Partial impulse observability of linear descriptor systems

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Abstract

A research paper in this journal vol. 61, no. 3, pp. 427–434, 2012, by M. Darouach, provides a functional observer design for linear descriptor systems under the partial impulse observability condition. The observer design is correct, but there was a flaw in the algebraic criterion characterizing partial impulse observability. In the present paper, we derive a novel characterization of partial impulse observability in terms of a simple rank condition involving the system coefficient matrices and an alternative characterization in terms of the Wong sequences.

Keywords: Linear descriptor systems, Differential-algebraic equations, Partial impulse observability, Wong sequences

1. Introduction

We consider linear time-invariant multivariable descriptor systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t), && \text{(1a)} \\ y(t) &= Cx(t), && \text{(1b)} \\ z(t) &= Lx(t), && \text{(1c)} \end{aligned}$$

where $E, A \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{p\times n}$, and $L \in \mathbb{R}^{r\times n}$ are known constant matrices. We call $x(t) \in \mathbb{R}^n$ the (unknown) semistate vector, $y(t) \in \mathbb{R}^p$ the measured output vector, and $z(t) \in \mathbb{R}^r$ the unknown output vector. The unknown output z contains those variables which cannot be measured and, therefore, observers are required to estimate them. The first order matrix polynomial $(\lambda E - A)$, in the indeterminate λ, is called matrix pencil for (1). Moreover, the system (1) is called regular, if the matrix pencil $(\lambda E - A)$ is regular, which means that $m = n$ and $\det(\lambda E - A)$ is not the zero polynomial in λ. If $(\lambda E - A)$ is not regular, then it is called singular. Notably, if $m = n$, then E may be singular (i.e., not invertible).

In [1], Darouach derived an algebraic test for partial impulse observability of (1) with respect to L and used this concept in designing a functional observer to estimate z. Under the same algebraic assumption, the observer designing approach of [1] has been improved by using a linear matrix inequality (LMI) formulation in [2]. The concept of partial impulse observability was first introduced in [1] as follows (see Definition 1 in [1]): The descriptor system (1), or the triplet (E, A, C), is said to be partially impulse observable with respect to L if $y(t)$ is impulse free for $t \geq 0$, only if $Lx(t)$ is impulse free for $t \geq 0$.

Roughly speaking, partial observability of (1) is related to the reconstruction of $z(t)$ from the knowledge of $y(t)$. However, in the presence of inconsistent initial values, z may exhibit impulses and hence partial impulse observability becomes relevant. For a fundamental analysis of this concept, it is essential to redefine partial impulse observability in a more rigorous way by considering a proper framework for distributional

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solutions of (1). Here, we consider the class of piecewise-smooth distributions $\mathscr{D}'_{pw\mathscr{C}^\infty}$ as introduced in [3]; for a thorough discussion of this class of distributions, we also refer to [4]. Motivated by [5], we denote the set of all distributional solutions of (1) on $[0,\infty)$ by

$$\mathscr{B} := \left\{ (x,y,z) \in (\mathscr{D}'_{pw\mathscr{C}^\infty})^{n+p+r} \,\middle|\, (x,y,z) \text{ satisfies (1) on } [0,\infty) \right\}.$$

$\mathscr{B}$ is called ITP-behavior in [5]. We stress that the equations (1) are only supposed to hold on $[0,\infty)$, and the solution is free on $(-\infty,0)$, which is different from considering solutions of (1) on $\mathbb{R}$ restricted to $[0,\infty)$. Here, it is important to note that the distributional restriction to any interval $M \subseteq \mathbb{R}$ is well defined for $\mathcal{D} \in \mathscr{D}'_{pw\mathscr{C}^\infty}$ [4]. Moreover, any $\mathcal{D} \in \mathscr{D}'_{pw\mathscr{C}^\infty}$ can be uniquely represented as a combination of a distribution induced by a locally integrable piecewise-smooth function f, Dirac delta distributions δ_{t_j} and their distributional derivatives $\delta_{t_j}^{(i)}$ [3]. The part of $\mathcal{D} \in \mathscr{D}'_{pw\mathscr{C}^\infty}$ corresponding to δ_{t_j} and its derivatives is called the *impulsive part* and denoted by $D[t_j]$, see also the definition in [5, Eq. (2)]. Since the class $\mathscr{D}'_{pw\mathscr{C}^\infty}$ allows to perform point evaluation of any element, throughout the article, $x[t]$, $y[t]$, and $z[t]$ stand for the impulsive part of the respective variables at time t.

We now exploit the behavior $\mathscr{B}$ to reformulate the definition of partial impulse observability as follows.

Definition 1. The descriptor system (1) or the triplet (E,A,C) is partially impulse observable with respect to L, if

$$\forall\, (x,y,z) \in \mathscr{B}: \ \big(\forall\, t \geq 0: \ y[t] = 0\big) \implies \big(\forall\, t \geq 0: \ z[t] = 0\big).$$

In [1, Lemma 5], the author claimed that the triplet (E,A,C) is partially impulse observable with respect to L if, and only if,

$$\operatorname{rank}\begin{bmatrix} E & A \\ 0 & E \\ 0 & C \\ 0 & L \end{bmatrix} = \operatorname{rank}\begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix}. \tag{2}$$

Moreover, to prove that (2) implies the partial impulse observability of (1), [1, Lemma 5] assumed without loss of generality that $E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, and $L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}$, and then system (1) is transformed into the following form (see (4) in [1]):

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t), \tag{3a}$$

$$\begin{bmatrix} A_{22} \\ C_2 \end{bmatrix} x_2(t) = -\begin{bmatrix} A_{21} \\ C_1 \end{bmatrix} x_1(t) + \begin{bmatrix} 0 \\ y(t) \end{bmatrix}, \tag{3b}$$

$$z(t) = L_1x_1(t) + L_2x_2(t). \tag{3c}$$

Further, by using the fact that (2) is equivalent to the existence of a matrix Ω such that $L_2 = \Omega \begin{bmatrix} A_{22} \\ C_2 \end{bmatrix}$, the author of [1, Lemma 5] showed that

$$z(t) = \left(L_1 - \Omega \begin{bmatrix} A_{22} \\ C_2 \end{bmatrix} \right) x_1(t) + \Omega \begin{bmatrix} 0 \\ I \end{bmatrix} y(t)$$

and, based on that, claimed that $z(t)$ is impulse free when $y(t)$ is. This claim is actually not correct in general, because $z(t)$ may have impulses due to the impulses in $x_1(t)$ that are not visible in $y(t)$ on $[0,\infty)$. As a specific example, consider system (1) with matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \text{ and } L = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \tag{4}$$

then (3) reduces to

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} x_3(t), \tag{5a}$$

$$\begin{bmatrix}0\\ 0\end{bmatrix} x_3(t) = -\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ y(t)\end{bmatrix}, \tag{5b}$$

$$z(t) = x_2(t). \tag{5c}$$

Moreover, for any nonzero $\alpha \in \mathbb{R}$, if we take $x_1(t) = -\alpha$ for $t < 0$ and $x_1(t) = 0$ for $t \geq 0$, $x_2 = \alpha\delta$, $x_3 = \alpha\dot{\delta}$, $y = x_1$, and $z = x_2$, then it is clear from (5) that $(x, y, z) \in \mathscr{B}$. Thus, $y[t] = 0$ for all $t \geq 0$, but $z[0] = \alpha\delta$, which is impulsive for nonzero α. Therefore, the system is not partially impulse observable with respect to L. Here, it is straightforward that the condition (2) is satisfied by this example, showing that [1, Lemma 5] is incorrect.

The present paper is organized as follows. Section 2 collects some preliminary results used in the remainder of the article. Section 3 contains the main contribution of the paper, where we provide a modified algebraic test to check the partial impulse observability of (1). Section 4 contains a few examples to illustrate the proposed theory. Finally, Section 5 concludes the paper.

We use the following notations: 0 and I stand for zero and identity matrices of appropriate dimension, respectively. Sometimes, for more clarity, the identity matrix of size $n \times n$ is denoted by I_n. In a block partitioned matrix, all missing blocks are zero matrices of appropriate dimensions. The symbols $\operatorname{im} A$ and $\ker A$ denote the image and kernel, respectively, of any matrix $A \in \mathbb{R}^{m\times n}$. The set $AM := \{\, Ax \mid x \in M \,\}$ is the image of a subspace $M \subseteq \mathbb{R}^n$ under $A \in \mathbb{R}^{m\times n}$ and $A^{-1}M := \{\, x \in \mathbb{R}^n \mid Ax \in M \,\}$ represents the pre-image of $M \subseteq \mathbb{R}^m$ under $A \in \mathbb{R}^{m\times n}$.

2. Preliminaries

First we collect some standard results for the characterization of solutions to the following homogeneous system:

$$\mathscr{E}\dot{x} = \mathscr{A}x, \tag{6}$$

where $\mathscr{E}$, $\mathscr{A} \in \mathbb{R}^{m\times n}$, and if $m = n$, $\mathscr{E}$ may be singular. By a solution of (6) we mean a piecewise-smooth distribution $x \in (\mathscr{D}'_{pw\mathscr{C}^\infty})^n$ which satisfies (6) on $[0, \infty)$. For any singular matrix pencil, the Kronecker canonical form (KCF) is the simplest decomposition which provides many useful theoretical tools for analyzing (6).

Lemma 1. The Kronecker Canonical Form (KCF) [6]: *For every matrix pencil $(\lambda\mathscr{E} - \mathscr{A})$ there exist non-singular matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that, for multi-indices ϵ, f, σ, and η,*

$$P(\lambda\mathscr{E} - \mathscr{A})Q = \begin{bmatrix}\lambda E_\epsilon - A_\epsilon & & & \\ & \lambda I_f - J_f & & \\ & & \lambda J_\sigma - I_\sigma & \\ & & & \lambda E_\eta - A_\eta\end{bmatrix}, \tag{7}$$

where $\lambda E_\epsilon - A_\epsilon$ and $\lambda E_\eta - A_\eta$ have block diagonal structure; each block takes the form $\lambda E_{\epsilon_i} - A_{\epsilon_i} = \lambda \begin{bmatrix} I_{\epsilon_i} & 0_{\epsilon_i\times 1}\end{bmatrix} - \begin{bmatrix} 0_{\epsilon_i\times 1} & I_{\epsilon_i}\end{bmatrix}$ and $\lambda E_{\eta_i} - A_{\eta_i} = \lambda \begin{bmatrix} I_{\eta_i} \\ 0_{1\times\eta_i}\end{bmatrix} - \begin{bmatrix} 0_{1\times\eta_i} \\ I_{\eta_i}\end{bmatrix}$, respectively; both J_f and J_σ are in Jordan canonical form; J_σ has zeros on its diagonal and thus is a nilpotent matrix; J_f contains, on its diagonal, all finite eigenvalues of $(\lambda\mathscr{E} - \mathscr{A})$.

Remark 1. The blocks in (7) appear only in pairs. For example, if E_ϵ vanishes, then A_ϵ also vanishes. Moreover, $\epsilon-$blocks with $\epsilon_i = 0$ and/or $\eta-$blocks with $\eta_i = 0$ are possible, which results in zero columns (for $\epsilon_i = 0$) and/or zero rows (for $\eta_i = 0$) in the KCF (7). The KCF structure (7) is unique up to the reordering of the diagonal blocks. The KCF (7) without $\epsilon-$ and $\eta-$blocks is also called the Weierstrass canonical form (WCF). In case of a regular matrix pencil $(\lambda\mathscr{E} - \mathscr{A})$, the KCF (7) reduces to the WCF.

Remark 2. In this paper, we use the KCF (7) to simplify the proof of some theoretical results. But, the determination of the KCF is not recommended because the computation is numerically ill-posed [7]. Furthermore, since J_f and J_σ are in Jordan canonical form, in general, the matrices P and Q in (7) are complex-valued matrices. This is computationally undesirable, because if the system matrices are real-valued, one would like to get real P and Q. To remove such difficulties in the computation of the KCF, based on the Wong sequences, a numerically stable quasi-Kronecker decomposition, which also reveals the KCF structure, can be found in [8, 9].

The solution theory of descriptor systems is a simple application of the KCF because it has a block diagonal structure and the associated variables can be considered separately. Setting

$$x = Q \begin{bmatrix} x_\epsilon^\top & x_f^\top & x_\sigma^\top & x_\eta^\top \end{bmatrix}^\top, \tag{8}$$

then in terms of the four different blocks in the KCF, (6) reduces to

$$\begin{aligned} E_\epsilon \dot{x}_\epsilon &= A_\epsilon x_\epsilon, && (9a)\\ \dot{x}_f &= J_f x_f, && (9b)\\ J_\sigma \dot{x}_\sigma &= x_\sigma, && (9c)\\ E_\eta \dot{x}_\eta &= A_\eta x_\eta. && (9d) \end{aligned}$$

Thus, the following solution analysis of (6), via (9), is now straightforward.

S1) Systems of the form (9a) can be written as

$$\begin{bmatrix} I_\epsilon & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{10}$$

where A_1 is a nilpotent matrix. Thus, any solution $x_\epsilon = \begin{bmatrix} x_1^\top & x_2^\top \end{bmatrix}^\top$ to (10) is given by

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} e^{A_1 t} x_1^0 + \int_0^t e^{A_1(t-\tau)} A_2 x_2(\tau) \mathrm{d}\tau \\ x_2(t) \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=0}^{h_1-1} \frac{(A_1 t)^i}{i!} x_1^0 + \sum_{i=0}^{h_1-1} A_1^i A_2 \int_0^t \frac{(t-\tau)^i}{i!} x_2(\tau) \mathrm{d}\tau \\ x_2(t) \end{bmatrix}, \ t \geq 0, \end{aligned} \tag{11}$$

for some x_1^0 of appropriate dimension, where h_1 is the nilpotency index of A_1 and x_2 is arbitrary. Hence, in general, x_ϵ satisfying (9a) is always impulsive, cf. [5, p. 26]. Moreover, by [4, Cor. 2.4] any solution x of (6) is uniquely determined if, and only if, the $\epsilon-$blocks in (7) are not present.

S2) Corresponding to any initial condition, the solution of the free homogeneous state space system (9b) exhibits no impulses, see [3, Thm. 3.3], i.e., $x_f[t] = 0$ for all $t \geq 0$.

S3) According to [10] the solution of (9c) is given by

$$x_\sigma|_{[0,\infty)} = -\sum_{i=1}^{h_2-1} \delta^{(i-1)} J_\sigma^i x_\sigma(0-), \tag{12}$$

where h_2 is the nilpotency index of the matrix J_σ. Therefore, the solution of (9c) is impulsive (i.e., $x_\sigma[0] \neq 0$) if, and only if, $x_\sigma(0-) \notin \ker J_\sigma$.

S4) Each block in (9d) can be written as

$$\begin{aligned} \dot{x}_{\eta_i} &= J_{\eta_i}^\top x_{\eta_i}, \\ 0 &= e_{\eta_i}^\top x_{\eta_i}, \end{aligned}$$

where $J_{\eta_i}^{\top}$ is a nilpotent matrix having nilpotency index η_i and e_{η_i} is the last column of I_{η_i}. The only solution for this block is $x_\eta = 0$ and, in particular, $x_\eta[t] = 0$ for all $t \geq 0$, cf. also [5, p. 25]. Consequently, there are no impulses in the solutions of (6) due to $\eta-$blocks. It is important to note that the $\eta-$blocks do not have any solution, not even in the distributional sense, with respect to nonzero initial conditions.

Remark 3. From above solution analysis, it is clear that the semistate x in (1) may have impulses only due to ϵ- and σ-blocks in the KCF of $(\lambda E - A)$.

The concept of partial impulse observability of (1) is a natural extension of impulse observability (I-observability) of system (1a)-(1b): (E, A, C) is impulse observable if, and only if, (E, A, C) is partially impulse observable with respect to $L = I_n$. Alternative definitions for impulse observability are given in [11, 12] for instance, see also the survey [5] for more details. To check the I-observability of system (1a)-(1b), the following algebraic criterion has been provided in the literature [5, 11, 12]:

$$\operatorname{rank} \begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} = n + \operatorname{rank} E. \tag{13}$$

Remark 4. Clearly, I-observability of (1a)-(1b) implies partial impulse observability of (1) with respect to any matrix L. But the opposite implication is not true in general. In Remark 5 below, we show that when $L = I_n$, the criteria for partial impulse observability of (1) developed in the following sections trivially reduce to I-observability of (1a)-(1b).

Now, we present some results from basic linear algebra, which play an important role in the further discussion. The following fundamental result can be found in any standard textbook on linear algebra.

Lemma 2. *Let X and Y be any two matrices of compatible dimensions. Then* $\operatorname{rank} \begin{bmatrix} X \\ Y \end{bmatrix} = \operatorname{rank} X$ *if, and only if,* $\ker X \subseteq \ker Y$.

Lemma 3. *[13] Let X, W, and Y be any matrices of compatible dimensions. If X has full row rank and/or Y has full column rank, then*

$$\operatorname{rank} \begin{bmatrix} X & W \\ 0 & Y \end{bmatrix} = \operatorname{rank} X + \operatorname{rank} Y.$$

Finally, we recall the concept of Wong sequences corresponding to (1a) from [8]; for our purposes we only need the second Wong sequence.

Definition 2. For matrices $E,\ A \in \mathbb{R}^{m\times n}$ the Wong sequence $\{\mathcal{W}^i_{[E,A]}\}_{i=0}^{\infty}$ is a sequence of subspaces, defined by

$$\mathcal{W}^0_{[E,A]} := \{0\}, \quad \mathcal{W}^{i+1}_{[E,A]} := E^{-1}(A\mathcal{W}^i_{[E,A]}),\ i \in \mathbb{N}.$$

The union $\mathcal{W}^*_{[E,A]} := \bigcup_{i\in\mathbb{N}} \mathcal{W}^i_{[E,A]}$ is called the limit of the Wong sequence.

We conclude this section by recalling the following result for I-observability of system (1a)-(1b) in terms of the Wong sequences.

Lemma 4. *[5] The triple (E, A, C) is I-observable if, and only if,*

$$\mathcal{W}^*_{[\bar{E},\bar{A}]} \cap \bar{A}^{-1}(\operatorname{im} \bar{E}) = \{0\},$$

where $\bar{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}$ *and* $\bar{A} = \begin{bmatrix} A \\ C \end{bmatrix}$.

3. Main result

The main aim of this section is to derive a simple rank criteria for partial impulse observability of (1) in terms of the original system matrices. In Theorem 1 below, we first derive this condition in terms of the KCF of the matrix pencil $(\lambda E - A)$. To prove Theorem 1, without loss of generality, we assume that the pencil $(\lambda E - A)$ is in KCF. Moreover, we use the notations

$$CQ = \begin{bmatrix} C_\epsilon & C_f & C_\sigma & C_\eta \end{bmatrix} \tag{14}$$

and

$$LQ = \begin{bmatrix} L_\epsilon & L_f & L_\sigma & L_\eta \end{bmatrix}, \tag{15}$$

where the sizes of the block matrices on the right hand side of (14) and (15) are compatible with the sizes of the blocks in the KCF of $(\lambda E - A)$. Furthermore, we assume that C_ϵ and L_ϵ are partitioned, corresponding to the decomposition (10) as follows:

$$C_\epsilon = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \quad \text{and} \quad L_\epsilon = \begin{bmatrix} L_1 & L_2 \end{bmatrix}. \tag{16}$$

Theorem 1. *Consider system (1). Then (E, A, C) is partially impulse observable with respect to L if, and only if, there exists an integer $q \geq 1$ such that*

$$\ker \begin{bmatrix} C_2 & C_1A_2 & C_1A_1A_2 & \dots & C_1A_1^{l-1}A_2 & -C_\sigma J_\sigma \\ & C_2 & C_1A_2 & \dots & C_1A_1^{l-2}A_2 & -C_\sigma J_\sigma^2 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & C_2 & C_1A_2 & -C_\sigma J_\sigma^l \\ & & & & C_2 & 0 \end{bmatrix} \subseteq \ker \begin{bmatrix} L_2 & L_1A_2 & L_1A_1A_2 & \dots & L_1A_1^{l-1}A_2 & -L_\sigma J_\sigma \\ & L_2 & L_1A_2 & \dots & L_1A_1^{l-2}A_2 & -L_\sigma J_\sigma^2 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & L_2 & L_1A_2 & -L_\sigma J_\sigma^l \\ & & & & L_2 & 0 \end{bmatrix} \tag{17}$$

for all integers $l \geq q$.

Proof. ($\Rightarrow$): Set $q := 1$, let $l \geq 1$ and

$$\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_l \\ v \end{bmatrix} \in \ker \begin{bmatrix} C_2 & C_1A_2 & C_1A_1A_2 & \dots & C_1A_1^{l-1}A_2 & -C_\sigma J_\sigma \\ & C_2 & C_1A_2 & \dots & C_1A_1^{l-2}A_2 & -C_\sigma J_\sigma^2 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & C_2 & C_1A_2 & -C_\sigma J_\sigma^l \\ & & & & C_2 & 0 \end{bmatrix}.$$

Define $x_\sigma(t) = v$ for $t < 0$, $x_\sigma|_{[0,\infty)}$ as in (12) and $x_2 = \sum_{j=0}^{l} \delta^{(j)} v_j$. Then, with x_1 as in (11) for $x_1^0 = 0$, $x_\epsilon = \begin{bmatrix} x_1^\top & x_2^\top \end{bmatrix}^\top$, $x_f = 0$, $x_\eta = 0$, $y = C_1x_1 + C_2x_2 + C_\sigma x_\sigma$, and $z = L_1x_1 + L_2x_2 + L_\sigma x_\sigma$, we have $(x, y, z) \in \mathscr{B}$. Now, using the convolution property

$$\int_0^t \frac{(t-\tau)^i}{i!} \delta_s^{(j)} \mathrm{d}\tau \quad = \quad \begin{cases} \frac{(t-s)^{i-j}}{(i-j)!}, & j = 0, \dots, i, \\ \delta_s^{(j-i-1)}, & j = i+1, \dots, l, \end{cases} \tag{18}$$

for any $s \geq 0$, the equation (11) implies

$$x_1(t) = \sum_{i=0}^{h_1-1} A_1^i A_2 \left\{ \sum_{j=0}^{i} \frac{t^{i-j}}{(i-j)!} v_j + \sum_{j=i+1}^{l} \delta^{(j-i-1)} v_j \right\}, \tag{19}$$

where h_1 is the nilpotency index of A_1. Thus, if h_2 is the nilpotency index of J_σ, by (19) and (12), we obtain

$$x_1[0] = \sum_{i=0}^{h_1-1}\sum_{j=i+1}^{l} \delta^{(j-i-1)} A_1^i A_2 v_j, \tag{20a}$$

$$x_\sigma[0] = -\sum_{i=0}^{h_2-1} J_\sigma^{i+1}\delta^{(i)} v, \tag{20b}$$

and clearly $x_1[t]=0$ and $x_\sigma[t]=0$ for all $t>0$. Since, by choices of v and v_i $(0\le i\le l)$,

$$\begin{aligned} y[0] &= \sum_{i=0}^{h_1-1}\sum_{j=i+1}^{l} \delta^{(j-i-1)} C_1 A_1^i A_2 v_j + \sum_{i=0}^{l}\delta^{(i)} C_2 v_i - \sum_{i=0}^{l}\delta^{(i)} C_\sigma J_\sigma^{i+1} v \\ &= \begin{bmatrix}\delta I & \delta^{(1)} I & \dots & \delta^{(l)} I\end{bmatrix} \begin{bmatrix} C_2 & C_1A_2 & C_1A_1A_2 & \dots & C_1A_1^{l-1}A_2 & -C_\sigma J_\sigma \\ & C_2 & C_1A_2 & \dots & C_1A_1^{l-2}A_2 & -C_\sigma J_\sigma^2 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & C_2 & C_1A_2 & -C_\sigma J_\sigma^l \\ & & & & C_2 & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_l \\ v \end{bmatrix} = 0. \end{aligned}$$

Therefore, partial impulse observability of the system implies $z[0]=0$. Hence,

$$\begin{aligned} 0 = z[0] &= \sum_{i=0}^{h_1-1}\sum_{j=i+1}^{l} \delta^{(j-i-1)} L_1 A_1^i A_2 v_j + \sum_{i=0}^{l}\delta^{(i)} L_2 v_i - \sum_{i=0}^{l}\delta^{(i)} L_\sigma J_\sigma^{i+1} v \\ &= \begin{bmatrix}\delta I & \delta^{(1)} I & \dots & \delta^{(l)} I\end{bmatrix} \begin{bmatrix} L_2 & L_1A_2 & L_1A_1A_2 & \dots & L_1A_1^{l-1}A_2 & -L_\sigma J_\sigma \\ & L_2 & L_1A_2 & \dots & L_1A_1^{l-2}A_2 & -L_\sigma J_\sigma^2 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & L_2 & L_1A_2 & -L_\sigma J_\sigma^l \\ & & & & L_2 & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_l \\ v \end{bmatrix}, \end{aligned} \tag{21}$$

which means that

$$\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_l \\ v \end{bmatrix} \in \ker \begin{bmatrix} L_2 & L_1A_2 & L_1A_1A_2 & \dots & L_1A_1^{l-1}A_2 & -L_\sigma J_\sigma \\ & L_2 & L_1A_2 & \dots & L_1A_1^{l-2}A_2 & -L_\sigma J_\sigma^2 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & L_2 & L_1A_2 & -L_\sigma J_\sigma^l \\ & & & & L_2 & 0 \end{bmatrix}.$$

($\Leftarrow$): Let $(x,y,z)\in\mathscr{B}$ be such that $y[t]=0$ for all $t\ge 0$ and $x=\begin{bmatrix}x_1^\top & x_2^\top & x_f^\top & x_\sigma^\top & x_\eta^\top\end{bmatrix}^\top$ as in (8) and (10). By definition of $\mathscr{D}'_{pw\mathscr{C}^\infty}$ there is a locally finite set $(t_k)_{k\in\mathbb{Z}}\subseteq\mathbb{R}$ such that $x_2[t_k]\ne 0$ and $x_2[t]=0$ for all $t\ne t_k$, see [3]. Let n_1 and n_2 be the number of components in x_2 and x_σ, respectively. Then by [3, Prop. 2.1.12] there exist $l_k\in\mathbb{N}$ and $v_{k,j}\in\mathbb{R}^{n_1}$, for $k\in\mathbb{Z}$ and $j=0,\dots,l_k$, such that

$$x_2[t_k] = \sum_{j=0}^{l_k} v_{k,j}\delta_{t_k}^{(j)}.$$

Fix $k \in \mathbb{Z}$. Without loss of generality we assume that $l_k \geq q$, otherwise we may add additional terms with $v_{k,j} = 0$. Then, by (11), (12), and (18) with $s = t_k$, we obtain

$$\begin{aligned} x_1[t_k] &= \sum_{i=0}^{h_1-1} \sum_{j=i+1}^{l_k} \delta_{t_k}^{(j-i-1)} A_1^i A_2 v_{k,j}, \\ x_2[t_k] &= \sum_{i=0}^{l_k} \delta_{t_k}^{(i)} v_{k,i}, \\ x_\sigma[0] &= -\sum_{i=0}^{h_2-1} \delta^{(i)} J_\sigma^{i+1} x_\sigma^0, \end{aligned}$$

where $x_\sigma^0 \in \mathbb{R}^{n_2}$. Thus, from $y[t_k] = 0$, it follows that

$$\begin{bmatrix} v_{k,0} \\ v_{k,1} \\ v_{k,2} \\ \vdots \\ v_{k,l_k} \\ x_\sigma^0 \end{bmatrix} \in \ker \begin{bmatrix} C_2 & C_1A_2 & C_1A_1A_2 & \dots & C_1A_1^{l_k-1}A_2 & -C_\sigma J_\sigma \\ & C_2 & C_1A_2 & \dots & C_1A_1^{l_k-2}A_2 & -C_\sigma J_\sigma^2 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & C_2 & C_1A_2 & -C_\sigma J_\sigma^{l_k} \\ & & & & C_2 & 0 \end{bmatrix}.$$

Then assumption (17) implies that

$$\begin{bmatrix} v_{k,0} \\ v_{k,1} \\ v_{k,2} \\ \vdots \\ v_{k,l_k} \\ x_\sigma^0 \end{bmatrix} \in \ker \begin{bmatrix} L_2 & L_1A_2 & L_1A_1A_2 & \dots & L_1A_1^{l_k-1}A_2 & -L_\sigma J_\sigma \\ & L_2 & L_1A_2 & \dots & L_1A_1^{l_k-2}A_2 & -L_\sigma J_\sigma^2 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & L_2 & L_1A_2 & -L_\sigma J_\sigma^{l_k} \\ & & & & L_2 & 0 \end{bmatrix}$$

which, by a similar calculation as in (21), implies $z[t_k] = 0$. Since k was arbitrary and $z[t] = 0$ for $t \neq t_k$ is obvious, this proves partial impulse observability of (1) with respect to L. □

Before investigating the algebraic criteria for partial impulse observability of (1) directly in terms of the system coefficient matrices, we define

$$\bar{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \ \bar{A} = \begin{bmatrix} A \\ C \end{bmatrix}, \ \bar{E}_1 = \begin{bmatrix} \bar{E} \\ 0 \end{bmatrix}, \ \bar{A}_1 = \begin{bmatrix} \bar{A} \\ L \end{bmatrix},$$

$$\mathcal{F}_l := \overbrace{\begin{bmatrix} \bar{E} & \bar{A} & & & \\ & \bar{E} & \bar{A} & & \\ & & \ddots & \ddots & \\ & & & \bar{E} & \bar{A} \\ & & & & \bar{E} \end{bmatrix}}^{l \text{ block columns}} \Big\} l \text{ block rows}, \quad \mathcal{F}_{l,L} := \overbrace{\begin{bmatrix} \bar{E}_1 & \bar{A}_1 & & & \\ & \bar{E}_1 & \bar{A}_1 & & \\ & & \ddots & \ddots & \\ & & & \bar{E}_1 & \bar{A}_1 \\ & & & & \bar{E}_1 \end{bmatrix}}^{l \text{ block columns}} \Big\} l \text{ block rows},$$

and introduce the following rank condition

$$\forall\, l \geq n+1 : \ \operatorname{rank} \mathcal{F}_l = \operatorname{rank} \mathcal{F}_{l,L}. \tag{22}$$

The above rank condition can be transformed in terms of the blocks of the KCF. For example, if $l = 2$, then

$$\operatorname{rank}\mathcal{F}_2 = \operatorname{rank}\begin{bmatrix}\bar{E} & \bar{A}\\ & \bar{E}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}E & A\\ & C\\ & E\end{bmatrix} = \operatorname{rank}\begin{bmatrix}E_\epsilon & & & & A_\epsilon & & & \\ & I_f & & & & J_f & & \\ & & J_\sigma & & & & I_\sigma & \\ & & & E_\eta & & & & A_\eta\\ & & & & C_\epsilon & C_f & C_\sigma & C_\eta\\ & & & & E_\epsilon & & & \\ & & & & & I_f & & \\ & & & & & & J_\sigma & \\ & & & & & & & E_\eta\end{bmatrix}.$$

Since E_ϵ has full row rank, E_η has full column rank, and I_f has full rank, it is a direct consequence of Lemma 3 (applied four times) that

$$\operatorname{rank}\mathcal{F}_2 = \operatorname{rank}E_\epsilon + 2\operatorname{rank}I_f + 2\operatorname{rank}E_\eta + \operatorname{rank}\begin{bmatrix}J_\sigma & & I_\sigma\\ & C_\epsilon & C_\sigma\\ & E_\epsilon & \\ & & J_\sigma\end{bmatrix}.$$

Then, using column operations corresponding to the multiplication of the last matrix with $\begin{bmatrix}-I_\sigma & & \\ & I & \\ J_\sigma & & I_\sigma\end{bmatrix}$ from the right, we obtain

$$\begin{aligned}\operatorname{rank}\mathcal{F}_2 &= \operatorname{rank}E_\epsilon + 2\operatorname{rank}I_f + 2\operatorname{rank}E_\eta + \operatorname{rank}\begin{bmatrix} & & I_\sigma\\ C_\sigma J_\sigma & C_\epsilon & C_\sigma\\ & E_\epsilon & \\ J_\sigma^2 & & J_\sigma\end{bmatrix}\\ &= \operatorname{rank}E_\epsilon + 2\operatorname{rank}I_f + 2\operatorname{rank}E_\eta + \operatorname{rank}I_\sigma + \operatorname{rank}\begin{bmatrix}C_\sigma J_\sigma & C_\epsilon\\ & E_\epsilon\\ J_\sigma^2 & \end{bmatrix}.\end{aligned}$$

Now, substituting $E_\epsilon = \begin{bmatrix}I_\epsilon & 0\end{bmatrix}$, $C_\epsilon = \begin{bmatrix}C_1 & C_2\end{bmatrix}$ and again using Lemma 3 due to full rank of I_ϵ, we obtain

$$\operatorname{rank}\mathcal{F}_2 = 2\operatorname{rank}E_\epsilon + 2\operatorname{rank}I_f + 2\operatorname{rank}E_\eta + \operatorname{rank}I_\sigma + \operatorname{rank}\begin{bmatrix}C_\sigma J_\sigma & C_2\\ J_\sigma^2 & \end{bmatrix}. \tag{23}$$

By a similar calculation as above, it is easy to show that

$$\begin{aligned}\operatorname{rank}\mathcal{F}_{2,L} &= \operatorname{rank}\begin{bmatrix}\bar{E}_1 & \bar{A}_1\\ & \bar{E}_1\end{bmatrix} = \operatorname{rank}\begin{bmatrix}E & A\\ & C\\ & L\\ & E\end{bmatrix}\\ &= 2\operatorname{rank}E_\epsilon + 2\operatorname{rank}I_f + 2\operatorname{rank}E_\eta + \operatorname{rank}I_\sigma + \operatorname{rank}\begin{bmatrix}C_\sigma J_\sigma & C_2\\ L_\sigma J_\sigma & L_2\\ J_\sigma^2 & \end{bmatrix}.\end{aligned} \tag{24}$$

Thus, in view of Lemma 2, (23) and (24) provide that $\operatorname{rank}\mathcal{F}_2 = \operatorname{rank}\mathcal{F}_{2,L}$ if, and only if,

$$\ker\begin{bmatrix}C_\sigma J_\sigma & C_2\\ J_\sigma^2 & \end{bmatrix} \subseteq \ker\begin{bmatrix}L_\sigma J_\sigma & L_2\end{bmatrix}. \tag{25}$$

We now expound the calculation for any $l \geq 3$. We first introduce the following three operations on $\mathcal{F}_l$:

(i) Write all the block rows in terms of the original system matrices and then substitute the decompositions (7), (14), and (15) for $E,\ A,\ C$, and L.

(ii) Since each block row contains the full column rank matrices E_η and I_f, apply Lemma 3 ($2l$-times) from the bottom to the top block row.

(iii) Apply Lemma 3 again to the full row rank matrix E_ϵ in the first block row.

Thus, we obtain

$$
\begin{aligned}
\operatorname{rank}\mathcal{F}_l \quad &= \quad \operatorname{rank}E_\epsilon + l\operatorname{rank}I_f + l\operatorname{rank}E_\eta \\
&\quad + \operatorname{rank}\begin{array}{cc}
\left[\begin{array}{cccccccccc}
J_\sigma & & I_\sigma & & & & & & & \\
0 & C_\epsilon & C_\sigma & & & & & & & \\
\hline
 & E_\epsilon & & A_\epsilon & & & & & & \\
 & & J_\sigma & & I_\sigma & & & & & \\
 & & & C_\epsilon & C_\sigma & & & & & \\
\hline
 & & & & & \ddots & & \ddots & & \\
\hline
 & & & & & & E_\epsilon & & A_\epsilon & \\
 & & & & & & & J_\sigma & & I_\sigma \\
 & & & & & & & & C_\epsilon & C_\sigma \\
\hline
 & & & & & & & & E_\epsilon & \\
 & & & & & & & & & J_\sigma
\end{array}\right] &
\begin{array}{l}
1 \\ \\ \\ 2 \\ \\ \vdots \\ \\ l-1 \\ \\ l \\ \\
\end{array}
\end{array}
\end{aligned}
\tag{26}
$$

Now, to simplify the rank of the last matrix in (26), we perform the following two operations:

(i) Use elementary column operations to obtain only zero entries to the left of I_σ in the first $(l-1)$ block rows. This can be achieved by multiplying the last matrix in (26) with

$$
\begin{bmatrix}
I_\sigma & & & & & & \\
 & I & & & & & \\
-J_\sigma & & I_\sigma & & & & \\
 & \ddots & & \ddots & & & \\
J_\sigma^2 & & \ddots & & I_\sigma & & \\
\vdots & \ddots & & \ddots & & I & \\
(-1)^{l-1}J_\sigma^{l-1} & \cdots & J_\sigma^2 & & -J_\sigma & & I_\sigma
\end{bmatrix}
\text{ from the right.}
$$

(ii) Apply Lemma 3 ($(l-1)-$times) to the full rank matrices I_σ in each of the first $(l-1)$ block rows.

Thus, (26) reduces to

$$
\begin{aligned}
\operatorname{rank}\mathcal{F}_l \quad &= \quad \operatorname{rank}E_\epsilon + l\operatorname{rank}I_f + l\operatorname{rank}E_\eta + (l-1)\operatorname{rank}I_\sigma \\
&\quad + \operatorname{rank}\begin{array}{cc}
\left[\begin{array}{cccccc}
C_\sigma J_\sigma & C_\epsilon & & & & \\
\hline
0 & E_\epsilon & A_\epsilon & & & \\
C_\sigma J_\sigma^2 & 0 & C_\epsilon & & & \\
\hline
\vdots & & \ddots & \ddots & & \\
\hline
0 & & & & E_\epsilon & A_\epsilon \\
C_\sigma J_\sigma^{l-1} & & & & 0 & C_\epsilon \\
\hline
0 & & & & & E_\epsilon \\
J_\sigma^l & & & & & 0
\end{array}\right] &
\begin{array}{l}
1 \\ 2 \\ \\ \vdots \\ l-1 \\ \\ l \\ \\
\end{array}
\end{array}
\end{aligned}
\tag{27}
$$

Now, for further simplification, we use the following three operations on the last matrix in (27):

(i) Write $E_\epsilon = \begin{bmatrix} I_\epsilon & 0 \end{bmatrix}$ and substitute the decomposition (16) for C_ϵ.

(ii) Use elementary row operations to obtain only zero entries above of I_ϵ in each of the corresponding block columns. This can be achieved by multiplying the last matrix in (27) with
$$\begin{bmatrix} I & -C_1 & C_1A_1 & \cdots & (-1)^{l-1}C_1A_1^{l-2} & \\ & I_\epsilon & -A_1 & \cdots & (-1)^{l-2}A_1^{l-2} & \\ & & I & -C_1 & \ddots & \vdots & \\ & & & I_\epsilon & \ddots & C_1A_1 & \\ & & & & \ddots & -A_1 & \\ & & & & & -C_1 & \\ & & & & & I_\epsilon & \\ & & & & & & I \end{bmatrix} \text{ from the left.}$$

(iii) Apply Lemma 3 ($(l-1)$-times) to the full rank matrices I_ϵ.

Thus, using the fact that $\operatorname{rank} E_\epsilon = \operatorname{rank} I_\epsilon$, we obtain

$$\begin{aligned} \operatorname{rank} \mathcal{F}_l &= l \operatorname{rank} E_\epsilon + l \operatorname{rank} I_f + l \operatorname{rank} E_\eta + (l-1) \operatorname{rank} I_\sigma \\ &\quad + \operatorname{rank} \begin{bmatrix} C_\sigma J_\sigma & C_2 & C_1A_2 & \dots & C_1A_1^{l-4}A_2 & C_1A_1^{l-3}A_2 \\ C_\sigma J_\sigma^2 & & C_2 & \dots & C_1A_1^{l-5}A_2 & C_1A_1^{l-4}A_2 \\ \vdots & & & \ddots & \vdots & \vdots \\ C_\sigma J_\sigma^{l-2} & & & & C_2 & C_1A_2 \\ C_\sigma J_\sigma^{l-1} & & & & & C_2 \\ J_\sigma^l & & & & & \end{bmatrix}. \end{aligned} \tag{28}$$

Using similar operations on $\operatorname{rank} \mathcal{F}_{l,L}$, it is straightforward to show that

$$\begin{aligned} \operatorname{rank} \mathcal{F}_{l,L} &= l \operatorname{rank} E_\epsilon + l \operatorname{rank} I_f + l \operatorname{rank} E_\eta + (l-1) \operatorname{rank} I_\sigma \\ &\quad + \operatorname{rank} \begin{bmatrix} C_\sigma J_\sigma & C_2 & C_1A_2 & C_1A_1A_2 & \dots & C_1A_1^{l-3}A_2 \\ C_\sigma J_\sigma^2 & & C_2 & C_1A_2 & \dots & C_1A_1^{l-4}A_2 \\ \vdots & & & \ddots & \ddots & \vdots \\ C_\sigma J_\sigma^{l-2} & & & & C_2 & C_1A_2 \\ C_\sigma J_\sigma^{l-1} & & & & & C_2 \\ L_\sigma J_\sigma & L_2 & L_1A_2 & L_1A_1A_2 & \dots & L_1A_1^{l-3}A_2 \\ L_\sigma J_\sigma^2 & & L_2 & L_1A_2 & \dots & L_1A_1^{l-4}A_2 \\ \vdots & & & \ddots & \ddots & \vdots \\ L_\sigma J_\sigma^{l-2} & & & & L_2 & L_1A_2 \\ L_\sigma J_\sigma^{l-1} & & & & & L_2 \\ J_\sigma^l & & & & & \end{bmatrix}. \end{aligned} \tag{29}$$

Finally, due to (28) and (29), we may infer from Lemma 2 that, for any integer $l \geq 3$, $\operatorname{rank} \mathcal{F}_l = \operatorname{rank} \mathcal{F}_{l,L}$ if, and only if,

$$\ker \begin{bmatrix} C_\sigma J_\sigma & C_2 & C_1A_2 & C_1A_1A_2 & \dots & C_1A_1^{l-3}A_2 \\ C_\sigma J_\sigma^2 & & C_2 & C_1A_2 & \dots & C_1A_1^{l-4}A_2 \\ \vdots & & & \ddots & \ddots & \vdots \\ C_\sigma J_\sigma^{l-2} & & & & C_2 & C_1A_2 \\ C_\sigma J_\sigma^{l-1} & & & & & C_2 \\ J_\sigma^l & & & & & \end{bmatrix} \subseteq \ker \begin{bmatrix} L_\sigma J_\sigma & L_2 & L_1A_2 & L_1A_1A_2 & \dots & L_1A_1^{l-3}A_2 \\ L_\sigma J_\sigma^2 & & L_2 & L_1A_2 & \dots & L_1A_1^{l-4}A_2 \\ \vdots & & & \ddots & \ddots & \vdots \\ L_\sigma J_\sigma^{l-2} & & & & L_2 & L_1A_2 \\ L_\sigma J_\sigma^{l-1} & & & & & L_2 \end{bmatrix}. \tag{30}$$

With these findings we are now ready to state the main result of this paper.

Theorem 2. *For a given system* (1), *the following statements are equivalent:*
(a) (E, A, C) *is partially impulse observable with respect to* L.
(b) *The condition* (22) *holds.*
(c) $\operatorname{rank} \mathcal{F}_{n+1} = \operatorname{rank} \mathcal{F}_{n+1,L}$.
(d) $\mathcal{W}^*_{[\bar{E},\bar{A}]} \cap \bar{A}^{-1}(\operatorname{im} \bar{E}) \subseteq \ker L$.

Proof. The equivalence of (a) and (b) is a direct consequence of Theorem 1 and the conditions (25) and (30). The statement (b) $\Rightarrow$ (c) is obvious. Thus, in order to complete the proof, it is sufficient to show that (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (b). Before proving these statements, we observe, by a simple permutation of rows, that

$$\mathcal{F}_{l,L} = P \left[\begin{array}{c} \mathcal{F}_l \\ \overset{l \text{ block columns}}{\longleftrightarrow} \\ \left[\begin{array}{cccc} 0 & L & & \\ & & \ddots & \\ & & & L \end{array}\right] \Big\updownarrow {\scriptstyle (l-1) \text{ block rows}} \end{array}\right],$$

where P is a suitable permutation matrix, and hence $\operatorname{rank} \mathcal{F}_l = \operatorname{rank} \mathcal{F}_{l,L}$ holds if, and only if,

$$\ker \mathcal{F}_l \subseteq \ker \begin{bmatrix} 0 & L & & \\ & & \ddots & \\ & & & L \end{bmatrix} = \mathbb{R}^n \times \underbrace{\ker L \times \ldots \times \ker L}_{(l-1) \text{ times}}. \tag{31}$$

(c) $\Rightarrow$ (d): Let $v_n \in \mathcal{W}^*_{[\bar{E},\bar{A}]} \cap \bar{A}^{-1}(\operatorname{im} \bar{E})$. Since the Wong sequences terminate after finitely many steps, and in each iteration before termination the dimension increases by at least one, it is clear that $\mathcal{W}^*_{[\bar{E},\bar{A}]} = \mathcal{W}^n_{[\bar{E},\bar{A}]}$. Hence there exists $v_{n-1} \in \mathcal{W}^{n-1}_{[\bar{E},\bar{A}]}$ such that $\bar{E}v_n = -\bar{A}v_{n-1}$. Successively, there exist $v_i \in \mathcal{W}^i_{[\bar{E},\bar{A}]}$ such that $\bar{E}v_{i+1} = -\bar{A}v_i$ for $i = n-2, \ldots, 1$ and $\bar{E}v_1 = 0$, since $v_1 \in \mathcal{W}^1_{[\bar{E},\bar{A}]} = \ker \bar{E}$. Furthermore, since also $v_n \in \bar{A}^{-1}(\operatorname{im} \bar{E})$ there exists $v_{n+1} \in \mathbb{R}^n$ such that $\bar{A}v_n = -\bar{E}v_{n+1}$. Therefore, we find that $(v_{n+1}^\top, v_n^\top, \ldots, v_1^\top)^\top \in \ker \mathcal{F}_{n+1}$ and from (31) it follows that $v_n \in \ker L$.

(d) $\Rightarrow$ (b): In order to show (b) we prove that (31) holds for all $l \geq n+1$. Let $x = (x_l^\top, \ldots, x_1^\top)^\top \in \ker \mathcal{F}_l$. Then $\bar{E}x_l = -\bar{A}x_{l-1}, \ \ldots, \ \bar{E}x_2 = -\bar{A}x_1, \ \bar{E}x_1 = 0$ and hence we have

$$\begin{aligned} x_1 &\in \ker \bar{E} = \mathcal{W}^1_{[\bar{E},\bar{A}]}, \\ x_2 &= \bar{E}^{-1}(-\bar{A}x_1) \in \bar{E}^{-1}(\bar{A}\mathcal{W}^1_{[\bar{E},\bar{A}]}) = \mathcal{W}^2_{[\bar{E},\bar{A}]}, \\ &\vdots \\ x_l &= \bar{E}^{-1}(-\bar{A}x_{l-1}) \in \bar{E}^{-1}(\bar{A}\mathcal{W}^{l-1}_{[\bar{E},\bar{A}]}) = \mathcal{W}^l_{[\bar{E},\bar{A}]}. \end{aligned}$$

Since $\mathcal{W}^i_{[\bar{E},\bar{A}]} \subseteq \mathcal{W}^*_{[\bar{E},\bar{A}]}$ for all $i \geq 1$ we have that $x_i \in \mathcal{W}^*_{[\bar{E},\bar{A}]}$ for all $i = 1, \ldots, l$. Furthermore, since $\bar{A}x_i = -\bar{E}x_{i+1}$ for $i = 1, \ldots, l-1$ we have that $x_i \in \bar{A}^{-1}(\operatorname{im} \bar{E})$, hence

$$\forall\, i = 1, \ldots, l-1 : \ x_i \in \mathcal{W}^*_{[\bar{E},\bar{A}]} \cap \bar{A}^{-1}(\operatorname{im} \bar{E}) \subseteq \ker L,$$

which shows (31). This completes the proof. $\square$

Remark 5. It is clear that if $L = I_n$, then the condition of statement (d) in Theorem 2 reduces to the criterion for I-observability from Lemma 4.

Remark 6. The condition of statement (c) in Theorem 2 is straightforward to implement by using a one-line command, for instance, in MATLAB. If s is the least positive integer such that $\mathcal{W}^{s+1}_{[\bar{E},\bar{A}]} = \mathcal{W}^s_{[\bar{E},\bar{A}]}$, then the number $(n+1)$ in statement (c) of Theorem 2 can be replaced by s. Here, we use $(n+1)$ blocks in $\mathcal{F}$ because the value of s is not known in advance and our main aim is to provide a condition directly in terms of the known data, i.e., the system coefficient matrices and the dimension n. Notably, using $(n+1)$ blocks does not make the condition of statement (c) in Theorem 2 less or more restrictive.

4. Illustrative examples

Example 1. *Consider system* (1) *described by the coefficient matrices*

$$E = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ \textit{and } L = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Then (E, A, C) *is not partially impulse observable with respect to* L*, because choosing* x_1 *as the Heaviside step function and* $x_2 = \delta$ *we obtain a solution with* $y[t] = 0$ *for all* $t \geq 0$*, but* $z[0] = x_2[0] = \delta \neq 0$*, thus* z *exhibits impulses while* y *is impulse free. On the other hand, it is easy to verify that*

$$\operatorname{rank} \mathcal{F}_3 = 4 \neq 5 = \operatorname{rank} \mathcal{F}_{3,L}.$$

Example 2. *Consider system* (1) *with the coefficient matrices as in the counterexample in Eq.* (4) *in Section 1. Then, as shown there,* (E, A, C) *is not partially impulse observable with respect to* L*. It is easy to see that*

$$\operatorname{rank} \mathcal{F}_4 = 9 \neq 10 = \operatorname{rank} \mathcal{F}_{4,L}.$$

Example 3. *Consider system* (1) *with the same matrices* E*,* A*, and* L *as in the counterexample in Eq.* (4)*, but with* $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$*. Here our purpose is to show that, by changing the matrix* C*, it is easily possible to modify the given system in such a way that it becomes partially impulse observable with respect to the same* L*. It is clear that*

$$x|_{[0,\infty)} = -\begin{bmatrix} 0 \\ x_1^0 \\ x_2^0 \end{bmatrix} \delta - \begin{bmatrix} 0 \\ 0 \\ x_1^0 \end{bmatrix} \dot{\delta},$$

for suitable x_1^0, x_2^0*, and hence*

$$\begin{aligned} y[0] &= -x_1^0 \delta - x_2^0 \dot{\delta}, \\ z[0] &= -x_1^0 \delta. \end{aligned}$$

Clearly $y[0] = 0$ *implies* $x_1^0 = x_2^0 = 0$ *and hence also* $z[0] = 0$*. Thus* (E, A, C) *is partially impulse observable with respect to* L*. We can also verify this fact by checking the rank condition*

$$\operatorname{rank} \mathcal{F}_4 = 11 = \operatorname{rank} \mathcal{F}_{4,L}.$$

This demonstrates the effectiveness of statement (c) in Theorem 2.

5. Conclusion

This paper has established necessary and sufficient conditions for the partial impulse observability of linear descriptor systems. The developed conditions in terms of a rank criterion involving the original system coefficient matrices and the Wong sequences, respectively, are very easy to implement.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] M. Darouach, On the functional observers for linear descriptor systems, Systems & Control Letters 61 (3) (2012) 427–434.

[2] M. Darouach, F. Amato, M. Alma, Functional observers design for descriptor systems via LMI: Continuous and discrete-time cases, Automatica 86 (2017) 216–219.

[3] S. Trenn, Distributional differential algebraic equations, Ph.D. thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Ilmenau, Germany (2009)

[4] S. Trenn, Solution concepts for linear DAEs: A survey, in: Surveys in differential-algebraic equations I, Springer, 2013, pp. 137–172.

[5] T. Berger, T. Reis, S. Trenn, Observability of linear differential-algebraic systems: A survey, in: Surveys in differential-algebraic equations IV, Springer, 2017, pp. 161–219.

[6] F. R. Gantmacher, The Theory of Matrices, vol.: 2, Chelsea Publishing Company, New York, 1959.

[7] P. Van Dooren, The computation of Kronecker's canonical form of a singular pencil, Linear Algebra and Its Applications 27 (1979) 103–140.

[8] T. Berger, S. Trenn, The quasi-Kronecker form for matrix pencils, SIAM Journal on Matrix Analysis and Applications 33 (2) (2012) 336–368.

[9] T. Berger, S. Trenn, Addition to "the quasi-Kronecker form for matrix pencils", SIAM Journal on Matrix Analysis and Applications 34 (1) (2013) 94–101.

[10] L. Dai, Singular control systems, vol. 118, Springer, 1989.

[11] M. Hou, P. Müller, Causal observability of descriptor systems, IEEE Transactions on Automatic Control 44 (1) (1999) 158–163.

[12] J. Y. Ishihara, M. H. Terra, Impulse controllability and observability of rectangular descriptor systems, IEEE Transactions on Automatic Control 46 (6) (2001) 991–994.

[13] G. Matsaglia, G. P. H. Styan, Equalities and inequalities for ranks of matrices, Linear and Multilinear Algebra 2 (3) (1974) 269–292.

[14] M. Hou, Controllability and elimination of impulsive modes in descriptor systems, IEEE Transactions on Automatic Control 49 (10) (2004) 1723–1729.