A non-linear Lie group

Let G be a Lie group and ρ a representation of G. Then ρ is said to be faithful if the map $G \ni g \mapsto \rho(g)$ is injective. A linear Lie group is a Lie group that can be realized as a Lie group consisting of matrices. In other words, a Lie group G is linear if there exists a finite dimensional faithful representation of G. Let

$$G = \begin{pmatrix} 1 & \mathbf{R} & \mathbf{R}/\mathbf{Z} \\ 0 & 1 & \mathbf{R} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{R} & \mathbf{R} \\ 0 & 1 & \mathbf{R} \\ 0 & 0 & 1 \end{pmatrix} \Big/ \begin{pmatrix} 1 & 0 & \mathbf{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Show that $\begin{pmatrix} 1 & 0 & \mathbf{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a closed normal subgroup of $\begin{pmatrix} 1 & \mathbf{R} & \mathbf{R} \\ 0 & 1 & \mathbf{R} \\ 0 & 0 & 1 \end{pmatrix}$. Conclude that G is a Lie group.

In this exercise we will prove that there exists no finite dimensional faithful representation ρ of G and hence that G is not linear. Working towards a contradiction, assume that (ρ, V) is a finite dimensional faithful representation of G. Let $p \in \mathbf{N}$ be prime and let

$$x = \begin{pmatrix} 1 & 1 & \mathbf{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 & \frac{1}{p} + \mathbf{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) Show that V decomposes as a direct sum of eigenspaces for $\rho(y)$. (Hint:

$$T = \begin{pmatrix} 1 & 0 & \mathbf{R}/\mathbf{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a compact abelian subgroup of G.)

(iii) Prove that $\rho(y)$ has at least one eigenvalue α that is a p^{th} root of unity that is not equal to 1.

Let V_α be the eigenspace of $\rho(y)$ for the eigenvalue α.

(iv) Prove that V_α is a G-invariant subspace.

(v) Prove that xy is conjugate to x, i.e. there exists an element $g \in G$ such that $gxyg^{-1} = x$.

(vi) Prove that the restriction of $\rho(x)$ to V_α is conjugate to the restriction of $\alpha\rho(x)$ to V_α. In other words, prove that there exists an $A \in \mathrm{Aut}(V_\alpha)$ such that $\rho(x) = A\big(\alpha\rho(x)\big)A^{-1}$ on V_α.

(vii) Show that the restriction of $\rho(x)$ to V_α has at least p distinct eigenvalues.

(viii) Prove that the dimension of V is at least p. Use this to reach a contradiction.