Flows, diffeomorphism groups, and regularity

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Overview

Flows for complete time-dependent vector fields give rise to curves in diffeomorphism groups.

How do the diffeomorphisms depend on the vector field?

Answer will involve

- ∞-dim calculus and ∞-dim Lie groups
- Lie groups $\mathrm{Diff}(M)$ and $\mathrm{Diff}^{\omega}(M)$ of smooth and real-analytic diffeomorphisms, respectively, for a compact manifold M

Answers also for ODEs with right-hand sides merely L^1 in time

§1 Infinite-dimensional calculus and Lie groups

E, F	locally convex topological vector spaces
$U \subseteq E$	open

Definition (Andrée Bastiani '64)

A map $f\colon U \to F$ is called C^1 if it is continuous, the directional derivative

$$df(x,y) := \lim_{t\to 0} \frac{f(x+ty)-f(x)}{t}$$

exists for each $x \in U$ and $y \in E$, and $df\colon U \times E \to F$ is continuous.

If f is C^1 and df is C^k, say that f is a C^{k+1}-map.

Call f a C^∞-map (or "smooth") if f is C^k for all finite k.

Then $f'(x) := df(x,\cdot)\colon E \to F$ is continuous linear.

Chain Rule: Compositions of composable C^k-maps are C^k.

Remark. In the special case of normed spaces E and F, let us compare with k times continuously Fréchet differentiable mappings (FC^k-maps). Then

$$C^{k+1} \;\Rightarrow\; FC^k \;\Rightarrow\; C^k\,.$$

Notably, same smooth maps, $FC^\infty = C^\infty$.

Returning to general locally convex spaces:

Chain Rule: $\rightsquigarrow$ can define smooth manifolds and Lie groups modeled on a locally convex space, as expected.

Smooth manifold modeled on locally convex space E:

Hausdorff topological space M with maximal atlas of local parametrizations (homeomorphisms from open subsets of E onto open subsets of M) which are C^∞-compatible

Lie group modeled on E:

group G with smooth manifold structure modeled on E turning group operations

$$G \times G \to G, \quad (x, y) \mapsto xy$$

$$G \to G, \quad x \mapsto x^{-1}$$

into smooth maps.

Remark. The main point of a Lie group structure is to have a local parametrization

$$\phi\colon V \to U$$

from an open subset $V \subseteq E$ onto an open e-neighbourhood $U \subseteq G$. Then the translates

$$g\phi\colon V \to gU, \quad x \mapsto g\phi(x)$$

provide an atlas of local parametrizations for G, for $g \in G$.

$\mathfrak{g} := L(G) := T_e G$ **Lie algebra** of G

Main classes of examples:

- **Linear Lie groups** like $A^\times$ for a Banach algebra A and its Lie subgroups (or other topological algebras)
- **Mapping groups** like $C^k(M, G)$ for G a Lie group, M a compact smooth manifold
- **Diffeomorphism groups** like the Lie group $\mathrm{Diff}(M)$ of all C^∞-diffeomorphisms $\psi\colon M \to M$
- **Direct limit groups:** Ascending unions $\bigcup_{n\in\mathbb{N}} G_n$ of finite-dimensional Lie groups $G_1 \subseteq G_2 \subseteq \cdots$, e.g. $\mathrm{GL}_\infty(\mathbb{R}) := \bigcup_{n\in\mathbb{N}} \mathrm{GL}_n(\mathbb{R})$, $A \,\hat{=} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$

General references for §1

Infinite-dimensional calculus, manifolds and Lie groups:

Hamilton '82, Milnor '84, G. '02a, Neeb '06, G.–Neeb '23, Schmeding '23; cf. also Keller '74 (where Bastiani's calculus is called C_c^k-theory)

Compare Kriegl-Michor '97 for an inequivalent approach to infinite-dimensional differential calculus, the so-called convenient differential calculus.

Lie group structure on the main classes of examples of infinite-dimensional Lie groups:

See, for example, Michor '80, Hamilton '82, Milnor '84, Kriegl–Michor '97, Omori '97, G. '02b, G. '02c, G. '05, Neeb '06, G.–Neeb '23.

If E is a locally convex space, give $C^k([0,1],E)$ the topology of uniform convergence of all derivatives of order $j \leq k$. It can be defined using the seminorms

$$C^k([0,1],E) \to [0,\infty[, \quad f \mapsto \sup_{t\in[0,1]} p(f^{(j)}(t))$$

for j as before and continuous seminorms $p\colon E \to [0,\infty[$.

Path groups

Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and G be a Lie group G modelled on a locally convex space E. Then $C^k([0,1],G)$ can be made a Lie group modelled on $C^k([0,1],E)$.

Idea. Pick a local parametrization $\phi\colon V \to U \subseteq G$ around e. Then $C^k([0,1],V)$ is open in $C^k([0,1],E)$ and the map

$$C^k([0,1],\phi)\colon C^k([0,1],V) \to C^k([0,1],U), \quad f \mapsto \phi \circ f$$

is a bijection which can be used as a local parametrization around the neutral element $t \mapsto e$ (cf., e.g., Milnor '84 or G. '02b).

§3 The diffeomorphism group of a compact C^∞-manifold

Let M be a compact smooth manifold and $\Gamma^\infty(TM)$ be the vector space of smooth vector fields $X\colon M \to TM$ on M, with its natural locally convex topology (see appendix).

Thus $X(p) \in T_pM$ for all $p \in M$.

Lie groups of smooth diffeomorphisms

The group $\mathrm{Diff}(M)$ of all C^∞-diffeomorphisms $\psi\colon M \to M$ can be made a Lie group modelled on $\Gamma^\infty(TM)$.

Idea. Pick a Riemannian metric g on M and let $\exp_g\colon TM \to M$ be the Riemannian exponential map. For an open 0-neighbourhood $V \subseteq \Gamma^\infty(TM)$, the map

$$V \to \mathrm{Diff}(M), \quad X \mapsto \exp_g \circ X$$

is a local parametrization around id_M, when considered as a map to its image (cf. Michor '80, Hamilton '82, Milnor '84, Kriegl–Michor '97).

§4 Diffeomorphism groups of real-analytic manifolds

Let M be a compact real-analytic manifold (changes between local parametrizations are locally given by convergent power series). Let $\Gamma^\omega(TM)$ be the locally convex space of real-analytic vector fields $X\colon M \to TM$ on M (see end of talk).

Lie groups of real-analytic diffeomorphisms

The group $\mathrm{Diff}^\omega(M)$ of all real-analytic diffeomorphisms $\psi\colon M \to M$ can be made a Lie group modelled on $\Gamma^\omega(TM)$.

Idea. Pick a real-analytic Riemannian metric g on M (which is possible as M embeds in some $\mathbb{R}^n$, see Grauert '58). Let $\exp_g\colon TM \to M$ be the Riemannian exponential map. For an open 0-neighbourhood $V \subseteq \Gamma^\omega(TM)$, the map

$$V \to \mathrm{Diff}^\omega(M), \quad X \mapsto \exp_g \circ X$$

is a local parametrization around id_M, when considered as a map to its image (cf. Kriegl–Michor '97, Dahmen–Schmeding '15, Leslie '82–'83).

§5 Flows and parameter-dependence

We are interested in differential equations on an open subset $U \subseteq \mathbb{R}^n$ with parameter in an open subset $P \subseteq \mathbb{R}^m$. Let $\mathbb{I} := [0,1]$ and

$$f\colon \mathbb{I} \times U \times P \to \mathbb{R}^n$$

be a C^k-function with $k \in \mathbb{N} \cup \{\infty\}$. For simplicity, assume completeness: For all $t_0 \in \mathbb{I}$, $y_0 \in U$, and $p \in P$, the inital value problem

$$y'(t) = f(t, y(t), p), \quad y(t_0) = y_0$$

has a (necessarily unique) solution $\phi_{t_0,y_0,p}\colon \mathbb{I} \to U$. We get a globally defined flow

$$\mathrm{Fl}\colon \mathbb{I} \times \mathbb{I} \times U \times P \to U, \quad (t, t_0, y_0, p) \mapsto \mathrm{Fl}^p_{t,t_0}(y_0) := \phi_{t_0,y_0,p}(t)\,.$$

It is classical that Fl is C^k. Since

$$\mathrm{Fl}^p_{t_0,t} \circ \mathrm{Fl}^p_{t,t_0} = \mathrm{Fl}^p_{t_0,t_0} = \mathrm{id}_U,$$

the map $\mathrm{Fl}^p_{t,t_0}\colon U \to U$ is a C^k-diffeomorphism for all $t, t_0 \in \mathbb{I}$.

We mention that P may be replaced by an open subset of an arbitrary locally convex space (G.-Neeb '23).

For a compact C^∞-manifold M, consider the differential equation

$$\dot{y}(t) = X_t(y(t))$$

on M with $X \in C^\infty(\mathbb{I}, \Gamma^\infty(TM))$ as a parameter (a time-dependent smooth vector field $t \mapsto X_t$ with smooth time dependence). We deduce that the flow

$$\mathrm{Fl}\colon \mathbb{I} \times \mathbb{I} \times M \times C^\infty(\mathbb{I}, \Gamma^\infty(M)) \to M, (t, t_0, y_0, X) \mapsto \mathrm{Fl}^X_{t,t_0}(y_0)$$

is smooth. Fixing t_0 and using twice an exponential law like

$$C^\infty(U \times V, F) \cong C^\infty(U, C^\infty(V, F)),\ f \mathrel{\hat{=}} (x \mapsto f(x, \cdot)),$$

one can reformulate smooth parameter-dependence as follows:

Theorem A. The following map is smooth:

$$C^\infty(\mathbb{I}, \Gamma^\infty(TM)) \to C^\infty(\mathbb{I}, \mathrm{Diff}(M)), \quad X \mapsto (t \mapsto \mathrm{Fl}^X_{t,t_0}(\cdot))\,.$$

Similar formulations are possible in situations where smoothness of $\mathrm{Fl}\colon \mathbb{I} \times \mathbb{I} \times M \times P \to M$ does not make sense.

If E is a Fréchet space, call $f\colon [0,1] \to E$ **absolutely continuous** if there exists $g \in L^1([0,1],E)$ with

$$f(t) = f(0) + \int_0^t g(s)\,ds \quad \text{for all} \;\; t \in [0,1].$$

Thus $g\colon [0,1] \to E$ is Borel-measurable, $g([0,1])$ is separable and $\|p \circ g\|_{L^1} < \infty$ for each continuous seminorm p on E.

For λ_1-almost all $t \in [0,1]$, the derivative $f'(t)$ exists and equals $g(t)$ (see, e.g., G. '15).

It therefore makes sense to consider absolutely continuous solutions to differential equations

$$y'(t) = f(t, y(t)),$$

requiring equality only λ_1-almost everywhere (cf. also Schechter '97 if E is Banach). Likewise in manifolds:

Definition

A function $f\colon [0,1] \to M$ to a Fréchet manifold is absolutely continuous if it is absolutely continuous piecewise in local charts.

The space $\mathrm{AC}([0,1],E)$ of absolutely continuous functions is a Fréchet space with seminorms

$$f \mapsto \|p \circ f\|_\infty + \|p \circ f'\|_{L^1} .$$

Given a Fréchet–Lie group G, obtain a Fréchet–Lie group

$$\mathrm{AC}([0,1],G)$$

as in the case of $C([0,1],G)$ (see G. '15).

Remark. Similar concepts are available for sequentially complete locally convex spaces (see Thomas '74, Florencio et al. '95 and Nikitin '21; cf. also Lewis '22).

Let M be a compact smooth manifold.

Theorem B (G. '15 and G. '20)

Each $X \in L^1(\mathbb{I}, \Gamma^\infty(TM))$, $t \mapsto X_t$ yields a globally defined flow

$$\mathbb{I} \times \mathbb{I} \times M \to M, \quad (t, t_0, y_0) \mapsto \mathrm{Fl}^X_{t,t_0}(y_0)$$

for the differential equation

$$\dot{y}(t) = X_t(y(t))$$

on M, and $\mathrm{Fl}^X_{t,t_0}(\cdot) \in \mathrm{Diff}(M)$ holds for all $t, t_0 \in \mathbb{I}$. For fixed t_0, the map $\mathbb{I} \to \mathrm{Diff}(M)$, $t \mapsto \mathrm{Fl}^X_{t,t_0}(\cdot)$ is absolutely continuous. For $t_0 = 0$, the following map is smooth

$$L^1(\mathbb{I}, \Gamma^\infty(TM)) \to AC(\mathbb{I}, \mathrm{Diff}(M)), \quad X \mapsto (t \mapsto \mathrm{Fl}^X_{t,t_0}(\cdot))\,.$$

G.'20: If M is a compact real-analytic manifold, can replace $\Gamma^\infty(TM)$ with $\Gamma^\omega(TM)$ and $\mathrm{Diff}(M)$ with $\mathrm{Diff}^\omega(M)$ (Theorem C).

Complementary approach also for non-compact M, emphasizing germs around a point: Jafarpour and Lewis '14.

Compare also Schuricht–von der Mosel '00 and Klose–Schuricht '11.

§8 Lie-theoretic background: regularity properties

Let G be a Lie group modeled on a locally convex space E, with neutral element e; let $\mathfrak{g} := T_e G \cong E$ be its Lie algebra.

For $g \in G$, consider the right translation $\rho_g \colon G \to G$, $x \mapsto xg$. Passing to tangent maps, we get a smooth right action

$$TG \times G \to TG, \quad (v, g) \mapsto v.g := T\rho_g(v)$$

of G on TG. Let $k \in \mathbb{N}_0 \cup \{\infty\}$.

The Lie group G is called ***C^k*-semiregular** if, for each $\gamma \in C^k([0,1], \mathfrak{g})$, there exists a (necessarily unique) C^1-function $\eta \colon [0,1] \to G$ such that

$$\dot{\eta}(t) = \gamma(t).\eta(t) \quad \text{and} \quad \eta(0) = e.$$

Then η is C^{k+1} and we call $\mathrm{Evol}(\gamma) := \eta$ the (right) evolution of γ.

If G is C^k-semiregular and $\mathrm{Evol} \colon C^k([0,1], \mathfrak{g}) \to C([0,1], G)$ is smooth, then G is called ***C^k*-regular** ($\Leftrightarrow C^\infty$ as map to $C^{k+1}([0,1], G)$).

C^k-regularity implies C^ℓ-regularity for all $\ell \geq k$.

Thus C^∞-regularity (introduced by John Milnor in 1984 and abbreviated "regularity") is the weakest property.

Definition (G. '15, Nikitin '21)

For $p \in [1, \infty]$, say that a Lie group G modelled on a sequentially complete locally convex space is ***L^p*-regular** if each $\gamma \in L^p([0,1], \mathfrak{g})$ has a right evolution $\mathrm{Evol}(\gamma) \in AC([0,1], G)$ and $\mathrm{Evol}\colon L^p([0,1], \mathfrak{g}) \to C([0,1], G)$ is smooth (equivalently, as a map to $AC([0,1], G)$).

Then

L^1-reg. $\Rightarrow$ L^p-reg. $\Rightarrow$ L^∞-reg. $\Rightarrow$ C^0-reg. $\Rightarrow$ C^k-reg. $\Rightarrow$ regular

Theorem (Milnor '84)

Let G and H be Lie groups and $\psi\colon \mathfrak{g} \to \mathfrak{h}$ be a continuous Lie algebra homomorphism. If G is 1-connected and H is regular, then $\psi = T_e\phi$ for a smooth group homomorphism $\phi\colon G \to H$.

Every regular Lie group has a smooth exponential function,

$$\exp_G\colon \mathfrak{g} \to G, \quad v \mapsto \mathrm{Evol}(t \mapsto v)(1);$$

i.e., $\exp_G((t+s)v) = \exp_G(tv)\exp_G(sv)$ for all $s, t \in \mathbb{R}$ and $\frac{d}{dt}\Big|_{t=0} \exp_G(tv) = v$.

Theorem (cf. G. '15)

If G is L^∞-regular, then the Trotter product formula holds, $\exp_G(x+y) = \lim_{n\to\infty}(\exp_G(x/n)\exp_G(y/n))^n$.

Hanusch '20: C^0-regularity suffices. Background: Hanusch '22

Useful for representation theory (see, e.g., Neeb-Salmasian '13)

Examples. (a) All Banach–Lie groups are L^1-regular (G. '15).
(b) $A^\times$ is L^1-regular for each locally m-convex Fréchet algebra with open unit group (G. '15).
(c) For $k \in \mathbb{N}_0$, the Banach–Lie group $C^k(M,G)$ is L^1-regular for each compact smooth manifold M and Banach–Lie group G, and also the Fréchet–Lie group $C^\infty(M,G) = \lim\limits_{\leftarrow} C^k(M,G)$ (G. '15).
(d) $\lim\limits_{\rightarrow} G_n$ is L^1-regular for all fin-dim Lie groups $G_1 \subseteq G_2 \subseteq \cdots$ (G. '15; C^0-regularity G. '05).
(e) $\mathrm{Diff}(M)$ is L^1-regular for each compact smooth manifold M, with Evol the map $X \mapsto (t \mapsto \mathrm{Fl}^X_{t,0}(\cdot))$ from Theorem B. Likewise for $\mathrm{Diff}_c(M)$ if M is a paracompact, fin-dim C^∞-mfd (G. '15; C^0-reg. Schmeding '15; regularity for compact M Milnor '84, Kriegl–Michor '97).
(f) $\mathrm{Diff}^\omega(M)$ is L^1-regular for each compact real-analytic manifold M (G. '20; C^1-regularity Dahmen–Schmeding '15).

$\mathrm{Diff}_c(M)$ Lie group of C^∞-diffeos $f\colon M \to M$ s.t. $f(x) = x$ off a compact set; modelled on the locally convex space $\Gamma^\infty_c(TM) = \lim\limits_{\rightarrow} \Gamma^\infty_K(TM)$ of compactly supported smooth vector fields (cf. Michor '80).

Helpful theoretical results available, for example:

Theorem (G. '15 + G. '20)

If Evol exists on a 0-neighbourhood in $L^1([0,1],\mathfrak{g})$ and is continuous at 0, then G is L^1-regular.

Theorem (Hanusch '19)

If a Fréchet–Lie group G is C^k-semiregular, then G is C^k-regular.

Thus: If Evol exists, its smoothness is automatic!

General references: Milnor '84, Kriegl–Michor '97, Neeb '06, G. '15, G. '16, G. '20, Nikitin '21, Hanusch '22, G.–Hilgert '23

Let M be a compact real-analytic manifold. Then the vector space $\Gamma^\omega(TM)$ of real-analytic vector fields can be made a locally convex space and is a so-called **Silva space** (cf. Floret '71 for this notion):

A locally convex space is called a **Silva space** if it is a locally convex direct limit

$$E = \bigcup_{n\in\mathbb{N}} E_n = \lim_{\to} E_n$$

for an ascending sequence $E_1 \subseteq E_2 \subseteq \cdots$ of Banach spaces, such that all inclusion maps $E_n \to E_{n+1}$ are compact operators.

Use seminorms $p\colon E \to [0,\infty[$ with all $p|_{E_n}$ continuous to topologize E

Analysis on Silva spaces works well: A map $f\colon E \to F$ is C^k if and only if $f|_{E_n}$ is C^k for each $n \in \mathbb{N}$ (see, e.g., G.-Neeb '23).

To prove L^1-regularity, one has to prove smoothness of a map on

$$L^1([0,1],E) = \lim_{\to} L^1([0,1],E_n) \qquad \text{(see Florencio et al. '95)}$$

which is not a Silva space; much harder!

Example: The unit circle $\mathbb{S}_1 \subseteq \mathbb{C}$

We describe Banach spaces E_n with $\Gamma^\omega(T\mathbb{S}_1) = \varinjlim E_n$.

As a real submanifold of $\mathbb{C}^\times$ with $T(\mathbb{C}^\times) = \mathbb{C}^\times \times \mathbb{C}$, we have

$$T_z(\mathbb{S}_1) = \mathbb{R}\, iz \quad \text{for all } z \in \mathbb{S}_1.$$

Pick $r_1 > r_2 > \cdots$ with $\lim_{n\to\infty} r_n = 1$ and consider

$$U_n := \left\{ z \in \mathbb{C} \colon \frac{1}{r_n} < |z| < r_n \right\}.$$

Let $(\mathrm{Hol}_b(U_n), \|\cdot\|_\infty)$ be the Banach space of bounded holomorphic functions on U_n and

$$E_n := \{ f \in \mathrm{Hol}_b(U_n) \colon (\forall z \in \mathbb{S}_1)\ f(z) \in \mathbb{R}iz \}$$

be the closed real vector subspace of those functions which restrict to a vector field on $\mathbb{S}_1$. The identity theorem implies that the real linear restriction maps

$$E_n \to \Gamma^\omega(T\mathbb{S}_1), \quad f \mapsto f|_{\mathbb{S}_1}$$

are injective; identifying E_n with its image, get $\Gamma^\omega(T\mathbb{S}_1) = \bigcup_{n\in\mathbb{N}} E_n$ with $E_1 \subseteq E_2 \subseteq \cdots$ and we give it the locally convex direct limit topology.

Pick $r_n' \in \,]r_{n+1}, r_n[$ and define

$$U_n' := \{z \in \mathbb{C} \colon 1/r_n' < |z| < r_n'\};$$

endow the closed vector subspace

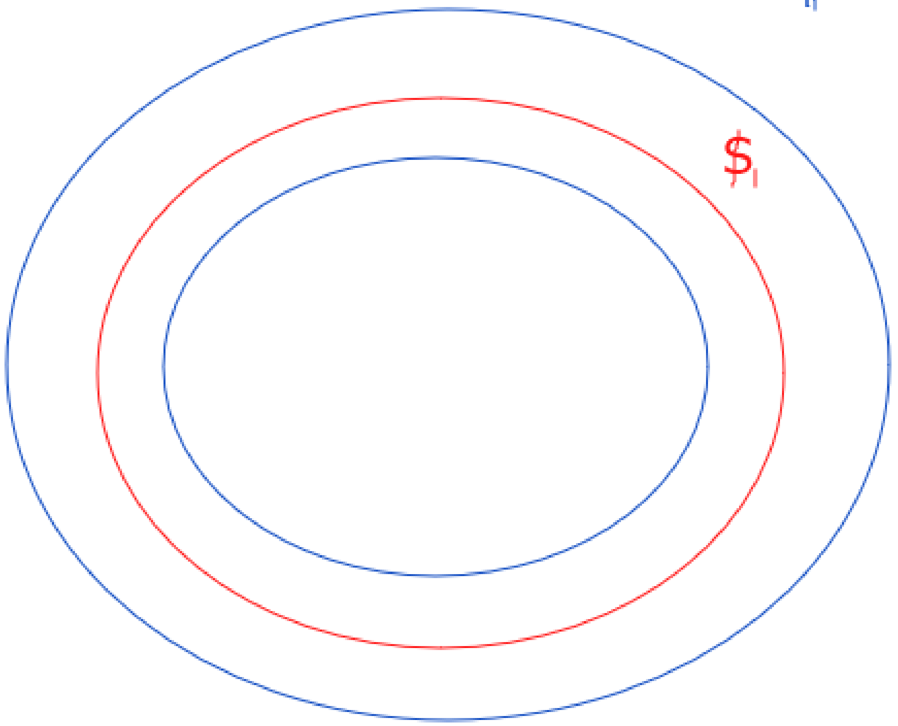
$$E_n' := \{f \in \mathrm{Hol}(U_n') \colon (\forall z \in \mathbb{S}_1)\; f(z) \in \mathbb{R}iz\}$$

of $\mathrm{Hol}(U_n')_{c.o.} \subseteq C(U_n', \mathbb{C})_{c.o.}$ with the compact-open topology. The restriction map $E_n \to E_{n+1}$ is the composition of the continuous linear restriction maps

$$E_n \to E_n' \to E_{n+1}$$

the first of which takes the unit ball to a relatively compact set by Ascoli's Theorem. As a consequence, the composition is a compact operator.

$$U_n = \{ z \in \mathbb{C} : \frac{1}{r_n} < |z| < r_n \}$$



Appendix A: Details concerning real-analytic vector fields*

Let M be an m-dimensional compact real-analytic manifold. We want to turn the space $\Gamma^\omega(TM)$ of real-analytic vector fields into a locally convex space.

There exists a complex m-dimensional complex manifold $\widetilde{M}$ such that $M \subseteq \widetilde{M}$ and $T_p\widetilde{M} = T_pM \oplus iT_pM$ for each $p \in M$. This **complexification** $\widetilde{M}$ can be chosen such that

$$M = \{z \in \widetilde{M} \colon \tau(z) = z\}$$

for an antiholomorphic involution $\tau \colon \widetilde{M} \to \widetilde{M}$ (Bruhat–Whitney '59).

Example. For the complex unit circle $M = \mathbb{S}_1$, the punctured plane $\widetilde{M} = \mathbb{C} \setminus \{0\}$ is a complexification and

$$z \mapsto 1/\overline{z}$$

an antiholomorphic involution.

We pick a basis $U_1 \supseteq U_2 \supseteq \cdots$ of open neighbourhoods of M in $\widetilde{M}$ such that $U_j = \tau(U_j)$.

We endow the space
$$\Gamma^{\mathcal{O}}(TU_j) \subseteq C(U_j, TU_j)$$
of holomorphic vector fields with the compact-open topology and its real vector subspace
$$\Gamma^{\mathcal{O}}(TU_j)_{\mathbb{R}} := \{X \in \Gamma^{\mathcal{O}}(TU_j) \colon T\tau \circ X \circ \tau = X\}$$
with the induced topology. Then
$$\Gamma^{\mathcal{O}}(TU_j) = \Gamma^{\mathcal{O}}(TU_j)_{\mathbb{R}} \oplus i\Gamma^{\mathcal{O}}(TU_j)_{\mathbb{R}}.$$
The maps
$$\Gamma^{\mathcal{O}}(TU_j) \to \Gamma^{\omega}(TM), \quad X \mapsto X|_M$$
being injective (if each component of U_j meets M), get vector subspaces
$$\Gamma^{\mathcal{O}}(TU_1)_{\mathbb{R}} \subseteq \Gamma^{\mathcal{O}}(TU_2)_{\mathbb{R}} \subseteq \cdots$$
of $\Gamma^{\omega}(TM)$. Give $\Gamma^{\omega}(TM)$ the topology of the locally convex direct limit
$$\varinjlim \Gamma^{\mathcal{O}}(TU_j)_{\mathbb{R}}.$$

Appendix B: Topology on the space of C^∞ vector fields*

If M and N are C^k-manifolds modelled on locally convex spaces, give $C^k(M,N)$ the initial topology with respect to the maps

$$C^k(M,N) \to C(T^jM, T^jN)_{c.o.}, \quad f \mapsto T^jf \quad \text{for } j \in \mathbb{N}_0 \text{ with } j \leq k,$$

where $T^jM := T(T^{j-1}M)$ are the iterated tangent bundles and $T^jf := T(T^{j-1}f)$ "compact-open C^k-topology"

Facts (see, e.g., Neeb '06 or G.–Neeb '23)

(a) If E is a locally convex space, then also $C^k(M,E)$.

(b) Let $k=\infty$ and $\pi_{TM}\colon TM \to M$, $T_xM \ni v \mapsto x$ be the bundle projection. The induced topology turns the vector space

$$\Gamma^\infty(TM) = \{X \in C^\infty(M,TM) \colon \pi_{TM} \circ X = \mathrm{id}_M\}$$

of smooth vector fields into a locally convex space.

If M is σ-compact and finite-dimensional, then $\Gamma^\infty(TM)$ is a Fréchet space.

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