,s) + \\
&\qquad U(t,s)^\top \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(t)^\top (P_1(t) - P_2(t))E(t)\big)U(t,s) + \\
&\qquad U(t,s)^\top E(t)^\top (P_1(t) - P_2(t))E(t)\tfrac{\mathrm{d}}{\mathrm{d}t}U(t,s) \\
&\overset{3.3.12\,(i)}{=} (A(t)U(t,s))^\top (P_1(t) - P_2(t))E(t)U(t,s) + \\
&\qquad U(t,s)^\top \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(t)^\top (P_1(t) - P_2(t))E(t)\big)U(t,s) + \\
&\qquad U(t,s)^\top E(t)^\top (P_1(t) - P_2(t))A(t)U(t,s).
\end{aligned}$$

Since, by Proposition 3.3.12 (ii), $U(t,s)x \in \mathcal{V}(t)$ holds for all $x \in \mathbb{R}^n$ and $P_1(\cdot), P_2(\cdot)$ both solve (3.55) it follows that

$$\forall\, t \geq s :\ \dot{M}(t) = 0.$$

Hence $M(\cdot)$ must be constant, which gives

$$\forall\, t \geq s :\ M(t) = M(s).$$

Repeatedly invoking Proposition 3.3.12 (ii) we deduce

$$\begin{aligned}
&\alpha_1 U(t,s)^\top U(t,s) - \beta_2 U(t,s)^\top U(t,s)\\
\overset{(3.68)}{\leq}\;& \underbrace{U(t,s)^\top E(t)^\top P_1(t)E(t)U(t,s) - U(t,s)^\top E(t)^\top P_2(t)E(t)U(t,s)}_{=M(t)}\\
\overset{(3.68)}{\leq}\;& \beta_1 U(t,s)^\top U(t,s) - \alpha_2 U(t,s)^\top U(t,s)
\end{aligned}$$

for all $t \geq s$. Since (3.2) is asymptotically stable we find, as in the proof of Proposition 3.5.2,

$$\lim_{t\to\infty} U(t,s) = 0, \quad \text{thus having} \quad \lim_{t\to\infty} M(t) = 0.$$

Hence we get $M(s) = 0$, i.e. $(E(s)U(s,s)x)^\top (P_1(s) - P_2(s))(E(s)U(s,s)x) = 0$ for all $x \in \mathbb{R}^n$, or, equivalently,

$$\forall\, x \in \mathcal{V}(s):\; x^\top E(s)^\top (P_1(s) - P_2(s))E(s)x = 0. \qquad \square$$

3.6.14 Example. We show that the solution to (3.55) is not unique on all of $(\tau,\infty)\times\mathbb{R}^n$ in general. Let

$$E(t) = \begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}, \quad A(t) = \begin{bmatrix} -1 & 0\\ 0 & 1\end{bmatrix}, \quad t \in \mathbb{R}.$$

Then (E,A) is already in SCF and we may choose $S(\cdot) = T(\cdot) = I$. Hence $n_1 = n_2 = 1$ and $\mathcal{V} = \mathbb{R} \times \operatorname{im} \begin{bmatrix} 1\\ 0\end{bmatrix} = \mathcal{EV}$. Let $Q(\cdot) \equiv I$. Then

$$P : \mathbb{R} \to \mathbb{R}^2,\; t \mapsto \begin{bmatrix} \frac{1}{2} & 0\\ 0 & p(t)\end{bmatrix}$$

solves (3.55) and fulfills (3.53) for any $p \in C(\mathbb{R}\to\mathbb{R})$ with $\exists\, p_1, p_2 > 0\ \forall\, t \in \mathbb{R} : p_1 \leq p(t) \leq p_2$. $\diamond$

3.6.15 Remark (Uniqueness condition)**.** By Proposition 3.6.13 the uniformly bounded solution of the Lyapunov-equation (3.55) is unique on $\mathcal{EV}$. To get a unique solution on all of $(\tau,\infty)\times\mathbb{R}^n$ we are somehow free to choose the behavior of $P(\cdot)$ on $(\tau,\infty)\times\mathbb{R}^n \setminus \mathcal{EV}$. For instance choose $V : (\tau,\infty) \to \mathbb{R}^{n\times n}$ such that $\operatorname{im} V(t) = \mathcal{V}(t)$

for all $t > \tau$, and let $Q(\cdot), P_1(\cdot), P_2(\cdot)$ be as in Proposition 3.6.13 and (3.2) be asymptotically stable. If the condition

$$\forall\, i \in \{1,2\}\ \forall\, t > \tau: \ P_i(t) = (E(t)V(t))^\top P_i(t)(E(t)V(t)) \tag{3.69}$$

is satisfied, then $P_1(t) = P_2(t)$ for all $t > \tau$.

Since Proposition 3.6.13 yields $P_1(\cdot) =_{\mathcal{E}\mathcal{V}} P_2(\cdot)$, i.e. $(E(t)V(t))^\top(P_1(t) - P_2(t))(E(t)V(t)) = 0$ for any $t > \tau$, the assertion immediately follows from (3.69). $\diamond$

The next corollary is a consequence of Proposition 3.6.13 and Remark 3.6.15 and states a representation of the unique solution of (3.55) under certain conditions. Note that symmetry of $P(\cdot)$ or $Q(\cdot)$ is not required.

3.6.16 Corollary. *Let* (3.2) *be transferable into SCF and use the notation from Definition 3.3.6. Furthermore, let $Q \in C((\tau,\infty) \to \mathbb{R}^{n\times n})$. Suppose $E(\cdot)$ and $N(\cdot)$ are continuously differentiable, $E(\cdot)^\top E(\cdot) \in \mathcal{P}_\mathcal{V}$, $Q(\cdot)$ satisfies $\gamma I_n \le_\mathcal{V} Q(\cdot) \le \delta I_n$ for some $\gamma, \delta > 0$, and $E(\cdot)$ and $\big(\dot E(\cdot) + A(\cdot)\big)$ are bounded. If* (3.2) *is exponentially stable, then*

$$P : (\tau,\infty) \to \mathbb{R}^{n\times n},\ t \mapsto S(t)^\top T(t)^\top \int_t^\infty U(s,t)^\top Q(s) U(s,t)\, ds\ T(t)S(t)$$

is the unique solution of

$$\left.\begin{aligned}
&A(\cdot)^\top P(\cdot)E(\cdot) + E(\cdot)^\top P(\cdot)A(\cdot) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big(E(\cdot)^\top P(\cdot)E(\cdot)\big) =_\mathcal{V} -Q(\cdot),\\
&\forall\, t > \tau: \ \left(S(t)^{-1}\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix} S(t)\right)^\top P(t) \left(S(t)^{-1}\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix} S(t)\right) = P(t),\\
&\exists\, \alpha,\beta > 0: \ \alpha I_n \le_\mathcal{V} E(\cdot)^\top P(\cdot)E(\cdot) \le_\mathcal{V} \beta I_n.
\end{aligned}\right\} \tag{3.70}$$

Proof: It follows from the assumptions, that $P(t)$ exists for all $t > \tau$, $E(\cdot)^\top P(\cdot)E(\cdot)$ is continuously differentiable and solves (3.55) and $\alpha I_n \le_\mathcal{V} E(\cdot)^\top P(\cdot)E(\cdot) \le_\mathcal{V} \beta I_n$ for some $\alpha, \beta > 0$, as it has been shown in the proof of Theorem 3.6.10 (iii). Furthermore, since

$$\begin{aligned}
& U(s,t)T(t)S(t)\left(S(t)^{-1}\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} S(t)\right) \\
= \; & T(s)\begin{bmatrix} \Phi_J(s,t) & 0 \\ 0 & 0 \end{bmatrix} T(t)^{-1}T(t)\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} S(t) \\
= \; & T(s)\begin{bmatrix} \Phi_J(s,t) & 0 \\ 0 & 0 \end{bmatrix} S(t) \\
= \; & U(s,t)T(t)S(t)
\end{aligned}$$

for all $s, t > \tau$ we find

$$\forall\, t > \tau : \quad \left(S(t)^{-1}\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} S(t)\right)^{\top} P(t) \left(S(t)^{-1}\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} S(t)\right) = P(t).$$

Then

$$\begin{aligned}
\forall\, t > \tau : \; E(t)\mathcal{V}(t) &= E(t)\operatorname{im} U(t,t) = \operatorname{im} E(t)U(t,t) \\
&= \operatorname{im}\left(S(t)^{-1}\begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t) \end{bmatrix} T(t)^{-1}T(t)\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T(t)^{-1}\right) \\
&= \operatorname{im}\left(S(t)^{-1}\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T(t)^{-1}\right) = \operatorname{im} S(t)^{-1}\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} S(t),
\end{aligned}$$

and Proposition 3.6.13 together with Remark 3.6.15 yield that $P(\cdot)$ is the unique solution of (3.70). □

3.6.17 Remark. Revisiting the main results of this section (Theorem 3.6.2, Corollary 3.6.4, Theorem 3.6.10, Corollary 3.6.11, Proposition 3.6.13) one observes that all (crucial) assumptions in theses results are made on the system magnitudes $E(\cdot)$ and $A(\cdot)$. It is also possible to obtain similar results by doing all calculations after transforming the system in SCF and reducing the problems to the case of ordinary differential equations. Calculations are more simple then, but the assumptions one has to state are on the transformation matrices $S(\cdot)$ and $T(\cdot)$ and also stronger than the assumptions in the above mentioned results. ◇

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