Flows, parameter dependence, and diffeomorphism groups

Helge Glöckner (Paderborn)

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Overview

Flows for complete time-dependent vector fields give rise to curves in diffeomorphism groups.

How do the diffeomorphisms depend on the vector field?

- differential calculus in locally convex spaces
- examples of flows and diffeomorphism groups
- evolution equations on Lie groups (regularity)
- differential equations on G-manifolds

E, F — locally convex topological vector spaces

$U \subseteq E$ — open

$k \in \mathbb{N}_0 \cup \{\infty\}$

Definition (Andrée Bastiani, 1964)

A map $f\colon U \to F$ is called C^k if it is continuous, the iterated directional derivatives

$$d^j f(x, y_1, \ldots, y_j) := (D_{y_j} \cdots D_{y_1} f)(x)$$

exists at $x \in U$ for all $j \in \mathbb{N}$ with $j \leq k$ and $y_1, \ldots, y_j \in E$, and $d^j f\colon U \times E^j \to F$ is continuous.

C^∞-maps are also called **smooth**.

Chain Rule $\rightsquigarrow$ can define smooth manifolds and Lie groups modeled on a locally convex space

Smooth manifold modeled on locally convex space E:

Hausdorff topological space M with maximal atlas of local parametrizations (homeomorphisms from open subsets of E onto open subsets of M) which are C^∞-compatible

Lie group modeled on E:

group G with smooth manifold structure modeled on E turning group operations into smooth maps

$\mathfrak{g} := L(G) := T_e G$ **Lie algebra** of G

Every C^1-map $f\colon E \supseteq U \to F$ is locally Lipschitz:

For each continuous seminorm q on F and $x_0 \in U$, there exists a continuous seminorm p on E such that

$$q(f(y) - f(x)) \leq p(y - x)$$

for all x, y in some x_0-neighbourhood U_q in U.

General references: Milnor 1984, G. 2002, Neeb 2006, G/Neeb 2021

§2 The diffeomorphism group of a compact convex set

$K \subseteq \mathbb{R}^n$ compact, convex subset with dense interior
e.g. $[0,1] \subseteq \mathbb{R}$, $[0,1]^2 \subseteq \mathbb{R}^2$, closed disc in $\mathbb{R}^2$, closed ball in $\mathbb{R}^3$

$C^\infty(K,\mathbb{R}^n)$ space of all $f\colon K \to \mathbb{R}^n$ having all partial derivatives on K° with continuous extensions to K

$C^\infty_{\partial K}(K,\mathbb{R}^n)$ vector subspace of all f such that $f|_{\partial K} = 0$

Locally convex spaces with seminorms $\|\partial^{|\alpha|} f/\partial x^\alpha\|_\infty$

$\mathrm{Diff}_{\partial K}(K)$ set of all C^∞-diffeomorphisms $f\colon K \to K$ such that $f(x) = x$ for all $x \in \partial K$

Theorem (G./Neeb 2017)

$\mathrm{Diff}_{\partial K}(K)$ is a group and $\Omega := \{f - \mathrm{id}_K \colon f \in \mathrm{Diff}_{\partial K}(K)\}$ is an open subset of $C^\infty_{\partial K}(K,\mathbb{R}^n)$. Make $\mathrm{Diff}_{\partial K}(K)$ a C^∞-manifold with

$$\Omega \to \mathrm{Diff}_{\partial K}(K), \quad f \mapsto \mathrm{id}_K + f$$

as a global parametrization. Then $\mathrm{Diff}_{\partial K}(K)$ is a Lie group.

Consider a time-dependent smooth vector field on K which vanishes on ∂K, with C^k-dependence on time, $k \in \mathbb{N}_0 \cup \{\infty\}$.

That is, we consider a C^k-map

$$X\colon [0,1] \to C^\infty_{\partial K}(K, \mathbb{R}^n), \quad t \mapsto X_t.$$

For all $t_0 \in [0,1]$ and $y_0 \in K$, the initial value problem

$$y'(t) = X_t(y(t)), \quad y(t_0) = y_0$$

on K has a unique solution $\gamma_{t_0,y_0}\colon [0,1] \to K$; **flow**

$$\mathrm{Fl}^X\colon [0,1] \times [0,1] \times K \to K, \quad (t, t_0, y_0) \mapsto \mathrm{Fl}_{t,t_0}(y_0) := \gamma_{t_0,y_0}(t).$$

For all $t, t_0 \in [0,1]$, consider $\mathrm{Fl}_{t,t_0}\colon K \to K$, $y_0 \mapsto \mathrm{Fl}_{t,t_0}(y_0)$.

Theorem (G/Neeb 2017)

For each $X \in C^k([0,1], C^\infty_{\partial K}(K, \mathbb{R}^n))$, we have $\mathrm{Fl}^X_{t,t_0} \in \mathrm{Diff}_{\partial K}(K)$ for all $t, t_0 \in [0,1]$. The map

$$\mathrm{Evol}(X)\colon [0,1] \to \mathrm{Diff}_{\partial K}(K), \quad t \mapsto \mathrm{Fl}^X_{t,t_0}$$

is C^{k+1} and the following map is smooth:

$$\mathrm{Evol}\colon C^k([0,1], C^\infty_{\partial K}(K, \mathbb{R}^n)) \to C^{k+1}([0,1], C^\infty(K, \mathbb{R}^n)).$$

In particular,

$$\mathrm{Evol}\colon C([0,1], C^\infty_{\partial K}(K,\mathbb{R}^n)) \to C^1([0,1], C^\infty(K,\mathbb{R}^n))$$

is smooth.

Now consider a time-dependent vector field $X\colon [0,1] \to C^\infty_{\partial K}(K,\mathbb{R}^n)$ which is **piecewise continuous** in time.

Thus, there exists a subdivision $0 = t_0 < \cdots < t_m = 1$ such that $X|_{]t_j,t_{j+1}[}$ has a continuous extension

$$X_j\colon [t_j, t_{j+1}] \to C^\infty_{\partial K}(K,\mathbb{R}^n).$$

For example, X may be **piecewise constant** (an $C^\infty_{\partial K}(K,\mathbb{R}^n)$-valued staircase function).

Flow for $t_0 := 0$:

$$\mathrm{Fl}^X_{t,0} = \mathrm{Fl}^{X_j}_{t,t_j} \circ \mathrm{Fl}^{X_{j-1}}_{t_j,t_{j-1}} \circ \cdots \circ \mathrm{Fl}^{X_0}_{t_1,t_0} \in \mathrm{Diff}_{\partial K}(K) \quad \text{if } \ t \in [t_j, t_{j+1}].$$

Then the following map is piecewise C^1:

$$\mathrm{Evol}(X)\colon [0,1] \to \mathrm{Diff}_{\partial K}(K), \ \ t \mapsto \mathrm{Fl}^X_{t,0}.$$

Identify piecewise continuous X and Y if $X(t) = Y(t)$ for all but finitely many $t \in [0,1]$; get locally convex space

$$C_{\mathrm{pw}}([0,1], C^\infty_{\partial K}(K, \mathbb{R}^n))$$

with seminorms $\|q \circ X\|_{L^1}$, for q in set of continuous seminorms on $C^\infty_{\partial K}(K, \mathbb{R}^n)$. **Lie theoretic facts imply:**

Evol: $C_{\mathrm{pw}}([0,1], C^\infty_{\partial K}(K, \mathbb{R}^n)) \to \mathrm{AC}([0,1], C^\infty(K, \mathbb{R}^n))$ is a smooth map to the Fréchet space of absolutely continuous $C^\infty(K, \mathbb{R}^n)$-valued functions.

Notably Evol is continuous (and locally Lipschitz). Thus:

If $X \in C([0,1], C^\infty_{\partial K}(K, \mathbb{R}^n))$ and $(X_n)_{n\in\mathbb{N}}$ is a sequence of piecewise constant vector fields $X_n\colon [0,1] \to C^\infty_{\partial K}(K, \mathbb{R}^n)$ such that $X_n \to X$ in the L^1-topology, then $\mathrm{Evol}(X_n) \to \mathrm{Evol}(X)$ uniformly

and even in $\mathrm{AC}([0,1], C^\infty(K, \mathbb{R}^n))$.

If E is a Fréchet space, call $f\colon [0,1]\to E$ **absolutely continuous** if there exists $g\in L^1([0,1],E)$ (a Bochner-integrable function) with

$$f(t)=f(0)+\int_0^t g(s)\,ds \quad \text{for all } \; t\in[0,1].$$

For λ_1-almost all $t\in[0,1]$, the derivative $f'(t)$ exists and equals $g(t)$.

The absolutely continuous E-valued functions form a vector space $\mathrm{AC}([0,1],E)$. It is a Fréchet space with respect to the seminorms

$$\|f\|_{\mathrm{AC},q}:=\max\{\|q\circ f\|_\infty,\|q\circ f'\|_{L^1}\},$$

for q in the set of continuous seminorms on E.

§3 Lie-theoretic background

Let G be a Lie group modeled on a locally convex space E, with neutral element e; let $\mathfrak{g} := T_e G \cong E$ be its Lie algebra.

For $g \in G$, consider the right translation $\rho_g \colon G \to G$, $x \mapsto xg$. Passing to tangent maps, we get a smooth right action

$$TG \times G \to TG, \ \ (v, g) \mapsto v.g := T\rho_g(v)$$

of G on TG. Let $k \in \mathbb{N}_0 \cup \{\infty\}$.

The Lie group G is called ***C^k*-semiregular** if, for each $\gamma \in C^k([0,1], \mathfrak{g})$, there exists a (necessarily unique) C^1-function $\eta \colon [0,1] \to G$ such that

$$\dot{\eta}(t) = \gamma(t).\eta(t) \quad \text{and} \quad \eta(0) = e.$$

Then η is C^{k+1} and we call $\mathrm{Evol}(\gamma) := \eta$ the (right) evolution of γ.

If G is C^k-semiregular and $\mathrm{Evol} \colon C^k([0,1], \mathfrak{g}) \to C^{k+1}([0,1], G)$ is smooth, then G is called ***C^k*-regular**.

C^k-regularity implies C^ℓ-regularity for all $\ell \geq k$.

Thus C^∞-regularity (introduced by John Milnor in 1984 and abbreviated "regularity") is the weakest property. We used:

For each $k \in \mathbb{N}_0 \cup \{\infty\}$ and Lie group G modeled on E, the group $C^k([0,1],G)$ can be made a Lie group modeled on $C^k([0,1],E)$.

Let $\phi\colon E \subseteq V \to U \subseteq G$ be a local parametrization with $e \in U$; then $C^k([0,1],V)$ is open in $C^k([0,1],E)$ and the bijection

$$\phi_*\colon C^k([0,1],V) \to C^k([0,1],U) \subseteq C^k([0,1],G), \;\; f \mapsto \phi \circ f,$$

together with it right translates $f \mapsto \phi_*(f)g$ with $g \in C^k([0,1],G)$, can be used as a C^∞-atlas of local parametrizations for $C^k([0,1],G)$.

Likewise, $\mathrm{AC}([0,1],G)$ is a Lie group modeled on $\mathrm{AC}([0,1],E)$ if G is modeled on a sequentially complete locally convex space E

Say that G is **L^1-regular** if each $\gamma \in L^1([0,1],\mathfrak{g})$ has a right evolution $\mathrm{Evol}(\gamma) \in AC([0,1],G)$ and $\mathrm{Evol}\colon L^1([0,1],\mathfrak{g}) \to AC([0,1],G)$ is smooth.

L^1-regularity implies C^0-regularity.

Theorem (Milnor 1984)

Let G and H be Lie groups and $\psi\colon \mathfrak{g} \to \mathfrak{h}$ be a continuous Lie algebra homomorphism. If G is 1-connected and H is regular, then $\psi = T_e\phi$ for a smooth group homomorphism $\phi\colon G \to H$.

Every regular Lie group has an exponential function,

$$\exp_G\colon \mathfrak{g} \to G, \quad v \mapsto \mathrm{Evol}(t \mapsto v)(1).$$

Theorem (G. 2015)

If G is L^1-regular, then the Trotter product formula holds, $\exp_G(x+y) = \lim_{n\to\infty}(\exp_G(x/n)\exp_G(y/n))^n$.

Hanusch 2020: C^0-regularity suffices. Background: Hanusch 2017.

Compare Theorem 7.6 in G/Hilgert 2020 and Theorem 1.6 in G. 2020 for the following fact; cf. also earlier work by Hanusch.

Theorem

If G is C^0-regular, then $\mathrm{Evol}\colon C_{\mathrm{pw}}([0,1],\mathfrak{g}) \to \mathrm{AC}([0,1],G)$ is smooth with respect to the L^1-topology.

Examples

(a) $\mathrm{Diff}_{\partial K}(K)$ is C^0-regular (G/Neeb 2017) with Evol as in §2

(b) $\mathrm{Diff}_c(M)$ (as in Michor 1980) is L^1-regular for each paracompact, finite-dimensional smooth manifold M (G. 2015) (C^0-regularity Schmeding 2015)

(c) $\mathrm{Diff}_\omega(M)$ (as in Kriegl-Michor '97 or Dahmen-Schmeding '15) is L^1-regular for each compact real-analytic manifold M (G. 2020)

$\mathrm{Diff}_c(M)$ Lie group of C^∞-diffeos $f\colon M \to M$ s.t. $f(x) = x$ off a compact set
$\mathrm{Diff}_\omega(M)$ Lie group of all real-analytic diffeomorphisms of M

General references: Milnor 1984, Kriegl-Michor 1997, Neeb 2006, G. 2015, G. 2016, G. 2020, Nikitin 2021

If G is a Lie group, M a smooth manifold (both modeled on locally convex spaces) and $G \times M \to M$, $(g, p) \mapsto g.p$ a smooth left action, then each $v \in \mathfrak{g}$ yields a **fundamental vector field**

$$v_\sharp \colon M \to TM, \quad v_\sharp(p) := (d/dt)\big|_{t=0} \exp_G(tv).p$$

Theorem (G/Hilgert 2020)

If $\gamma \colon [0,1] \to \mathfrak{g}$ admits a right evolution $\mathrm{Evol}(\gamma) \colon [0,1] \to G$, then the ODE

$$\dot{y}(t) = (\gamma(t))_\sharp(y(t))$$

on M satisfies local existence and uniqueness of solutions with flow $[0,1] \times [0,1] \times M \to M$, $(t, t_0, y_0) \mapsto \mathrm{Evol}(\gamma)(t)\,\mathrm{Evol}(\gamma)(t_0)^{-1}.y_0$.

Let $y_0 \in M$, $p \in M$ and $U \subseteq M$ be a p-neighborhood. Fix $T > 0$. Call U **reachable** for controls in a subset $S \subseteq L^1([0,T], \mathfrak{g})$ if, for some $\gamma \in S$, we have $\eta(T) \in U$ for the solution $\eta \colon [0,T] \to M$ to

$$\dot{\eta}(t) = \gamma(t)_\sharp(\eta(t)), \quad \eta(t_0) = y_0.$$

Theorem (G/Hilgert 2020)

If G is C^0-regular, then the following are equivalent:

(a) U can be reached using continuous controls;

(b) U can be reached using piecewise continuous controls;

(c) U can be reached using piecewise constant controls.

If G is L^1-regular, then also the following condition is equivalent:

(d) U can be reached using L^1-controls.

If $K \subseteq \mathfrak{g}$ is a compact convex set, then (a)–(c) (resp. (a)–(d)) remain valid for functions with values in K, and equivalenty

(e) U can be reached using a piecewise constant control function with values in the set $\mathrm{ex}(K)$ of extreme points of K ("bang-bang-principle").

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