Homotopy groups of topological spaces containing a dense directed union of manifolds

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Consider a topological space X which is a union $X = \bigcup_{n=1}^{\infty} X_n$ of topological spaces

$$X_1 \subseteq X_2 \subseteq \cdots ,$$

such that all inclusion maps $X_n \to X_{n+1}$ and $X_n \to X$ are continuous. If every compact subset of X is a compact subset of some X_n (i.e., in the case of *compact regularity*), it is easy to see that the homotopy groups of X can be expressed as direct limits of those of the steps:

$$\pi_k(X,p) \; = \; \varinjlim \, \pi_k(X_n,p), \tag{1}$$

for all $k \in \mathbb{N}$ and $p \in X$. In the talk, I discuss situations where (1) still holds, although compact regularity need not be available. The alternative approach is based on approximation arguments, and assumes (among other things) that each X_n is a (possibly infinite-dimensional) manifold. It applies just as well if the union $\bigcup_{n=1}^{\infty} X_n$ is merely dense in X, and also in the case of uncountable directed unions.

Fundamental groups and second homotopy groups are needed in the extension theory of infinite-dimensional Lie groups (as developed recently by K.-H. Neeb). Many infinite-dimensional Lie groups can be expressed as ascending unions of better-understood Lie groups, or contain such a union as a dense subgroup.

The results entail a solution to a long-standing open problem. Given a Lie group H with Lie algebra $L(H)$, one can consider a certain "weighted mapping group" $\mathcal{S}(\mathbb{R}^\ell, H)$ modelled on the Schwartz space $\mathcal{S}(\mathbb{R}^\ell, L(H))$ of $L(H)$-valued rapidly decreasing functions. Answering a question by Boseck, Czichowski and Rudolph from 1984 in the affirmative, one deduces that always

$$\pi_k(\mathcal{S}(\mathbb{R}^\ell, H)) \; \cong \; \pi_{k+\ell}(H)\,.$$