Tilmann WURZBACHER (Université P.Verlaine-Metz)

Hecke operators on inverse limits of modular curves

This talk presents work in progress, joint with Mike Dostert (Metz).

Motivated by the idea –coming from theoretical physics– of viewing modular forms and their asymptotics via "neo-classical large N-limits" (compare [1]), we consider the modular group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod [N] \right\},$$

the associated moduli spaces $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}_+$ and their compactifications $X_0(N) = \Gamma_0(N)\backslash\mathbb{H}_+^*$, the so-called "modular curves", given by adjoining the appropriate cusps. Modular forms of degree k and weight N are now given as homorphic sections of the k-th power of the canonical line bundle of $X_0(N)$. It is well known that "going with k to infinity" corresponds to the semi-classical limit of letting the Planck constant going to zero. Here we want to study the limit of N going to infinity in a geometric way. Our first result, analogous to known results (see [2]) in the case of

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, a, d \equiv 1 \mod [N] \text{ and } b, c \equiv 0 \mod [N] \right\} \text{ is}$$

Theorem 1. The inverse limit $\varprojlim Y_0(N)$ over N in $\mathbb{N}$ is isomorphic to the space $SL_2(\mathbb{Z})\backslash\{\mathbb{H}_+ \times \mathbb{P}_1(\hat{\mathbb{Z}})\}$ where $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is the product of the p-adic integers for all prime numbers p.

Motivated by the related idea of treating Hecke operators quantum mechanically (as discrete multi-time evolution evolution operators), we construct Hecke operators directly at the limit:

Theorem 2. On an appropriate space of $\mathbb{C}$-valued functions on a principal $\mathbb{C}^*$-bundle over the inverse limit $\varprojlim Y_0(N)$, containing the modular forms for $\Gamma(N)$ for all N and all k, there are infinite weight Hecke operators via index-n lattices and torsion data.

The construction of the Hecke operators is given in terms of summimg over sublattices decorated with sequences fixing "half of the N-torsion of the associated elliptic curves for all N simultaneously". The construction of a "standard Hecke algebra" in Proposition 3.87 in [3] is completely different and relates only to the case of the congruence groups $\Gamma(N)$.

References

[1] S. G. Rajeev, *New Classical Limits of Quantum Theories*, in Infinite Dimensional Groups and Manifolds, *IRMA Lect. Math. Theor. Phys.*, de Gruyter, Berlin, 2002.

[2] David Mumford, *Tata lectures on theta. I*, Progress in Mathematics, Birkhäuser Boston Inc., Boston, MA, 1983.

[3] Alain Connes and Matilde Marcolli, *Noncommutative geometry, quantum fields and motives*, American Mathematical Society, Providence, RI, 2008.