Lie group structures on groups of holomorphic maps

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This is a report on joint work with Karl-Hermann Neeb (TU Darmstadt).

The question we address is on the existence and uniqueness of regular Lie group structures on the topological groups of smooth maps of the form $\mathcal{C}^\infty(M, K)$, where M denotes a non-compact smooth manifold and K a possibly infinite dimensional Lie group, and of holomorphic maps of the form $\mathcal{O}(M, K)$, where now M is a complex manifold and K a possibly infinite dimensional complex Lie group. Concerning the same question about a compact manifold M, the answer is (well known and) positive with respect to the natural smooth compact open topology for $\mathcal{C}^\infty(M, K)$.

About the unicity, we show that in case K is regular, a regular Lie group structure on $\mathcal{C}^\infty(M, K)$ which is compatible with evaluations (i.e. imposing smoothness of all evaluation maps $\mathrm{ev}_m : \mathcal{C}^\infty(M, K) \to K$, $m \in M$) and with given Lie algebra $\mathfrak{g} = \mathcal{C}^\infty(M, \mathfrak{k})$ (with $\mathfrak{k}$ the Lie algebra of K) is indeed unique.

We then show existence of such a Lie group structure on $\mathcal{C}^\infty(M, K)$ and $\mathcal{O}(M, K)$ for a large class of 1-dimensional manifolds M. In the smooth case, this permits via products to show the existence for manifolds of the form $M = \mathbb{R}^k \times C$ for some compact manifold C.

In the complex case, we show existence for a non-compact complex curve M with finitely generated fundamental group and K a complex Banach Lie group. The proof shows that in this case the subgroup of based maps $\mathcal{O}_*(M, K)$ is a submanifold in the infinite dimensional Fréchet space $\Omega^1_h(M, \mathfrak{k})$ of holomorphic 1-forms with values in $\mathfrak{k}$, embedded via the logarithmic derivative. We use a regular value theorem in infinite dimensions which is based on Glöckner's parametrized implicit function theorem [1].

References

[1] Helge Glöckner, Implicit functions from a topological vector space to Banach spaces. Isr. J. Math. **155** (2006) 205–252

[2] Karl-Hermann Neeb, Friedrich Wagemann, Lie group structures on groups of smooth and holomorphic maps on non-compact manifolds. Geom. Dedicata **134** (2008) 17–60