A PERTURBATION THEOREM FOR OPERATOR SEMIGROUPS IN HILBERT SPACES

CORNELIA KAISER AND LUTZ WEIS

Abstract. We prove a perturbation result for C_0 semigroups on Hilbert spaces and use it to show that certain operators of the form $Au = iu^{(2k)} + V \cdot u^{(l)}$ on $L^2(\mathbb{R})$ generate a semigroup that is strongly continuous on $(0, \infty)$.

1. Introduction

Perturbation theory of C_0-semigroups is an important tool in applications to differential equations. A minimal condition in many of the known perturbation theorems is the relative boundedness of the perturbation B in terms of the given semigroup generator A. Often these relative boundedness conditions are expressed as

$$\|B(\lambda - A)^{-1}\| \leq M < 1 \tag{1}$$

or

$$\|(\lambda - A)^{-1}Bx\| \leq M\|x\| \tag{2}$$

on a certain subset of the complex plane. E.g., in the proof of the well-known result for bounded perturbations (see e.g. [5, Chapter III, Theorem 1.3], [7, Chapter 3, Theorem 1.1]) condition (1) is one of the main ideas. The Miyadera-Voigt, respectively Desch-Schappacher, perturbation theorem uses (1), respectively (2) (see [5, Chapter III, Section 3]). If A generates a bounded analytic semigroup, then condition (1), satisfied for all λ in the right half plane, is sufficient to show that $A + B$ again generates an analytic semigroup. Clearly, this cannot be true for general C_0-semigroups. But in this paper we want to explore what can be said about $A + B$ if we only assume the relative boundedness conditions (1) and (2) on a halfplane. If the underlying space is a Hilbert space, we can show that $(A + B, D(A))$ generates a semigroup that is strongly continuous on $(0, \infty)$.

This paper is organized as follows. In the second section we collect some facts about semigroups that are strongly continuous on $(0, \infty)$. Section 3 contains our main results which are proved in Sections 4 and 5. In Section 6 we apply our theorem to certain differential operators.

2. Semigroups that are strongly continuous on $(0, \infty)$

Let X be a Banach space. By $\mathcal{L}(X)$ we denote the Banach space of all bounded linear operators from X to X. If $T : (0, \infty) \to \mathcal{L}(X)$ is a strongly continuous mapping (i.e., $t \mapsto T(t)x$ is continuous on $(0, \infty)$ for each $x \in X$) that satisfies the *semigroup property* $T(t)T(s) = T(t + s)$ for all $t, s > 0$, then we say that the family

2000 *Mathematics Subject Classification.* Primary 47A55, 47D06. Secondary 47D62, 47E05.
Key words and phrases. C_0-semigroup, Perturbation.
The research is supported in part by the *Landesforschungsschwerpunkt Evolutionsgleichungen des Landes Baden-Württenberg.*

$(T(t))_{t>0}$ is a *(operator) semigroup that is strongly continuous on* $(0,\infty)$. Examples for such semigroups can be found in [3], [6, Section I.8] and [5, Chapter I, 5.9 (7)].

In this paper we want to use Laplace transform methods. Therefore we will assume from now on that the mapping T is locally integrable on $(0,\infty)$ (i.e., $T \in L^1((0,b);\mathcal{L}(X))$ for every $b>0$) and

$$\left\| \int_0^t T(s)\,ds \right\| \leq Me^{\omega t}, \qquad t>0, \tag{3}$$

for some constants M and ω. Then, due to [2, Proposition 1.4.5], we can define the Laplace transform for $\lambda > \omega$. Using integration by parts and the semigroup property, we find that $(R(\lambda))_{\lambda>\omega}$ satisfies the resolvent equation $R(\lambda) - R(\mu) = (\mu-\lambda)R(\lambda)R(\mu)$. Therefore the following definition makes sense.

Definition 2.1. Let $(T(t))_{t>0}$ be a semigroup on a Banach space X that is strongly continuous and locally integrable on $(0,\infty)$ and satisfies the norm estimate (3). If there exists a linear operator $(A, D(A))$ in X, where $D(A) \subseteq X$ is the domain of A, such that (ω,∞) is contained in the resolvent set $\rho(A)$ of A and

$$R(\lambda, A) := (\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} T(t)\,dt, \qquad \lambda > \omega,$$

then $(A, D(A))$ is called the *generator* of $(T(t))_{t>0}$.

Using this definition, one can show easily the following properties of the semigroup $(T(t))_{t>0}$ and its generator A:

(a) if $x \in D(A)$, then $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$ for every $t>0$,

(b) if $x \in D(A)$ and $t>0$, then $x = T(t)x - \int_0^t T(s)Ax\,ds$.

The properties (a) and (b) imply that for $x \in D(A)$ the function u_x, defined by $u_x(t) := T(t)x$ $(t>0)$ and $u_x(0) = x$, is a solution of the abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t), & t>0, \\ u(0) = x. \end{cases} \tag{4}$$

Here, by a solution of (4) we mean a function $u \in C([0,\infty);X) \cap C^1((0,\infty);X)$ such that $u(t) \in D(A)$ and $u'(t) = Au(t)$ for every $t>0$ and $u(0) = x$ (see [6, Chapter 1, Definition 3.1], [8, Section 1]).

3. Main result

Our main result is the following perturbation theorem for C_0-semigroups on Hilbert spaces.

Theorem 3.1. *Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X and let $(B, D(B))$ be a closed operator in X such that $D(B) \supseteq D(A)$. We assume that there exist constants $M \in [0,1)$ and $\lambda_0 \in \mathbb{R}$ such that the set $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \lambda_0\}$ is contained in the resolvent set of A and the estimates*

$$\|BR(\lambda, A)x\| \leq M\|x\| \tag{5}$$

and

$$\|R(\lambda, A)By\| \leq M\|y\| \tag{6}$$

are satisfied are satisfied for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \lambda_0$ and all $x \in X$, $y \in D(B)$. Then $(A+B, D(A))$ generates a semigroup $(S(t))_{t\geq 0}$ that is strongly continuous on $(0,\infty)$.

The following example is a modification of the examples in [1, Section 3] and illustrates that it is not possible to drop condition (5) in the theorem. Using duality one can construct a similar example showing that the same is true for condition (6).

Example 3.2. Let $X = L^2(0, \infty)$. We define linear operators $(A, D(A))$ and $(B, D(B))$ by $(Af)(x) := f'(x)$ and $(Bf)(x) := \frac{1}{3x} f(x)$ with maximal domains. Using Hardy's Inequality, we can show that $\|R(\lambda, A)Bx\|_2 \leq \frac{2}{3}\|x\|_2$ for all $\operatorname{Re}\lambda > 0$, i.e. condition (6) is satisfied. The "candidate" for the perturbed semigroup is $S(t)f(x) := x^{-1/3}(x+t)^{1/3} f(x+t)$. But $S(t)$ is not a bounded operator on X.

It is still an open question whether the result of Theorem 3.1 is optimal, i.e. whether one can show strong continuity at 0 of the perturbed semigroup.

To prove Theorem 3.1 we will use the following result about generators for semigroups that are strongly continuous on $(0, \infty)$.

Theorem 3.3. *Let $(A, D(A))$ be a closed, densely defined operator on a Hilbert space X such that the resolvent $R(\lambda, A)$ exists and is uniformly bounded on $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\}$. Further, we assume that there exists a constant $C \geq 0$ such that*

$$\left(\int_{-\infty}^{\infty} \|R(is, A)x\|^2 ds \right)^{1/2} \leq C\|x\| \tag{7}$$

and

$$\left(\int_{-\infty}^{\infty} \|R(is, A^*)x\|^2 ds \right)^{1/2} \leq C\|x\| \tag{8}$$

for all $x \in X$. Then $(A, D(A))$ generates a semigroup $(T(t))_{t \geq 0}$ that is strongly continuous on $(0, \infty)$.

In this case, we see by the following example due to Kreĭn ([6]) that in general the operator A in Theorem 3.3 is not the generator of a C_0-semigroup.

Example 3.4. We consider the space $X = L^2(\mathbb{R}) \times L^2(\mathbb{R})$ which is a Hilbert space if we choose the norm $\|(u, v)\|_X := (\|u\|_2^2 + \|v\|_2^2)^{1/2}$. For $k \in \mathbb{N}$ and $\alpha \in [0, 4k)$ we define the function $a : \mathbb{R} \to \mathbb{R}^2$ by

$$a(x) := \begin{pmatrix} -1 - x^{2k} & x^{\alpha} \\ 0 & -1 - x^{2k} \end{pmatrix}. \tag{9}$$

Then the multiplication operator, given by

$$A(u, v) = a\tbinom{u}{v}, \qquad D(A) = \{(u, v) \in X : \ A(u, v) \in X\}, \tag{10}$$

satisfies the conditions of Theorem 3.3, hence A generates a semigroup that is strongly continuous on $(0, \infty)$. But if $\alpha \in (2k, 4k)$, then A is not strongly continuous at 0.

4. Proof of Theorem 3.3

In this section we give a proof of Theorem 3.3. We first state two technical lemmas.

Lemma 4.1. *Let $(A, D(A))$ be a closed operator in a Banach space X with $0 \in \rho(A)$. If we can find a subset G of $\rho(A)$ and a constant $M \geq 0$ such that $\|R(\lambda, A)\| \leq M$ on G, then there is a constant $c \geq 0$ such that*

$$\|R(\lambda, A)x\| \leq \tfrac{c}{1+|\lambda|}\|Ax\| \qquad \textit{and} \qquad \|R(\lambda, A)^2 y\| \leq \tfrac{c}{1+|\lambda|^2}\|A^2 y\|$$

for every $\lambda \in G$ and every $x \in D(A)$, $y \in D(A^2)$.

Proof. For $\lambda \in G \setminus \{0\}$ and $x \in D(A)$ the resolvent $R(\lambda, A)x$ can be written as $R(\lambda, A)x = \frac{1}{\lambda}(x + R(\lambda, A)Ax)$. If $y \in D(A^2)$ we obtain $R(\lambda, A)^2 y = \frac{1}{\lambda^2}(y + 2R(\lambda, A)Ay + R(\lambda, A)^2 A^2 y)$. Since 0 is in the resolvent set of A and the resolvent is uniformly bounded on G, the lemma is proved. □

Lemma 4.2. *Let $(A, D(A))$ be a closed operator in a Banach space X such that $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subseteq \rho(A)$ and $\|R(\lambda, A)\| \leq M$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$. For $x \in X$, $t > 0$ and $a \geq 0$ we define*

$$U(t)x := \frac{1}{2\pi i t} \int_{a-i\infty}^{a+i\infty} e^{\mu t} R(\mu, A)^2 x \, d\mu. \tag{11}$$

Then,

(a) if $x \in D(A^2)$, the integral in (11) is absolutely convergent and does not depend on $a \geq 0$,

(b) for all $x \in D(A^2)$ and all $t > 0$, the limit

$$\lim_{r \to \infty} \frac{1}{2\pi i} \int_{a-ir}^{a+ir} e^{\mu t} R(\mu, A)x \, d\mu \tag{12}$$

exists and is equal to $U(t)x$,

(c) for $x \in D(A^2)$ and $\operatorname{Re}\lambda > 0$, we have that

$$R(\lambda, A)x = \frac{x}{\lambda} + \int_0^\infty e^{-\lambda t}(U(t)x - x)dt, \tag{13}$$

(d) the semigroup property

$$U(t)U(s)x = U(t+s)x$$

holds for all $t, s > 0$ and all $x \in D(A^4)$.

Proof. Let $x \in D(A^2)$ and $t, s > 0$.
(a) Lemma 4.1 implies that the integral in (11) converges absolutely. The independence of $a \geq 0$ is a consequence of Cauchy's Theorem.
(b) Integration by parts yields that for $r > 0$

$$\int_{a-ir}^{a+ir} e^{\mu t} R(\mu, A)x \, d\mu = \frac{1}{t}(e^{a+irt} R(a+ir, A)x - e^{a-irt} R(a-ir, A)x) \\ + \frac{1}{t} \int_{a-ir}^{a+ir} e^{\mu t} R(\mu, A)^2 x \, d\mu.$$

By Lemma 4.1, $\|R(ir, A)x\|$ converges to 0 if $|r| \to \infty$. Therefore we have that the limit (12) exists and is equal to $U(t)x$.
(c) Let $\operatorname{Re}\lambda > 0$. If $x \in D(A)$, $t > 0$ and $0 < a < \operatorname{Re}\lambda$, we find

$$U(t)x - x = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\mu t} \left(R(\mu, A)x - \frac{x}{\mu} \right) d\mu = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\mu t} R(\mu, A)Ax \, \frac{d\mu}{\mu}.$$

For $x \in D(A^2)$, Lemma 4.1 yields $\|R(\mu, A)Ax\| \leq \frac{c}{1+|\mu|}\|A^2 x\|$. Therefore the above integral is absolutely convergent and $\|U(t)x - x\| \leq c'\|A^2 x\|$ for all $t > 0$. So we can form the Laplace transform of $U(t)x - x$ and obtain

$$\begin{aligned}
\lambda \int_0^\infty e^{-\lambda t}(U(t)x - x)dt &= \frac{\lambda}{2\pi i}\int_0^\infty e^{-\lambda t}\int_{a-i\infty}^{a+i\infty} e^{\mu t} R(\mu, A)Ax \,\frac{d\mu}{\mu}\,dt \\
&= \frac{\lambda}{2\pi i}\int_{a-i\infty}^{a+i\infty}\int_0^\infty e^{(\mu-\lambda)t}dt \; R(\mu, A)Ax \,\frac{d\mu}{\mu} \\
&= \frac{\lambda}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{\lambda-\mu}R(\mu, A)Ax \,\frac{d\mu}{\mu} \\
&= R(\lambda, A)Ax \;=\; \lambda R(\lambda, A)x - x,
\end{aligned}$$

using Fubini's and Cauchy's Theorems.
(d) Let $\mu > \lambda > 0$. Then integration by parts yields

$$\begin{aligned}
\frac{R(\lambda, A)x - R(\mu, A)x}{\mu - \lambda} &= \int_0^\infty e^{(\lambda-\mu)t} R(\lambda, A)x\,dt - \frac{x}{\mu(\mu-\lambda)} \\
&\qquad - \frac{1}{\mu-\lambda}\int_0^\infty e^{(\lambda-\mu)t}e^{-\lambda t}(U(t)x - x)\,dt \\
&= \int_0^\infty e^{(\lambda-\mu)t}\frac{x}{\lambda}\,dt + \int_0^\infty e^{(\lambda-\mu)t}\int_0^\infty e^{-\lambda s}(U(s)x - x)\,ds\,dt \\
&\qquad - \frac{x}{\mu(\mu-\lambda)} - \int_0^\infty e^{(\lambda-\mu)t}\int_0^t e^{-\lambda s}(U(s)x - x)\,ds\,dt \\
&= \frac{x}{\lambda(\mu-\lambda)} - \frac{x}{\mu(\mu-\lambda)} + \int_0^\infty e^{(\lambda-\mu)t}\int_t^\infty e^{-\lambda s}(U(s)x - x)\,ds\,dt \\
&= \frac{\mu x - \lambda x}{\lambda\mu(\mu-\lambda)} + \int_0^\infty e^{-\mu t}\int_t^\infty e^{\lambda(t-s)}(U(s)x - x)\,ds\,dt \\
&= \frac{x}{\lambda\mu} + \int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}(U(t+s)x - x)\,ds\,dt.
\end{aligned}$$

On the other hand, if $x \in D(A^4)$, then $U(t)x \in D(A^2)$ and

$$\begin{aligned}
R(\mu, A)R(\lambda, A)x &= \frac{R(\lambda, A)x}{\mu} + \int_0^\infty e^{-\mu t}(U(t)R(\lambda, A)x - R(\lambda, A)x)\,dt \\
&= \frac{x}{\lambda\mu} + \frac{1}{\mu}\int_0^\infty e^{-\lambda s}(U(s)x - x)\,ds + \int_0^\infty e^{-\mu t}\left(U(t)\frac{x}{\lambda} - \frac{x}{\lambda}\right)dt \\
&\qquad + \int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}(U(t)(U(s)x - x) - (U(s)x - x))\,ds\,dt \\
&= \frac{x}{\lambda\mu} + \int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}(U(t)U(s)x - x)\,ds\,dt.
\end{aligned}$$

By the uniqueness theorem for the Laplace transform we obtain that

$$U(t+s)x - x = U(t)U(s)x - x \tag{14}$$

for almost all $s, t > 0$ and for all $x \in D(A^4)$. For fixed s, the functions $t \mapsto U(t+s)x$ and $t \mapsto U(t)U(s)x$ both are continuous. So the equation (14) holds for all $t > 0$ and almost all $s > 0$. By exchanging the roles of s and t we obtain

$$U(t+s)x = U(t)U(s)x$$

for all $s, t > 0$ and all $x \in D(A^4)$. $\square$

We now are able to prove Theorem 3.3.

Proof of Theorem 3.3. We prove the theorem in four steps. Here, c is always an appropriate constant, and by $\langle \cdot, \cdot \rangle$ we denote the inner product on X.

Step 1: A "candidate" for the semigroup
We apply the inverse Fourier transform to $R(i\cdot, A)x \in L^2(\mathbb{R}, X)$: Take $t > 0$ and $x \in X$ and define

$$T(t)x := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} R(is, A)x \, ds = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\lambda t} R(\lambda, A)x \, d\lambda.$$

Since X is a Hilbert space, Plancherel's theorem yields $T(\cdot)x \in L^2((0, \infty), X)$ and $(\int_0^\infty \|T(t)x\|^2 ds)^{1/2} \leq c\|x\|$ for each $x \in X$. Obviously, T is linear in x, and from Lemma 4.2 (d) we know that the semigroup property $T(t)T(s)x = T(t + s)x$ is satisfied whenever $x \in D(A^4)$ and $t, s > 0$.

Step 2: Boundedness of $T(t)$
First we consider the adjoint operator A^*. As in step 1 we can show that $T^*(\cdot)x$, defined by

$$T^*(t)x := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} R(is, A^*)x ds = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\lambda t} R(\lambda, A^*)x d\lambda, \quad t > 0,$$

is in $L^2((0, \infty), X)$ for each $x \in X$ and $(\int_0^\infty \|T^*(t)x\|^2 ds)^{1/2} \leq c\|x\|$. It is easy to see that $\langle y, T(t)x \rangle = \langle T^*(t)y, x \rangle$ for $x, y \in X$ and $t > 0$.

Now let $t > 0$, $x \in D(A^4)$ and $y \in X$. Then

$$\begin{aligned} t\langle y, T(t)x \rangle &= \int_0^t \langle y, T(t)x \rangle ds = \int_0^t \langle y, T(t-s)T(s)x \rangle ds \\ &= \int_0^t \langle T^*(t-s)y, T(s)x \rangle ds \leq \int_0^t \|T^*(t-s)y\| \, \|T(s)x\| ds \end{aligned}$$

and we can estimate

$$\begin{aligned} \int_0^t \|T^*(t-s)y\| \, \|T(s)x\| ds &\leq \left(\int_0^t \|T^*(t-s)y\|^2 ds \right)^{1/2} \left(\int_0^t \|T(s)x\|^2 ds \right)^{1/2} \\ &\leq \left(\int_0^\infty \|T^*(s)y\|^2 ds \right)^{1/2} \left(\int_0^\infty \|T(s)x\|^2 ds \right)^{1/2} \\ &\leq c \, \|x\| \, \|y\|. \end{aligned}$$

This yields $\|T(t)x\| \leq \frac{c}{t}\|x\|$ for $x \in D(A^4)$. Since $(A, D(A))$ is densely defined and injective, $D(A^4)$ is dense in X. So we have proved that $T(t) \in \mathcal{L}(X)$. Moreover, the semigroup property $T(t)T(s) = T(t + s)$ is satisfied for all $s, t > 0$.

Step 3: The generator of $(T(t))_{t>0}$
Let $\operatorname{Re}\lambda > 0$. We want of prove that $R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t) dt$.

In Lemma 4.2 (c) we have already shown that $R(\lambda, A)x = \frac{x}{\lambda} + \int_0^\infty e^{-\lambda t}(U(t)x - x)dt$ for all $x \in D(A^2)$. Since $(\int_0^\infty \|T(t)x\|^2 ds)^{1/2} \leq c\|x\|$ and $D(A^2)$ is dense in X, the assertion is proved.

Step 4: Strong continuity on $(0, \infty)$
Finally, we show that $t \mapsto T(t)x$ is continuous on $(0, \infty)$ for each $x \in X$.

For $x \in D(A^2)$, Lemma 4.1 yields that $T(t)x - x = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\lambda t} R(\lambda, A)Ax \frac{d\lambda}{\lambda}$ converges absolutely and uniformly on compact intervals. Therefore $t \mapsto T(t)x$ is continuous on $[0, \infty)$ if $x \in D(A^2)$. Since $D(A^2)$ is dense in X and $tT(t)$ is uniformly bounded (Step 2), the mapping $t \mapsto T(t)x$ is continuous on $(0, \infty)$ for each $x \in X$. This proves the theorem. $\square$

5. Proof of Theorem 3.1

The following lemma is implicit in the standard presentations of perturbation theory ([5, Chapter III], [7, Chapter 3]).

Lemma 5.1. *Let $(A, D(A))$ and $(B, D(B))$ be closed operators on a Banach space X where $D(A) \subseteq D(B)$. Suppose that A and A^* are densely defined and that the resolvent set of A is nonempty . If there exists $M \in [0, 1)$ and $\emptyset \neq G \subseteq \rho(A)$ such that*

$$\|BR(\lambda, A)x\| \leq M\|x\| \quad \text{for every } x \in X \text{ and every } \lambda \in G \tag{15}$$

and

$$\|R(\lambda, A)Bx\| \leq M\|x\| \quad \text{for every } x \in D(B) \text{ and every } \lambda \in G, \tag{16}$$

then the operator $(A+B, D(A))$ is closed and $G \subseteq \rho(A+B)$. Furthermore we have that

$$R(\lambda, A+B) = [I - R(\lambda, A)B]^{-1}R(\lambda, A) \tag{17}$$

and

$$R(\lambda, (A+B)^*) = [I - R(\lambda, A^*)B^*]^{-1}R(\lambda, A^*) \tag{18}$$

for every $\lambda \in G$.

Using this lemma and Theorem 3.3, we can show Theorem 3.1.

Proof of Theorem 3.1. We can assume that $\max\{\omega(T), \lambda_0\} < 0$. Otherwise we consider $(A - \omega, D(A))$ instead of $(A, D(A))$, where $\omega > \max\{\omega(T), \lambda_0\}$.

For $x \in X$ we define the function $u_x : \mathbb{R} \to X$ by

$$u_x(t) := \begin{cases} T(t)x, & t \geq 0, \\ \quad 0, & t < 0. \end{cases}$$

Since $\omega(T) < 0$, the function u_x is in $L^2(\mathbb{R}, X)$ and there is a constant $c \geq 0$ such that $(\int_{-\infty}^{\infty} \|u_x(t)\|^2 dt)^{1/2} \leq c\|x\|$. By Plancherel's Theorem the Fourier transform $\mathcal{F}u_x$ of u_x is also $L^2(\mathbb{R}, X)$ and $\|\mathcal{F}u_x\|_2 = \sqrt{2\pi}\|u_x\|_2$. On the other hand, we know that $(\mathcal{F}u_x)(s) = \int_{-\infty}^{\infty} e^{-ist}u_x(t)dt = \int_0^{\infty} e^{-ist}T(t)xdt = R(is, A)x$ for every $s \in \mathbb{R}$. Therefore

$$\left(\int_{-\infty}^{\infty} \|R(is, A)x\|^2 ds\right)^{1/2} \leq c\sqrt{2\pi}\,\|x\|. \tag{19}$$

Using Lemma 5.1, it follows that

$$\begin{aligned} \left(\int_{-\infty}^{\infty} \|R(is, A+B)x\|^2 ds\right)^{1/2} &= \left(\int_{-\infty}^{\infty} \|[I - R(is, A)B]^{-1}R(is, A)x\|^2 ds\right)^{1/2} \\ &\leq \frac{1}{1-M}\left(\int_{-\infty}^{\infty} \|R(is, A)x\|^2 ds\right)^{1/2} \\ &\leq \frac{c\sqrt{2\pi}}{1-M}\|x\| \end{aligned}$$

for all $x \in X$.

We now consider $(A+B)^*$. As before we can show that

$$\left(\int_{-\infty}^{\infty} \|R(i\cdot, (A+B)^*)x\|\right)^2 \leq \frac{c\sqrt{2\pi}}{1-M}\|x\|$$

for each $x \in X$. So we can apply Theorem 3.3. $\square$

6. Application to ordinary differential operators

Let X be the Hilbert space $L^2(\mathbb{R})$ and $k \in \mathbb{N}$. We consider the operator $(A, D(A))$ in X defined by

$$Au := iu^{(2k)}, \qquad D(A) := W^{2k,2}(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : u^{(2k)} \in L^2(\mathbb{R})\}. \tag{20}$$

Here $u^{(2k)}$ denotes the $2k^{\text{th}}$ (distributional) derivative of the function u. It is well known that $(A, D(A))$ generates a C_0-semigroup on X (see, e.g, [2, Section 8.1]).

One can compute that $\mathbb{C} \setminus (i\mathbb{R}) \subseteq \rho(A)$ and that for $\lambda \in \mathbb{C} \setminus (i\mathbb{R})$ the resolvent of A is given by

$$R(\lambda, A)f(x) = \frac{i}{2k} \int_{-\infty}^{\infty} \sum_{j=1}^{k} \frac{e^{-\mu_j |x-s|}}{(-\mu_j)^{2k-1}} f(s)\, ds, \quad x \in \mathbb{R},$$

where f is a function in $L^2(\mathbb{R})$ and μ_j $(j = 1, \ldots, k)$ are the k solutions of the equation $\lambda - i\mu^{2k} = 0$ with $\operatorname{Re} \mu_j > 0$.

We now define the operator $(B, D(B))$ by

$$Bf := V \cdot f^{(l)}, \qquad D(B) := \{f \in X : \; V \cdot f^{(l)} \in X\}, \tag{21}$$

where V is a potential in $L^2(\mathbb{R})$ and $l \in \mathbb{N}_0$ such that $l < k$.

We want to look at $BR(\lambda, A)$. Take $f \in C_c^\infty(\mathbb{R})$, i.e., f is in $C^\infty(\mathbb{R})$ and has compact support. For $\lambda \in \mathbb{C} \setminus (i\mathbb{R})$ we compute

$$BR(\lambda, A)f(x) = V(x) \cdot \frac{i}{2k} \sum_{j=1}^{k} \left(\int_{-\infty}^{x} \frac{e^{-\mu_j (x-s)}}{(-\mu_j)^{2k-l-1}} f(s)\, ds - \int_{x}^{\infty} \frac{e^{\mu_j (x-s)}}{\mu_j^{2k-l-1}} f(s)\, ds \right).$$

Now, if $g \in C_c^\infty(\mathbb{R})$ we find

$$\begin{aligned}
&|\langle g, BR(\lambda, A)f\rangle| \\
&\leq \frac{1}{2k|\lambda|^{1-(l+1)/(2k)}} \sum_{j=1}^{k} \int_{-\infty}^{\infty} |g(x)|\, |V(x)| \int_{-\infty}^{\infty} e^{-\operatorname{Re}\mu_j |x-s|}\, |f(s)|\, ds\, dx \\
&\leq \frac{1}{2k|\lambda|^{1-(l+1)/(2k)}} \sum_{j=1}^{k} \int_{-\infty}^{\infty} |g(x)|\, |V(x)| \left(\int_{-\infty}^{\infty} e^{-2\operatorname{Re}\mu_j |x-s|} ds \right)^{1/2} dx\, \|f\|_2 \\
&= \frac{1}{2k|\lambda|^{1-(l+1)/(2k)}} \sum_{j=1}^{k} \left(\frac{1}{\operatorname{Re}\mu_j} \right)^{1/2} \int_{-\infty}^{\infty} |g(x)|\, |V(x)| dx\, \|f\|_2 \\
&\leq \frac{\|V\|_2}{2|\lambda|^{1-(l+1)/(2k)}} \frac{1}{k} \sum_{j=1}^{k} \left(\frac{1}{\operatorname{Re}\mu_j} \right)^{1/2} \|g\|_2 \|f\|_2.
\end{aligned}$$

Since $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, we have shown the estimate

$$\begin{aligned}
\|BR(\lambda, A)\| &\leq \frac{\|V\|_2}{2|\lambda|^{1-(l+1)/(2k)}} \frac{1}{k} \sum_{j=1}^{k} \left(\frac{1}{\operatorname{Re}\mu_j} \right)^{1/2} \\
&\leq \frac{\|V\|_2}{2|\lambda|^{1-(l+1)/(2k)} \min\{(\operatorname{Re}\mu_j)^{1/2} : j = 1, ..., k\}}.
\end{aligned}$$

If $\lambda = re^{i\varphi}$ with $r > 0$ and $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then a careful computation yields

$$\min\{(\operatorname{Re}\mu_j)^{1/2} : j = 1, .., k\} = |\lambda|^{1/(4k)} (\cos \psi_k)^{1/2},$$

where

$$\psi_k = \begin{cases} \frac{\varphi}{2k} - \frac{\pi}{4k} + \frac{\pi}{2}, & \text{if } k \text{ is even,} \\ \frac{\varphi}{2k} + \frac{\pi}{4k} - \frac{\pi}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

Since $|\lambda| = \operatorname{Re}\lambda(1+\tan^2\varphi)^{1/2} = \frac{\operatorname{Re}\lambda}{\cos\varphi}$, we have

$$\begin{aligned} |\lambda|^{1-(l+1)/(2k)} \min\{(\operatorname{Re}\mu_j)^{1/2} : j=1,...,k\} &= |\lambda|^{1-(l+1)/(2k)+1/(4k)}(\cos\psi_k)^{1/2} \\ &= (\operatorname{Re}\lambda)^{1-l/(2k)-1/(4k)} \frac{(\cos\psi_k)^{1/2}}{(\cos\varphi)^{1-l/(2k)-1/(4k)}} \\ &= (\operatorname{Re}\lambda)^{1-l/(2k)-1/(4k)} \left(\frac{\cos\psi_k}{\cos\varphi}\right)^{1/2} (\cos\varphi)^{-1/2+l/(2k)+1/(4k)}. \end{aligned}$$

But $-\frac{1}{2} + \frac{l}{2k} + \frac{1}{4k} = \frac{1}{2k}(l-k+\frac{1}{2}) \le 0$, and $\frac{\cos\psi_k}{\cos\varphi}$ is bounded from below by a constant $c > 0$ for all $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore

$$|\lambda|^{1-(l+1)/(2k)} \min\{(\operatorname{Re}\mu_j)^{1/2} : j=1,..,k\} \ge c(\operatorname{Re}\lambda)^{1-l/(2k)-1/(4k)}.$$

This shows the estimate

$$\|BR(\lambda,A)\| \le \frac{\|V\|_2}{2c(\operatorname{Re}\lambda)^{1-l/(2k)-1/(4k)}}. \tag{22}$$

We now can prove the following proposition.

Proposition 6.1. *Let $X = L^2(\mathbb{R})$ and let $(A, D(A))$ be defined as in* (20). *If $(B, D(B))$ is given by*

$$Bf := V \cdot f^{(l)}, \qquad D(B) := \{f \in X : \ V \cdot f^{(l)} \in X\},$$

where V is a potential in $L^2(\mathbb{R})$ and $l \in \mathbb{N}_0$ such that $l < k$, then $(A+B, D(A))$ generates a semigroup on X that is strongly continuous on $(0,\infty)$.

Proof. Since $1 - l/(2k) - 1/(4k) > 0$ by assumption, we obtain from (22) that there is $M < 1$ such that

$$\|BR(\lambda,A)\| \le M$$

if $\operatorname{Re}\lambda$ is large enough. It is easy to see that the same is true for A^* and B^* instead of A and B. This yields $\|R(\lambda,A)Bf\| \le M\|f\|$ for $f \in D(B)$ and we can apply Theorem 3.1. □

Corollary 6.2. *Let $X = L^2(\mathbb{R})$ and let $(A, D(A))$ be defined as in* (20). *If $V \in L^2(\mathbb{R}) + L^\infty(\mathbb{R})$ and $(B, D(B))$ is defined as*

$$Bf := V \cdot f, \qquad D(B) := \{f \in X : \ V \cdot f \in X\},$$

then $(A+B, D(A))$ generates a semigroup on X that is strongly continuous on $(0,\infty)$.

Proof. We split V into an L^2-part and a bounded part. The bounded part can be estimated by the Hille-Yosida theorem. For the L^2-part, we use again (22) as in the proof of Proposition 6.1 □

References

[1] Wolfgang Arendt, *Resolvent positive operators*, Proc. London Math. Soc. (3) **54** (1987), 321–349.

[2] Wolfgang Arendt, Charles J. K. Batty, Matthias Hieber, and Frank Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Birkhäuser, Basel Boston Berlin, 2001.

[3] Richard Beals, *On the abstract Cauchy problem*, J. Functional Analysis **10** (1972), 281–299.

[4] Adam Bobrowski, *On the generation of non-continuous semigroups*, Semigroup Forum **54** (1997), 237–252.

[5] Klaus-Jochen Engel and Rainer Nagel, "One-parameter semigroups for linear evolution equations", Graduate Texts in Mathematics, 194, Springer-Verlag, New York, 2000.

[6] Selim Grigorevich Kreĭn, "Linear differential equations in Banach spaces", Translated from the Russian by J. M. Danskin, Translations of Mathematical Monographs, Vol. 29, American Mathematical Society, Providence, R.I., 1971.

[7] A. Pazy, "Semigroups of linear operators and applications to partial differential equations", Applied Mathematical Sciences, 44, Springer-Verlag, New York, 1983.

[8] Kazuaki Taira, *The theory of semigroups with weak singularity and its application to partial differential equations*, Tsukuba J. Math. **13** (1989), no. 2, 513–562.

Mathematisches Institut I, Universität Karlsruhe, Englerstrasse 2, 76128 Karlsruhe, Germany

E-mail address: cornelia.kaiser@math.uni-karlsruhe.de

Mathematisches Institut I, Universität Karlsruhe, Englerstrasse 2, 76128 Karlsruhe, Germany

E-mail address: lutz.weis@math.uni-karlsruhe.de