Prof. Dr. Helge Glöckner

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3. Übungsblatt zur „Analysis II“

Hausübungen

Aufgabe H7 (Integrale ungerader Funktionen; 5 Punkte)

Seien $a \in]0,\infty[$ und $f \in R_a^{-a}$ mit $f(-x) = -f(x)$ für alle $x \in [-a,a]$. Zeigen Sie $\int_{-a}^{a} f(t)dt = 0$.

Lösung:

Es gilt $\int_{-a}^{a} f(t)\,dt = \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt$.

Zudem erhalten wir

$$\int_{-a}^{0} f(t)\,dt \underset{\text{Subst.}}{=} \int_{a}^{0} f(-t)\cdot(-1)\,dt = \int_{0}^{a} f(t)\,dt$$

$$= -\int_{0}^{a} f(t).$$

$\square$

Aufgabe H8 (Integrale II; 5 Punkte)

Berechnen Sie die folgenden Integrale:

(a) $\int_0^{\cancel{x}\ \pi/4} \cos^2(x)dx$ (b) $\int_0^1 \frac{1}{(1+x^2)^2}dx$ (c) $\int_\alpha^\beta \frac{1}{\sqrt{\sin(x)\cos^3(x)}}dx$ für $\alpha, \beta \in]0, \frac{\pi}{2}[$

(d) $\int_0^1 \frac{x^2}{\sqrt{1-x^6}}$ (Rückwärtssubstitution mit $\varphi(t) = t^{\frac{1}{3}}$).

Lösung:

(a) $\int_0^{\pi/4} \cos^2(x)dx = \left[\cos(x)\cdot\sin(x)\right]_0^{\pi/4} - \int_0^{\pi/4}(-\sin(x))\sin(x)\,dx$

$= \cos(\frac{\pi}{4})\sin(\frac{\pi}{4}) + \int_0^{\pi/4} 1-\cos^2(x)\,dx$

$\Rightarrow 2\int_0^{\pi/4}\cos^2(x)dx = \cos(\frac{\pi}{4})\sin(\frac{\pi}{4}) + \pi/4 = \frac{1}{2}+\frac{\pi}{4}$

$\Rightarrow \int_0^{\pi/4}\cos^2(x)\,dx = \frac{2+\pi}{8}$

(b) $\int_0^1 \frac{1}{(1+x^2)^2}dx = \int_0^{\pi/4}\frac{1}{(1+\tan^2(x))^2}\cdot\frac{1}{\cos^2(x)}\,dx$

$= \int_0^{\pi/4}\cos^2(x) = \frac{2+\pi}{8}$

(c) $\int_\alpha^\beta \frac{1}{\sqrt{\sin(x)\cos^3(x)}}dx = \int_\alpha^\beta \frac{1}{\sqrt{\frac{\sin(x)}{\cos(x)}\cdot\cos^4(x)}}\,dx$

$= \int_\alpha^\beta \frac{1}{\sqrt{\tan(x)}}\cdot\frac{1}{\cos^2(x)}\,dx = \int_{\tan(\alpha)}^{\tan(\beta)}\frac{1}{\sqrt{x}}\,dx$

$= \left[\sqrt{x}\right]_{\tan(\alpha)}^{\tan(\beta)} = \sqrt{\tan(\beta)} - \sqrt{\tan(\alpha)}$.

(d) $\int_0^1 \frac{x^2}{\sqrt{1-x^6}}dx = \frac{1}{3}\int_0^1 \frac{x^{2/3}}{\sqrt{1-x^2}}\cdot x^{-\frac{2}{3}}dx = \frac{1}{3}\int_0^1\frac{1}{\sqrt{1-x^2}}dx$

$= \frac{1}{3}\left[\arcsin(x)\right]_0^1 = \frac{1}{3}\arcsin(1) = \frac{\pi}{6}$

Aufgabe H9 (Verbesserte Intervalladditivität; 5 Punkte)

Aus der Vorlesung wissen wir, dass wenn $a \leq b \leq c$ und $f \in R_a^c$, so gilt $\int_a^c f = \int_a^b f + \int_b^c f$. Sei nun $D \subseteq \mathbb{R}$ ein nicht entartetes Intervall und $f\colon D \to \mathbb{R}$ stetig. Zeigen Sie: Sind $a, x \in D$ und $h \in \mathbb{R}$ mit $x + h \in D$, so gilt

$$\int_a^{x+h} f = \int_a^x f + \int_x^{x+h} f.$$

Lösung:

Fall 1: $h \geq 0$

Fall 1.1: $a \leq x$

Dann $a \leq x \leq x+h$. Es folgt die Aussage.

Fall 1.2: $a \geq x$

Fall 1.2.1: $a \leq x+h$

Dann $x \leq a \leq x+h$

Also $\int_x^{x+h} f = \int_x^a f + \int_a^{x+h} f$

$\Rightarrow \int_a^{x+h} f = \int_x^{x+h} f + \int_a^x f$

Fall 1.2.2 $x+h \leq a$

Dann $x \leq x+h \leq a$

Also $\int_x^a f = \int_x^{x+h} f + \int_{x+h}^a f$

$\int_a^{x+h} f = -\int_{x+h}^a f = -\int_x^a f + \int_x^{x+h} f = \int_a^x f + \int_x^{x+h} f$

Fall 2: $h \leq 0$

Fall 2.1: $a \leq x$

Fall 2.1.1: $a \leq x+h$

Dann $a \leq x+h \leq x$

Also $\int_a^x f = \int_a^{x+h} f + \int_{x+h}^x f$

$$\Rightarrow \int_a^{x+h} f = \int_a^x f - \int_{x+h}^x f = \int_a^x f + \int_x^{x+h} f$$

Fall 2.1.2 $x+h \leq a$

Dann $x+h \leq a \leq x$

Also $\int_{x+h}^x f = \int_{x+h}^a f + \int_a^x f$

$$\Rightarrow \int_a^{x+h} f = -\int_{x+h}^a f = \int_a^x f - \int_{x+h}^x f = \int_a^x f + \int_x^{x+h} f$$

Fall 2.2: $a \geq x$

Dann $x+h \leq x \leq a$

Also $\int_{x+h}^a f = \int_{x+h}^x f + \int_x^a f$

$$\Rightarrow \int_a^{x+h} f = -\int_{x+h}^a = \int_x^{x+h} f + \int_a^x f$$

$\square$