Some Remarks on Positive/Negative Feedback

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Abstract

In the context of unstable systems with control, a commonly-held precept is that negative and positive feedback cannot both be stabilizing. The canonical linear prototype is the scalar system $\dot{x} = u$ which, under negative linear feedback $u = -kx$ ($k > 0$) is exponentially stable for all $k > 0$, whereas the inherent lack of exponential instability of the uncontrolled system is amplified by positive feedback $u = kx$ ($k > 0$). By contrast, for nonlinear systems it is shown that this intuitively-appealing dichotomy may fail to hold.

Keywords: Nonlinear control systems; positive and negative feedback.

1. Nonlinear scalar system

Consider a scalar system, with state x, control u, and (unknown) bounded perturbation p, of the form

$$\dot{x}(t) = f(p(t), x(t)) + g(u(t)), \quad x(0) = x^0 \in \mathbb{R}, \tag{1.1}$$

where $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g\colon \mathbb{R} \to \mathbb{R}$, and $p\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ are continuous. The control objective is to determine a continuous feedback function $h\colon \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$, with an associated parameter $\eta \in \{-1, +1\}$, such that, for every $x^0 \in \mathbb{R}$ and bounded $p \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$, application of the feedback $u(t) = -\eta h(t, x(t))$ to (1.1) yields a feedback-controlled initial-value problem

$$\dot{x}(t) = f(p(t), x(t)) + g(-\eta h(t, x(t)), \quad x(0) = x^0$$

has a solution, every solution has a global extension to a solution on $\mathbb{R}_{\geq 0}$, every global solution is such that $x(t) \to 0$ as $t \to \infty$, and the control input $u\colon t \mapsto -\eta h(t, x(t))$ is bounded.

If $\eta = +1$ (respectively, $\eta = -1$), then the feedback $u(t) = -\eta h(t, x(t))$ is deemed to be *negative* (respectively, *positive*). If negative feedback ensures benign behaviour in the form of convergence of the state to zero, then intuition might suggest that positive feedback causes malign behaviour in the form of non-convergence of the state to zero.

This intuition finds voice in, for example, [1]: *Feedback can upregulate or downregulate a process. We refer to upregulation as positive feedback and downregulation as negative feedback. Positive feedback … acts as a destabilizing mechanism for the process. Negative feedback … acts as a stabilizing mechanism for the process.*

We show, by a simple example, that this positive/negative feedback dichotomy can be invalid. In particular, we construct an example (with arbitrary f and p) which, under either negative or positive feedback, ensures convergence of the state to zero whilst maintaining boundedness of the control input.

2. Feedback function

Let $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuously differentiable bijection with the property that, for some $c_\varphi > 0$, $\dot{\varphi}(t) \leq c_\varphi\big(1 + \varphi(t)\big)$ for all $t \geq 0$ (the simplest example being the identity function with $c_\varphi = 1$). Define the set

$$\mathcal{F} := \{(t, \xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid \ \varphi(t)|\xi| < 1\}, \tag{2.1}$$

the continuous function

$$\begin{aligned} &h\colon \mathcal{F} \to \mathbb{R}, \quad (t, \xi) \mapsto \alpha(\varphi(t)|\xi|)\varphi(t)\xi, \\ &\text{with } \alpha\colon [0, 1) \to [1, \infty), \ s \mapsto \frac{1}{1 - s}, \end{aligned} \tag{2.2}$$

and the feedback-controlled initial-value problem, on the domain $\mathcal{F}$,

$$\dot{x}(t) = f(p(t), x(t)) + g(-\eta h(t, x(t)), \quad x(0) = x^0 \tag{2.3}$$

with $p \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R})$. By a solution we mean a continuous function x on a left-closed interval I (with left endpoint 0) and graph$(x) \subset \mathcal{F}$, satisfying (2.3). A solution is *maximal* if it has no proper right extension that is also a solution.

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By the standard theory of ordinary differential equations, for each $x^0 \in \mathbb{R}$, there exists a solution, and every solution has a maximal extension; moreover, the closure of graph of a maximal solution is not a compact subset of $\mathcal{F}$. Loosely speaking, the role of the feedback is to maintain the system's evolution away from the boundary of the domain $\mathcal{F}$ (wherein singularity resides), in which case the evolution continues indefinitely to the right, with transient and asymptotic behaviour determined by the choice of φ.

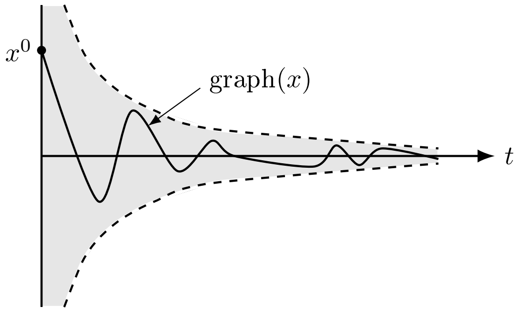

Figure 1: Domain $\mathcal{F}$.

3. Efficacy of both negative and positive feedback

Let g be given by

$$g\colon \mathbb{R} \to \mathbb{R}, \quad v \mapsto v \sin\big(\ln(1+|v|)\big). \tag{3.1}$$

The following technicality will play a central role.

Proposition 3.1. *Let $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous, g as in (3.1), and $K, P \subset \mathbb{R}$ compact. Define $V := [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$. The function*

$$\chi\colon \mathbb{R} \to \mathbb{R},$$
$$\quad s \mapsto \min\{vf(\rho,\xi) + vg(sv) \mid \rho \in P,\ \xi \in K,\ v \in V\}$$

is continuous and is such that, for each $\eta \in \{-1,+1\}$, $\sup_{s\geq 0} \chi(\eta s) = \infty$.

Proof. For notational convenience, introduce the function

$$F\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, \quad (z,s) = \big((\rho,\xi,v),s\big) \mapsto vf(\rho,\xi) + vg(sv)$$

and write $Z := P \times K \times V$, a compact set. Observe that

$$\chi\colon s \mapsto \min_{z\in Z} F(z,s)$$

and so, for each $s \in \mathbb{R}$, there exists $z_s \in Z$ such that $\chi(s) = F(z_s, s)$. Fix $s \in \mathbb{R}$ arbitrarily. We will show that χ is continuous at s. Let $\varepsilon > 0$. By continuity of F, for each $z \in Z$, there exists $\delta(z) > 0$ such that, for all $\zeta, \sigma \in \mathbb{R}$ we have

$$\|\zeta - z\| + |\sigma - s| < \delta(z) \implies |F(\zeta,\sigma) - F(z,s)| < \varepsilon$$

and so $\cup_{z\in Z}\{(\zeta,\sigma) \mid \|\zeta - z\| + |\sigma - s| < \delta(z)\}$ is an open cover of $Z \times \{s\}$. By compactness of the latter, there exists a finite sub-cover. In particular, there exist finitely many points $z_1, \ldots, z_n \in Z$ such that

$$Z \times \{s\} \subset \cup_{i=1}^n \{(\zeta,\sigma) \mid \|\zeta - z_i\| + |\sigma - s| < \delta(z_i)\}.$$

Define $\delta^* := \min\{\delta(z_1), \ldots, \delta(z_n)\}$ and observe that

$$\forall\, \zeta,\sigma \in \mathbb{R}\colon\ \ z \in Z,\ |\sigma - s| < \delta^* \implies |F(z,\sigma) - F(z,s)| < \varepsilon.$$

It follows that, for all $\sigma \in \mathbb{R}$ with $|\sigma - s| < \delta^*$ we have

$$\varepsilon > |F(z_s,\sigma) - F(z_s,s)| \geq F(z_s,\sigma) - F(z_s,s) \geq \chi(\sigma) - \chi(s)$$

and, similarly,

$$\varepsilon > |F(z_\sigma,\sigma) - F(z_\sigma,s)| \geq F(z_\sigma,s) - F(z_\sigma,\sigma) \geq \chi(s) - \chi(\sigma).$$

Therefore, $|\chi(\sigma) - \chi(s)| < \varepsilon$ for all $\sigma \in \mathbb{R}$ with $|\sigma - s| < \delta^*$, and so χ is continuous at s.

It remains to show that, for each $\eta \in \{-1,+1\}$, $\sup_{s\geq 0}\chi(\eta s) = +\infty$. Define the increasing and unbounded sequence (s_n) by

$$s_n := \tfrac{1}{2} e^{(n+1)\pi} - 1 > 0 \quad \forall\, n \in \mathbb{N},$$

whence

$$\ln(1+s_n) = (n+1)\pi - \ln 2 \quad \forall\, n \in \mathbb{N}.$$

Also, invoking the inequality $e^{\pi/2} > 4$,

$$1 + \tfrac{1}{2}s_n = \tfrac{1}{2} + \tfrac{1}{4}e^{(n+1)\pi} > e^{n\pi}(\tfrac{1}{4}e^{\pi}) > e^{(n+\frac{1}{2})\pi}$$

and so

$$\ln(1+\tfrac{1}{2}s_n) > n\pi + \tfrac{\pi}{2} \quad \forall\, n \in \mathbb{N}.$$

We may now infer

$$n\pi + \tfrac{\pi}{2} < \ln(1 + s_n|v|) \leq (n+1)\pi - \ln 2 \quad \forall\, v \in V\ \ \forall\, n \in \mathbb{N}.$$

First consider the case of n even and write $n = 2k$. Since the function sin is decreasing on the interval $\left[\frac{\pi}{2}, \pi - \ln 2\right]$, for all $v \in V$ and all $k \in \mathbb{N}$ we have

$$\begin{aligned} 1 = \sin(2k\pi + \tfrac{\pi}{2}) &> \sin(\ln(1 + s_{2k}|v|)) \\ &\geq \sin((2k+1)\pi - \ln 2) = \sin(\pi - \ln 2) = \sin(\ln 2) > 0, \end{aligned}$$

whence

$$vg(s_{2k}v) = s_{2k}v^2 \sin(\ln(1 + s_{2k}|v|)) \geq \tfrac{1}{4}s_{2k}\sin(\ln 2).$$

Therefore, writing

$$c_1 := \min\{vf(\rho,\xi) \mid \rho \in P,\ \xi \in K,\ v \in V\},$$

we have

$$\chi(s_{2k}) \geq c_1 + \min\{vg(s_{2k}v) \mid v \in V\} \geq c_1 + \tfrac{1}{4}s_{2k}\sin(\ln 2)$$

for all $k \in \mathbb{N}$ and so $\sup_{s\geq 0}\chi(s) = +\infty$.

Now consider the case of n odd and write $n = 2k - 1$.

Since the function sin is increasing on the interval $\big[-\frac{\pi}{2}, -\ln 2\big]$, we have

$$-1 = \sin\big((2k-1)\pi + \tfrac{\pi}{2}\big) < \sin(\ln(1+s_{2k-1}|v|)) \\ \leq \sin(2k\pi - \ln 2) = -\sin(\ln 2) \quad \forall\, v \in V, \quad \forall\, k \in \mathbb{N},$$

whence

$$vg(-s_{2k-1}v) = -s_{2k-1}v^2 \sin(\ln(1+s_{2k-1}|v|)) \\ \geq \tfrac{1}{4}s_{2k-1}\sin(\ln 2) \qquad \forall\, v \in V, \quad \forall\, k \in \mathbb{N}.$$

Therefore,

$$\chi(-s_{2k-1}) > c_1 + \tfrac{1}{4}s_{2k-1}\sin(\ln 2) \quad \forall\, k \in \mathbb{N}$$

and so $\sup_{s\geq 0}\chi(-s) = +\infty$. □

We arrive at the main focus of the note.

Proposition 3.2. *Let $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$ be arbitrary, g as in (3.1), and h as in (2.2). For each $\eta \in \{-1,+1\}$, every $x^0 \in \mathbb{R}$, and every bounded $p \in \mathcal{C}(\mathbb{R}_{\geq 0},\mathbb{R})$, (2.3) has a solution and every solution has a global extension. Every global solution $x\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is such that $x(t) \to 0$ as $t \to \infty$, and the control function $u\colon t \mapsto -\eta h(t,x(t))$ is bounded.*

Proof. Let $\eta \in \{-1,+1\}$, $x^0 \in \mathbb{R}$ and bounded $p \in \mathcal{C}(\mathbb{R}_{\geq 0},\mathbb{R})$ be arbitrary. Let compact $P \subset \mathbb{R}$ be such that $p(t) \in P$ for all $t \geq 0$. As already noted, by the standard theory of ordinary differential equations, we know that (2.3) has a solution and every solution can be maximally extended. Let $x\colon [0,\omega) \to \mathbb{R}$ be a maximal solution. It suffices to show existence of $\varepsilon > 0$ such that $\varphi(t)|x(t)| < 1-\varepsilon$ for all $t \in [0,\omega)$, in which case it immediately follows that $\omega = \infty$ (as, otherwise, graph(x) has compact closure in $\mathcal{F}$ which is impossible), $x(t) \to 0$ as $t \to \infty$ (since φ is unbounded), and $|u(t)| = |h(t,x(t))| \leq (1-\varepsilon)\alpha(1-\varepsilon) < \infty$ for all $t \geq 0$. The task of establishing the existence of such $\varepsilon > 0$ is, in turn, equivalent to proving boundedness of the function $k\colon t \mapsto \alpha(\varphi(t)|x(t)|)$. This we proceed to show.

Choose $\tau \in (0,\omega)$ arbitrarily, fix $\mu > 1/\varphi(\tau)$ such that $\mu \geq \max_{t\in[0,\tau]}|x(t)|$, and write $\varphi_\mu\colon t \mapsto \max\{\mu^{-1},\varphi(t)\}$. Since graph$(x) \in \mathcal{F}$ and φ (bijective) is strictly increasing, we may infer that

$$|x(t)| \leq \frac{1}{\varphi_\mu(t)} \quad \forall\, t \in [0,\omega)$$

and so, *a fortiori*, $x(t) \in K := [-\mu,\mu]$ for all $t \in [0,\omega)$. As before, define

$$\chi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}, \\ s \mapsto \min\{vf(\rho,\xi) + vg(sv) \mid \rho \in P,\ \xi \in K,\ v \in V\}, \\ V := [-1,-\tfrac{1}{2}] \cup [\tfrac{1}{2},1].$$

Let ν be the function

$$\nu\colon \quad \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad s \mapsto \nu(s) := \max\{\chi(\eta s), 0\}.$$

By Proposition 3.1, χ is continuous with $\sup_{s\geq 0}\chi(\eta s) = +\infty$ and so ν is continuous and surjective. Fix $\kappa_0 > \alpha(1/2)$ arbitrarily and define an increasing sequence (κ_n) in (κ_0,∞) as follows: choose $\kappa_1 > \kappa_0$ such that $\nu(\kappa_1) = \kappa_0 + 1$ and set

$$\kappa_n := \inf\{\kappa > \kappa_{n-1} \mid \nu(\kappa) = \kappa_0 + n\} \quad \forall\, n \geq 2.$$

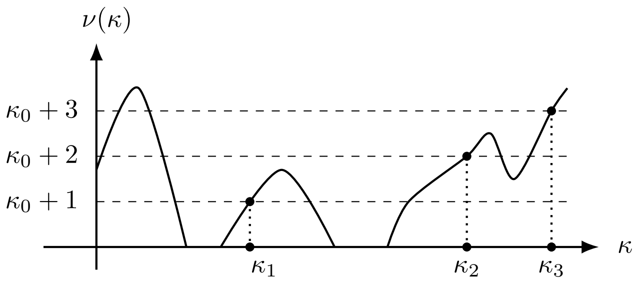

Figure 2: Schematic construction of the sequence (κ_n).

Now, we address the question of boundedness of $k(\cdot)$. Seeking a contradiction, suppose that $k(\cdot)$ is unbounded. This supposition, together with the observation that $k(0) = 1 < \kappa_0$, ensures that the following is a well-defined sequence (τ_n) in $(0,\omega)$

$$\tau_n := \inf\{t \in (0,\omega) \mid k(t) = \kappa_n\}, \quad n \in \mathbb{N}.$$

with the properties

$$\forall\, n \in \mathbb{N}: \ \tau_n < \tau_{n+1}, \quad k(\tau_n) = \kappa_n, \quad \nu(k(\tau_n)) = \kappa_0 + n.$$

Now define the sequence (σ_n) in $(0,\omega)$ by

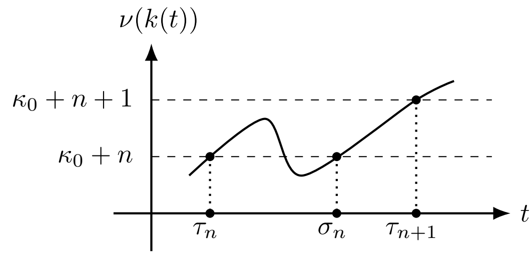

Figure 3: Schematic construction of the sequence (σ_n).

$$\sigma_n := \sup\{t \in [\tau_n,\tau_{n+1}] \mid \nu(k(t)) = \kappa_0 + n\}, \quad n \in \mathbb{N}.$$

Observe that $\sigma_n < \tau_{n+1}$, $k(\sigma_n) < k(\tau_{n+1})$ and

$$\kappa_0 + n < \nu(k(t)) = \chi(\eta k(t)) \leq \nu(k(\tau_{n+1})) = \kappa_0 + n + 1$$

for all $t \in (\sigma_n,\tau_{n+1}]$ and all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be arbitrary and suppose that, for some $t \in [\sigma_n,\tau_{n+1}]$, $\varphi(t)x(t) \notin V$.

Then $\varphi(t)|x(t)| < 1/2$ and

$$\alpha(\varphi(t)|x(t)|) = k(t) < \alpha(1/2) < \kappa_0 < \kappa_n < \kappa_{n+1} = k(\tau_{n+1})$$

Therefore, there exists $t^* \in (\sigma_n, \tau_{n+1})$ such that $k(t^*) = \kappa_n$ whence the contradiction

$$\kappa_0 + n < \nu(k(t^*)) = \nu(\kappa_n) = \kappa_0 + n.$$

Thus, we have shown that

$$\varphi(t)x(t) \in V \quad \forall\, t \in [\sigma_n, \tau_{n+1}] \quad \forall\, n \in \mathbb{N}.$$

By symmetry of V, we know that $\varphi(t)x(t) \in V$ if, and only if, $-\varphi(t)x(t) \in V$, and so we may infer that

$$\begin{aligned}
&-\varphi(t)x(t)f(p(t),x(t)) - \varphi(t)x(t)g(-\eta k(t)\varphi(t)x(t)) \\
&\geq \min\{vf(\rho,\xi) + vg(\eta k(t)v) \mid \rho \in P,\ \xi \in K,\ v \in V\} \\
&= \chi(\eta k(t)) \quad \forall\, t \in [\sigma_n, \tau_{n+1}] \quad \forall\, n \in \mathbb{N}
\end{aligned}$$

Also (noting that, by bijectivity, φ is strictly increasing), we have

$$\begin{aligned}
\varphi(t)\dot\varphi(t)x(t)^2 &\leq \frac{\dot\varphi(t)}{\varphi(t)} \leq c_\varphi\big((1/\varphi(t)) + 1\big) \\
&\leq c_\varphi\big((1/\varphi(\tau_1)) + 1\big) =: \gamma \qquad \forall\, t \in [\tau_1, \omega).
\end{aligned}$$

We may now deduce that

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\varphi(t)x(t))^2 &= \varphi(t)\dot\varphi(t)x(t)^2 \\
&\quad + \varphi(t)^2 x(t)\big(f(p(t),x(t)) + g(-\eta h(t,x(t))\big) \\
&\leq \gamma - \varphi(t)\big(-\varphi(t)x(t)f(p(t),x(t)) \\
&\quad - \varphi(t)x(t)g(-\eta k(t)\varphi(t)x(t))\big) \\
&\leq \gamma - \varphi(t)\chi(\eta k(t)) \\
&\leq \gamma - \varphi(\tau_1)\nu(k(t)) \leq \gamma - (\kappa_0 + n)\varphi(\tau_1)
\end{aligned}$$

for all $t \in [\sigma_n, \tau_{n+1}]$ and all $n \in \mathbb{N}$. Choose $n \in \mathbb{N}$ sufficiently large so that $\gamma - (\kappa_0 + n)\varphi(\tau_1) < 0$, in which case

$$\big(\varphi(\tau_{n+1})x(\tau_{n+1})\big)^2 < \big(\varphi(\sigma_n)x(\sigma_n)\big)^2$$

and we arrive at a contradiction

$$\begin{aligned}
k(\sigma_n) < k(\tau_{n+1}) &= \alpha(\varphi(\tau_{n+1})|x(\tau_{n+1})|) \\
&< \alpha(\varphi(\sigma_n)|x(\sigma_n)|) = k(\sigma_n).
\end{aligned}$$

Therefore, the supposition of unboundedness of $k(\cdot)$ is false. □

4. Conclusion

The claim in this note is purely existential in nature in the sense that a commonly-perceived dichotomy of negative and positive feedback is shown not to be valid in general. This has been done by proving the existence of a nonlinear counter-example: the practical utility and quantitative behaviour of the feedback employed therein has not been addressed. Whilst the counter-example is open to a criticism of artificiality, it does, however, serve to illustrate that the dichotomy "negative/positive feedback" equates to "good/bad behaviour" may be inaccurate. The counter-example is based on a wider body of work in the area of "funnel control" (a terminology consistent with Figure 1). The interested reader is referred to [2] for a review of this area.

References

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