Manifolds of bounded geometry and their diffeomorphism groups

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Let

$$Au = 0 \qquad \text{(D)}$$

be a (linear or non–linear) PDE on a Riemannian manifold (M^n, g), $\mathcal{G}$ the automorphism group of (D), $\mathcal{L}$ the set of all possible solutions of (D), $\mathcal{S}$ the set of all solutions,

$$\mathcal{C} = \mathcal{L}/\mathcal{G}$$

the configuration space,

$$\mathcal{M} = \mathcal{S}/\mathcal{G}$$

the moduli space of (D).

The main task of global analysis consists in establishing $\mathcal{S} \neq \emptyset$ and in "calculating" the topology and geometry of $\mathcal{M}$.

Examples are Einstein equations

$$\mathrm{Ric}\,(g) = \kappa g, \quad \mathcal{G} = \mathrm{Diff}\ (M),$$

equations of gauge theory

$$\delta R^{\omega} = 0, \quad \mathcal{G} = \text{gauge group}.$$

To attack these problems, we need reasonable and natural topologies in $\mathcal{L}$, $\mathcal{G}$, $\mathcal{C}$, $\mathcal{S}$, $\mathcal{M}$, in particular $\mathcal{G}$ as a "good" completed group, with a "good" action and – if any possible – a slice.

For M^n compact, to introduce such topologies is not a serious problem. This has been done by Eells/Palais already 40 years ago.

For M^n open, their approach fails, and the construction of Banach manifolds of maps was for a long time an open problem.

We considered manifolds of bounded geometry and defined appropriate Sobolev uniform structures and by means of them completed spaces of maps

$$\Omega^{p,r}((M, g), (N, h)),$$

the components of which are Sobolev manifolds, for $p = 2$ even Hilbert manifolds.

In the case $(N, h) = (M, g)$, we obtain by restriction completed diffeomorphism groups

$$\mathcal{D}^{p,r}_{\omega}(M, g, \omega),$$

ω a symplectic or volume form, and other ones. The main problem is to establish the uniform structure, i.e. to prove that the defined family of neighbourhoods of the diagonal is a basis for a metrizable uniform structure. This amounts to Sobolev estimates of the derivatives of Jacobi fields which are really very terrible.

References

[1] Eichhorn, J., Global analyis on open manifolds, New York 2007, 664 pp.

[2] Eichhorn, J., The manifold structure of maps between open manifolds, *Annals of Global Analysis and Geometry* 11 (1993), 253–300.

[3] Eichhorn, J., Schmid, R., Form preserving diffeomorphisms on open manifolds, *Annals of Global Analysis and Geometry* 14 (1996), 147–206.

[4] Eichhorn, J., Heber, G., The configuration space of gauge theory on open manifolds of bounded geometry, *Banach Center Publ.* 39 (1997), 269–286.

[5] Eichhorn, J., A slice theorem for open manifolds, *Geom. Dedicata* 91 (2002), 85–116.

[6] Eichhorn, J., Schmid, R., Lie groups of Fourier integral operators on open manifolds, *Comm. Anal. Geom.* 9 (2001), 983–1040.