INTEGRATED SEMIGROUPS AND LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. We study existence and uniqueness of solutions for linear partial differential equations with delay in L^p-spaces using an approach of Batkai and Piazzera and a recent perturbation result for integrated semigroups. We apply our result to an equation with delay in the highest-order derivatives.

1. Introduction

Partial differential equations with delay play an important role in different fields, e.g. in control theory. They have been studied for many years, using different methods.

We want to study equations that can be written as

$$\begin{cases} u'(t) = Au(t) + \Phi u_t, & t \geq 0 \\ u(0) = x \\ u_0 = f. \end{cases} \tag{1}$$

We use the standard notations (see [18]): A is an (unbounded) operator on a Banach space X, Φ is the delay operator, u_t is the history function and $x \in X$, $f \in L^p([-1,0],Z)$ are the initial values at time 0. Here Z is a Banach space such that $D(A) \hookrightarrow Z \hookrightarrow X$ with continuous embeddings.

Batkai and Piazzera [2, 3] showed that the delay equation (1) is equivalent to an abstract Cauchy problem

$$\begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t), & t \geq 0 \\ \mathcal{U}(0) = \binom{x}{f}. \end{cases}$$

on the space $\mathcal{X} = X \times L^p([-1,0],Z)$. Then, using perturbation theory of C_0-semigroups, they give sufficient conditions for $(\mathcal{A}, D(\mathcal{A}))$ to be the generator of a C_0-semigroup on $\mathcal{X}$. One of these conditions is that $(A, D(A))$ generates a C_0-semigroup on X. But there are important operators that do not have this property, e.g. the Schrödinger operator $i\Delta$ on $L^p(\mathbb{R}^n)$ for $p \neq 2$ ([14]). A weaker condition on A is that it generates an α-times integrated semigroup. We use the approach of Batkai and Piazzera and a new perturbation theorem for α-times integrated semigroups from [15] to show existence results for solutions of the delay equation (1).

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Another restriction in the results of Batkai and Piazzera is that one can not treat equations of the kind

$$u'(t) = Au(t) + \gamma Au(t-1) + \int_{-1}^{0} a(s)Au(t+s)ds \tag{2}$$

where A generates an analytic semigroup on X. On the other hand, in [7] there are wellposedness results for such equations in the case that X is a Hilbert space. With our approach we also can treat equations of the kind (2) on Banach spaces.

This paper is organized as follows. In Section 2 we recall the definition of α-times integrated semigroups and state the perturbation result we will use later. In the third section we prove our main result: If $(A, D(A))$ is densely defined and generator of an α-times integrated semigroup on X and if there are numbers $M \in [0,1)$ and $\lambda_0 > 0$ such that for all $\operatorname{Re}\lambda = \lambda_0$

$$\left\|\Phi(e^{\lambda \cdot} R(\lambda, A)x + R(\lambda, A_0)f)\right\|_X \leq M \max\{\|x\|_X, \|f\|_{L^p([-1,0],Z)}\}, \tag{3}$$

then $(\mathcal{A}, D(\mathcal{A}))$ generates a β-times integrated semigroup for some $\beta \geq \alpha$. This means that the delay equation has a unique solution for all pairs $\binom{x}{f}$ of initial values in the domain of $\mathcal{A}^n$ for $n \in \mathbb{N}$ large enough. Finally we give conditions under which (3) is satisfied. In the last section we apply our result to equation (2).

2. A Perturbation Theorem for α-times Integrated Semigroups

We recall the definition of an α-times integrated semigroup.

Definition 2.1. Let $\alpha \geq 0$ and $(A, D(A))$ be a linear operator on a Banach space X. A is called *generator of an α-times integrated semigroup* if there are nonnegative numbers ω, M and a mapping $S : [0,\infty) \to \mathcal{L}(X)$ such that

- $(S(t))_{t\geq 0}$ is strongly continuous and $\| \int_0^t S(s)\, ds\| \leq Me^{\omega t}$ for all $t \geq 0$,
- $(\omega, \infty) \subseteq \rho(A)$ and
- $R(\lambda, A) = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)\, dt$ for $\lambda > \omega$.

In this case, the family $(S(t))_{t\geq 0}$ is the *α-times integrated semigroup* generated by A.

Remarks (1) If $(A, D(A))$ generates an α-times integrated semigroup $(S(t))_{t\geq 0}$, then the halfplane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\}$ is contained in $\rho(A)$ and $R(\lambda, A) = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)dt$ if $\operatorname{Re}\lambda > \omega$.

(2) By uniqueness of the Laplace transform $(S(t))_{t\geq 0}$ is uniquely determined.

(3) If $\alpha = 0$, the definition above is consistent with the definition of a C_0-semigroup (see [1, Theorem 3.1.7]). In this case the generator A is densely defined and $(S(t))_{t\geq 0}$ is exponentially bounded. For $\alpha > 0$ this may not be true in general. If $(S(t))_{t\geq 0}$ is exponentially bounded, then the *growth bound* of $(S(t))_{t\geq 0}$ is defined by

$$\omega(S) := \inf\{\omega \in \mathbb{R} : \exists K \geq 0 \text{ such that } \|S(t)\| \leq Ke^{\omega t}\}.$$

(4) If A generates an α-times integrated semigroup $(S_\alpha(t))_{t\geq 0}$ then A also generates a β-times integrated semigroup $(S_\beta(t))_{t\geq 0}$ for each $\beta > \alpha$.

(5) If A generates an α-times integrated semigroup $(S(t))_{t\geq 0}$ then the abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t), & t \in [0,\tau], \\ u(0) = x, \end{cases} \tag{4}$$

has a unique classical solution for each $x \in D(A^{n+1})$ where $n \in \mathbb{N}_0$ such that $n-1 < \alpha \leq n$ ([12]). By a classical solution of (4) we mean a function $u \in C^1([0,\infty),X)$ such that $u(t) \in D(A)$ for all $t \geq 0$ and (4) is satisfied.

We will also need the following definition:

Definition 2.2. A Banach space X has *Fourier type* $p \in [1,2]$ if the Fourier transform extends to a bounded linear operator from $L^p(\mathbb{R},X)$ to $L^{p'}(\mathbb{R},X)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Each Banach space has Fourier type 1. A Banach space has Fourier type 2 if and only if it is isomorphic to a Hilbert space ([16]). If X has Fourier type p, then it has Fourier type r for each $r \in [1,p]$. Each closed subspace, each quotient space and the dual space X^* of a Banach space X have the same Fourier type as X. If X and Y both have Fourier type p, then $X \times Y$ also has Fourier type p. The space $L^r(\Omega,\mu)$ has Fourier type $\min\{r, \frac{r}{r-1}\}$ ([17]). If X has Fourier type p and $q \in [p,p']$, then $L^q(\mathbb{R},X)$ has Fourier type p ([10]). Each uniformly convex Banach space has Fourier type $p > 1$ ([4, 5]).

Then the following perturbation theorem holds. For the proof we refer to [15].

Theorem 2.3. *Let X be a Banach space of Fourier type $p \in [1,2]$. Let $(A, D(A))$ be the generator of an exponentially bounded α-times integrated semigroup $(S(t))_{t\geq 0}$ on X and let $(B, D(B))$ be a linear operator in X. Choose $\beta > \alpha + \frac{1}{p}$. If A is densely defined, $D(B) \supseteq D(A)$ and there are constants $\lambda_0 > \max\{0, \omega(S)\}$ and $M < 1$ such that*

$$\|BR(\lambda, A)\| \leq M$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda = \lambda_0$, then $(A + B, D(A))$ generates a β-times integrated semigroup.

3. Main Results

We consider the equation

$$\begin{cases} u'(t) = Au(t) + \Phi u_t, & t \geq 0 \\ u(0) = x \\ u_0 = f. \end{cases} \tag{5}$$

where

- X is a Banach space and $x \in X$,
- $(A, D(A))$ is a closed linear operator in X,
- $f \in L^p([-1,0],Z)$, $1 \leq p < \infty$, where Z is a Banach space such that $D(A) \hookrightarrow Z \hookrightarrow X$ with continuous embeddings,
- the delay operator $\Phi : W^{1,p}([-1,0],Z) \to X$ is linear and bounded,
- $u : [-1,\infty) \to X$ and
- $u_t : [-1,0] \to X$ is defined by $u_t(\sigma) = u(t+\sigma)$ for $\sigma \in [-1,0]$.

We say that a function $u : [-1,\infty) \to X$ is a *(classical) solution* of (5) if

(i) $u \in C([-1,\infty),Z) \cap C^1([0,\infty),X)$,
(ii) $u(t) \in D(A)$ and $u_t \in W^{1,p}([-1,0],Z)$ for all $t \geq 0$ and
(iii) u satisfies (5) for all $t \geq 0$.

We now want to investigate existence and uniqueness of solutions of (5) in the space X. To do this we introduce, as in [3], the Banach space

$$\mathcal{X} := X \times L^p([-1,0],Z)$$

and the operator

$$\mathcal{A} := \begin{pmatrix} A & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \tag{6}$$

in $\mathcal{X}$ with domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times W^{1,p}([-1,0], Z) : f(0) = x \right\}.$$

The matrix operator $\mathcal{A}$ is closed, since A is, and it is densely defined, if A is. If (5) has a solution u, then $u_0 = f \in W^{1,p}([-1,0], Z)$ and $u(0) = x \in D(A)$ i.e., $\begin{pmatrix} x \\ f \end{pmatrix} \in D(A)$.

The problem (5) and the abstract Cauchy problem associated to the operator $(\mathcal{A}, D(\mathcal{A}))$

$$\begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t), & t \geq 0 \\ \mathcal{U}(0) = \begin{pmatrix} x \\ f \end{pmatrix}. \end{cases} \tag{7}$$

on the space $\mathcal{X}$ are "equivalent": there is a natural correspondence between the solutions of the two problems (5) and (7). For a proof of the following Proposition see [2, Proposition 2.3 and 2.4].

Proposition 3.1. *Let* $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A})$.

(a) If $u : [-1, \infty) \to X$ *is a solution of the delay equation* (5) *then*

$$\mathcal{U} : \begin{cases} [0, \infty) & \to & \mathcal{X} \\ t & \mapsto & \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \end{cases}$$

is a classical solution of the abstract Cauchy problem (7).

(b) Let

$$\mathcal{U} : \begin{cases} [0, \infty) & \to & \mathcal{X} \\ t & \mapsto & \begin{pmatrix} z(t) \\ v(t) \end{pmatrix} \end{cases}$$

be a classical solution of the abstract Cauchy problem (7) *and* $u : [-1, \infty) \to X$ *defined by*

$$u(t) := \begin{cases} z(t), & t \geq 0 \\ f(t), & t \in [-1, 0]. \end{cases}$$

Then $u_t = v(t)$ *for all* $t \geq 0$ *and* u *is a solution of the delay equation* (5).

We now want to give sufficient conditions such that $(\mathcal{A}, D(\mathcal{A}))$ generates an α-times integrated semigroup on X. Then the abstract Cauchy problem (7) has a unique solution for $x \in D(\mathcal{A}^{n+1})$ where $n \geq \alpha$ (see [12]). To be able to use the perturbation result 2.3 we write $\mathcal{A}$ as sum $\mathcal{A}_0 + \mathcal{B}$, where

$$\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$$

with domain $D(\mathcal{A}_0) := D(\mathcal{A})$ and

$$\mathcal{B} = \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(D(\mathcal{A}_0), \mathcal{X}).$$

We first look at $\mathcal{A}_0$.

Proposition 3.2. *Let* $\alpha \geq 0$ *and* $(A, D(A))$ *be the generator of an* α*-times integrated semigroup* $(S(t))_{t \geq 0}$ *on* X *and let* $D(A) \hookrightarrow Z \hookrightarrow X$. *If*

(G) *for each $t \geq 0$ and each $x \in X$ the function $s \mapsto S(s)x$ is in $L^p([0,t];Z)$ and there is a positive constant c_t such that*

$$\left(\int_0^t \|S(s)x\|_Z^p ds\right)^{1/p} \leq c_t\|x\|,$$

then $(\mathcal{A}_0, D(\mathcal{A}_0))$ generates an α-times integrated semigroup on $\mathcal{X}$.

Remarks. (1) Condition (G) in Proposition 3.2 is automatically satisfied if $Z = X$.
(2) If $(A, D(A))$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on X and if there is $\delta > \omega(T)$ and $0 < \theta < \frac{1}{p}$ such that $D((\delta - A)^\theta) \hookrightarrow Z \hookrightarrow X$ with continuous and dense embeddings, then condition (G) is satisfied for $(T(t))_{t\geq 0}$ (see [3]).
(3) Let $(A, D(A))$ be the generator of an α-times integrated semigroup $(S(t))_{t\geq 0}$ on X and consider $\tilde{S}(t) := \int_0^t S(s)ds$. Then condition (G) is always satisfied for $(\tilde{S}(t))_{t\geq 0}$: We have that

$$\begin{aligned}\left(\int_0^t \|\tilde{S}(s)x\|_Z^p ds\right)^{1/p} &\leq C\left(\int_0^t \|\tilde{S}(s)x\|_{D(A)}^p ds\right)^{1/p}\\ &\leq C\left(\int_0^t \|\tilde{S}(s)x\|_X^p ds\right)^{1/p} + C\left(\int_0^t \|A\tilde{S}(s)x\|_X^p ds\right)^{1/p}.\end{aligned}$$

Now we use the Uniform Boundedness Principle and the formula

$$A\tilde{S}(s)x = S(s)x - \frac{s^\alpha}{\Gamma(\alpha+1)}x$$

(see [13, Proposition 2.4]). But $(\tilde{S}(t))_{t\geq 0}$ is the $(\alpha+1)$-times integrated semigroup generated by A. So we always can achieve (G) by enlarging the integration rate α.

For the proof of Proposition 3.2 we first compute the resolvent of $(\mathcal{A}_0, D(\mathcal{A}_0))$. For $\lambda \in \mathbb{C}$ and $\binom{y}{g} \in \mathcal{X}$ we look for $\binom{x}{f} \in D(\mathcal{A}_0)$ such that

$$(\lambda - \mathcal{A})\binom{x}{f} = \binom{y}{g}.$$

This is the case if and only if there are $x \in D(A)$ and $f \in W^{1,p}([-1,0], Z)$ such that

$$(\lambda - A)x = y, \quad f(0) = x \quad \text{and} \quad \lambda f - f' = g.$$

If $(A, D(A))$ generates an α-times integrated semigroup on X, then the set $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \lambda_0\}$ is contained in the resolvent set of A for some $\lambda_0 \in \mathbb{R}$. Therefore $x = R(\lambda, A)y$ for $\operatorname{Re}\lambda > \lambda_0$. The differential equation $\lambda f - f' = g$ with initial condition $f(0) = x$ has a unique solution $f \in W^{1,p}([-1,0], Z)$ for all $\lambda \in \mathbb{C}$ which is given by

$$f(\sigma) = e^{\lambda\sigma}\left(x + \int_\sigma^0 e^{-\lambda t}g(t)dt\right).$$

On the other hand, the operator

$$A_0 f = f'$$

with domain $D(A_0) = \{f \in W^{1,p}([-1,0], Z) : f(0) = 0\}$ has empty spectrum and

$$(R(\lambda, A_0)g)(\sigma) = e^{\lambda\sigma}\int_\sigma^0 e^{-\lambda t}g(t)dt.$$

Therefore the resolvent of $(\mathcal{A}_0, D(\mathcal{A}_0))$ is given by

$$
\begin{aligned}
R(\lambda, \mathcal{A}_0)\left(\begin{smallmatrix} y \\ g \end{smallmatrix}\right) &= \begin{pmatrix} R(\lambda, A)x \\ e^{\lambda\cdot} R(\lambda, A)y + R(\lambda, A_0)g \end{pmatrix} \\
&= \begin{pmatrix} R(\lambda, A) & 0 \\ e^{\lambda\cdot} R(\lambda, A) & R(\lambda, A_0) \end{pmatrix} \left(\begin{smallmatrix} y \\ g \end{smallmatrix}\right)
\end{aligned} \tag{8}
$$

for $\operatorname{Re}\lambda > \lambda_0$.

We fix some notations. It is well known ([9, I.4.17, II.2.11]), that $(A_0, D(A_0))$ generates the nilpotent shift semigroup $(T_0(t))_{t\geq 0}$ in $L^p([-1,0], Z)$ given by

$$
(T_0(t)f)(\sigma) := \begin{cases} f(\sigma + t), & \sigma + t \leq 0, \\ 0, & \sigma + t > 0. \end{cases}
$$

In particular, $(A_0, D(A_0))$ generates an α-times integrated semigroup $(S_0(t))_{t\geq 0}$, where

$$
S_0(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha - 1)} T_0(t)dt.
$$

Moreover we define

$$
(S_t x)(\tau) := \begin{cases} S(t+\tau)x, & -t < \tau \leq 0 \\ 0, & -1 \leq \tau \leq -t. \end{cases}
$$

where $(S(t))_{t\geq 0}$ is the α-times integrated semigroup generated by $(A, D(A))$. If (G) is true, then it easily follows that $S_t : X \to L^p([-1,0], Z)$ is bounded for all $t \geq 0$.

Now we prove Proposition 3.2:

Proof of Proposition 3.2 . Let

$$
\mathcal{S}_0(t) := \begin{pmatrix} S(t) & 0 \\ S_t & S_0(t) \end{pmatrix}, \qquad t \geq 0.
$$

For each $t \geq 0$, $\mathcal{S}_0(t) \in \mathcal{L}(\mathcal{X})$. Moreover $(\mathcal{S}_0(t))_{t\geq 0}$ is strongly continuous since $(S(t))_{t\geq 0}$ and $(S_0(t))_{t\geq 0}$ are strongly continuous.

Then, for $\lambda \in \mathbb{R}$ large,

$$
\lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)dt = R(\lambda, A) \quad \text{bzw.} \quad \lambda^\alpha \int_0^\infty e^{-\lambda t} S_0(t)dt = R(\lambda, A_0).
$$

Hence we get for $x \in X$ and $\tau \in [-1, 0]$

$$
\begin{aligned}
\lambda^\alpha \int_0^\infty e^{-\lambda t}(S_t x)(\tau)dt &= \lambda^\alpha \int_{-\tau}^\infty e^{-\lambda t} S(t+\tau)xdt \\
&= e^{\lambda\tau}\lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)xdt \\
&= e^{\lambda\tau} R(\lambda, A)x.
\end{aligned}
$$

This shows

$$
R(\lambda, \mathcal{A}_0) = \lambda^\alpha \int_0^\infty e^{-\lambda t} \mathcal{S}_0(t)dt,
$$

i.e., $(\mathcal{S}_0(t))_{t\geq 0}$ the α-times integrated semigroup generated by $(\mathcal{A}_0, D(\mathcal{A}_0))$. □

From now on we assume that $(A, D(A))$ generates an α-times integrated semigroup $(S, D(S))$ on X such that condition (G) in Proposition 3.2 holds. We use the perturbation results from Section 2 to find conditions on the delay operators Φ such that $(\mathcal{A}, D(\mathcal{A}))$ generates a β-times integrated semigroup on $\mathcal{X}$ for some $\beta \geq \alpha$.

If $\operatorname{Re}\lambda > \lambda_0$ then

$$\begin{aligned}\left\|\mathcal{B}R(\lambda,\mathcal{A}_0)\binom{x}{f}\right\|_{\mathcal{X}} &= \left\|\begin{pmatrix} 0 & \Phi \\ 0 & 0\end{pmatrix}\begin{pmatrix} R(\lambda,A)x \\ e^{\lambda\cdot}R(\lambda,A)x + R(\lambda,A_0)f\end{pmatrix}\right\|_{\mathcal{X}} \\ &= \left\|\Phi(e^{\lambda\cdot}R(\lambda,A)x + R(\lambda,A_0)f)\right\|_X\end{aligned}$$

für $\binom{x}{f} \in \mathcal{X}$. Here $\|\cdot\|_{\mathcal{X}}$ is one of the equivalent norms on $X \times L^p([-1,0],Z)$. Using Theorem 2.3 we get

Theorem 3.3. *Let X be a Banach space and let $(A,D(A))$ be densely defined and generator of an α-times integrated semigroup on X, where α is chosen such that condition (G) in Proposition 3.2 holds. Let $D(A) \hookrightarrow Z \hookrightarrow X$ and $\Phi \in \mathcal{L}(W^{1,p}([-1,0],Z),X)$, $1 \le p < \infty$. If there are numbers $M \in [0,1)$ and $\lambda_0 > 0$ such that for all $\operatorname{Re}\lambda = \lambda_0$*

$$\left\|\Phi(e^{\lambda\cdot}R(\lambda,A)x + R(\lambda,A_0)f)\right\|_X \le M\left\|\binom{x}{f}\right\|_{\mathcal{X}}, \tag{9}$$

then $(\mathcal{A},D(\mathcal{A}))$ generates a β-times integrated semigroup for all $\beta > \alpha + \frac{1}{r}$, where $\mathcal{X} = X \times L^p([-1,0],Z)$ is of Fourier type r. In particular, if X is of Fourier-type $q \in [1,2]$, then one can choose $r = \min\{p, 1-\frac{1}{p}, q\}$.

In the following we give examples for delay operators Φ that satisfy the condition (9).

Example 3.4. Let $D(A) \hookrightarrow Z \hookrightarrow X$ and $\Phi \in \mathcal{L}(L^p([-1,0],Z),X)$. Then

$$\|\Phi(e^{\lambda\cdot}R(\lambda,A)x + R(\lambda,A_0)f\|_X \le \|\Phi\|\Big(\|R(\lambda,A)x\|_Z + \frac{C}{\operatorname{Re}\lambda}\|f\|_p\Big)$$

for all $x \in X$, all $f \in L^p([-1,0],Z)$ and all $\operatorname{Re}\lambda$ large enough. Hence there is $\lambda_0 \in \mathbb{R}$ such that (9) is satisfied for $\operatorname{Re}\lambda = \lambda_0$ provided that $\|\Phi\|$ is small enough and $\|R(\lambda,A)\|_{\mathcal{L}(X,Z)}$ is uniformly bounded for λ in a half-plane. $\square$

Example 3.5. Let $D(A) \hookrightarrow Z \hookrightarrow X$ and let $\eta : [-1,0] \to \mathcal{L}(Z,X)$ be of bounded variation. If $\Phi : C([-1,0],Z) \to X$ is given by

$$\Phi f := \int_{-1}^0 d\eta f, \qquad f \in C([-1,0],Z)$$

then $\Phi \in \mathcal{L}(C([-1,0],Z),X)$. In particular, Φ is bounded from $W^{1,p}([-1,0],Z)$ to X since $W^{1,p}([-1,0],Z)$ is continuously embedded $C([-1,0],Z)$. If the real part of λ is large enough we get

$$\begin{aligned}\|\Phi(e^{\lambda\cdot}R(\lambda,A)x)\|_X &= \left\|\int_{-1}^0 d\eta(\sigma)e^{\lambda\sigma}R(\lambda,A)x\right\|_X \\ &\le \int_{-1}^0 e^{\operatorname{Re}\lambda\sigma}d|\eta|(\sigma)\,\|R(\lambda,A)x\|_Z \\ &\le |\eta|([-1,0])\|R(\lambda,A)x\|_Z\end{aligned}$$

and

$$\begin{aligned}
\|\Phi R(\lambda, A_0) f\|_X &= \left\| \int_{-1}^0 d\eta(\sigma)(R(\lambda, A_0)f)(\sigma) \right\|_X \\
&= \left\| \int_{-1}^0 d\eta(\sigma) \int_\sigma^0 e^{-\lambda(t-\sigma)} f(t) dt \right\|_X \\
&\leq \int_{-1}^0 \int_\sigma^0 e^{-\operatorname{Re}\lambda(t-\sigma)} \|f(t)\|_Z dt\, d|\eta|(\sigma) \\
&\leq \int_{-1}^0 \left(\int_\sigma^0 e^{-\operatorname{Re}\lambda(t-\sigma)q} dt \right)^{1/q} dt\, d|\eta|(\sigma) \|f\|_p \\
&\leq (\operatorname{Re}\lambda q)^{-1/q}\, |\eta|([-1,0])\, \|f\|_p,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $|\eta|$ is the positive Borel measure on $[-1, 0]$ defined by η. Hence there is $\lambda_0 \in \mathbb{R}$ such that (9) is satisfied for $\operatorname{Re}\lambda = \lambda_0$ provided that $|\eta|([-1, 0])$ is small enough and $\|R(\lambda, A)\|_{\mathcal{L}(X,Z)}$ is uniformly bounded for λ in a half-plane. $\square$

Example 3.6. Let $D(A) \hookrightarrow Z \hookrightarrow X$, $\Psi \in \mathcal{L}(C([-1, 0], X), X)$ and $B \in \mathcal{L}(Z, X)$ such that $\|BR(\lambda, A)\| \leq M < \|\Psi\|^{-1}$ for $\operatorname{Re}\lambda$ large enough. We define the delay operator $\Phi \in \mathcal{L}(C([-1, 0], Z), X)$ by

$$\Phi f = \Psi(s \mapsto B(f(s))).$$

Then, for all $x \in S$, $f \in L^p([-1, 0], Z)$ and all $\operatorname{Re}\lambda$ large enough,

$$\begin{aligned}
\|\Phi(e^{\lambda\cdot} R(\lambda, A)x)\|_X &= \|\Psi(e^{\lambda\cdot} BR(\lambda, A)x)\|_X \leq \|\Psi\| \sup_{-1\leq s\leq 1} \|e^{\lambda s} BR(\lambda, A)x\|_X \\
&\leq \|\Psi\| \|BR(\lambda, A)x\| \leq M\|\Psi\| < 1
\end{aligned}$$

and

$$\begin{aligned}
\|\Phi R(\lambda, A_0) f\|_X &= \|\Psi(s \mapsto B((R(\lambda, A_0)f)(s))\|_X \\
&= \left\| \Psi\left(s \mapsto B \int_s^0 e^{-\lambda(t-s)} f(t) dt \right) \right\|_X \\
&\leq \|\Psi\| \sup_{-1\leq s\leq 0} \left\| \int_s^0 e^{-\lambda(t-s)} B(f(t)) dt \right\|_X \\
&\leq \|\Psi\| \sup_{-1\leq s\leq 0} \int_s^0 e^{-\operatorname{Re}\lambda(t-s)} \|B(f(t))\|_X dt \\
&\leq \|\Psi\| \|B\| \sup_{-1\leq s\leq 0} \int_s^0 e^{-\operatorname{Re}\lambda(t-s)} \|f(t)\|_Z dt \\
&\leq \|\Psi\| \|B\| (\operatorname{Re}\lambda q)^{-1/q} \|f\|_{L^p([-1,0],Z)}
\end{aligned}$$

Using the triangle inequality we obtain that there is $\lambda_0 \in \mathbb{R}$ such that (9) is satisfied for $\operatorname{Re}\lambda = \lambda_0$. $\square$

4. Applications

We consider the equation

$$\begin{cases} u'(t) = Au(t) + \gamma Bu(t-1) + \displaystyle\int_{-1}^0 a(s) Bu(t+s) ds, \\ u(0) = x, \\ u_0 = f, \end{cases} \tag{10}$$

where $\gamma \in \mathbb{C}$ and $a \in L^1([-1, 0])$.

First we assume that $(A, D(A))$ generates a bounded analytic semigroup on a Banach space X and that $B \in \mathcal{L}(D(A), X)$ is relatively bounded with respect to A, i.e., there are nonnegative α, β with

$$\|Bx\| \leq \alpha\|Ax\| + \beta\|x\|, \qquad x \in D(A).$$

Let $K := \sup_{\operatorname{Re}\lambda>0} \|\lambda R(\lambda, A)\|$. Then

$$\|BR(\lambda, A)\| \leq \alpha(K+1) + \frac{\beta K}{|\lambda|}, \qquad \operatorname{Re}\lambda > 0.$$

Define $\Psi \in \mathcal{L}(C([-1,0], X), X)$ by

$$\Psi f := \gamma f(-1) + \int_{-1}^{0} a(s) f(s) ds$$

and let

$$\Phi f := \Psi(s \mapsto Bf(s)) = \gamma Bf(-1) + \int_{-1}^{0} a(s) Bf(s) ds.$$

Now we apply Example 3.6 which yields that condition (9) is satisfied for $\operatorname{Re}\lambda$ large enough if

$$\alpha(K+1) < \|\Psi\|^{-1}.$$

In particular, this is the case if $\alpha(K+1) < (|\gamma| + \|\alpha\|_1)^{-1}$.

Now Theorem 3.3 yields the following proposition:

Proposition 4.1. *Let $(A, D(A))$ be the generator of a bounded analytic semigroup on a Banach space X with $K := \sup_{\operatorname{Re}\lambda>0} \|\lambda R(\lambda, A)\|$ and let B be a linear operator in X bounded from $D(A)$ to X such that there are numbers $\alpha, \beta > 0$ with*

$$\|Bx\| \leq \alpha\|Ax\| + \beta\|x\|$$

for all $x \in D(A)$. Let $\gamma \in \mathbb{C}$ and $a \in L^1([-1,0])$ such that $\alpha(K+1) < (|\gamma| + \|a\|_1)^{-1}$. Then there is $n \in \mathbb{N}$ such that (10) has a unique solution for all $\binom{x}{f} \in D(\mathcal{A}^n)$, where $\mathcal{A}$ is the matrix operator associated to (10).

For a second application let $X = L^p(\mathbb{R})$ where $1 < p < \infty$ and let $m \geq 2$ be an integer. We define the operator $(A_m, D(A_m))$ by

$$A_m f := i f^{(m)} \qquad \text{if } m \text{ is even,}$$

and by

$$A_m f := f^{(m)} \qquad \text{if } m \text{ is odd,}$$

with domain $D(A) := W^{m,p}(\mathbb{R})$ in $L^p(\mathbb{R})$.

Then $(A_m, D(A_m))$ generates a C_0-semigroup on X if and only if $p = 2$ ([12]). For $m = 2$ this was proved first by Hörmander [14] in 1960. If $p \neq 2$, $(A_m, D(A_m))$ generates an α-times integrated semigroup on X for $\alpha > \left|\frac{1}{2} - \frac{1}{p}\right|$ ([12]).

We again consider the Delay equation (10) where $A = A_m$ and $(B, D(B))$ is given by

$$Bf := V \cdot f^{(l)}$$

with maximal domain

$$D(B) := \{f \in L^p(\mathbb{R}) : V \cdot f^{(l)} \in L^p(\mathbb{R})\}$$

in $L^p(\mathbb{R})$. Here, V is a potential and $l \in \mathbb{N} \cup \{0\}$.

We will use the following lemma from [15].

Lemma 4.2. *Let $X = L^p(\mathbb{R})$, $1 < p < \infty$, and let one of the conditions*

(i) $l \le \frac{1}{p}(m-1)$ *and* $V \in L^p(\mathbb{R})$

or

(ii) $l = 0$ *and* $V \in L^p(\mathbb{R}) + L^\infty(\mathbb{R})$

hold true. Let $(A_m, D(A_m))$ and $(B, D(B))$ be defined as above. Then $D(A) \subseteq D(B)$ and for each $M \ge 0$ there is a $\lambda_M \in \mathbb{R}$ such that $\|BR(\lambda, A_m)\| \le M$ for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \ge \lambda_M$.

In particular, $D(A) \hookrightarrow D(B) \hookrightarrow X$ and $B \in \mathcal{L}(D(B), X)$, since B is closed. If we define Ψ and Φ as above, then Example 3.6 yields that condition (9) is satisfied for $\operatorname{Re}\lambda$ large enough. We again use Theorem 3.3 and obtain

Proposition 4.3. *Let $X = L^p(\mathbb{R})$ and let $A = A_m$ and B be defined as above, where*

(i) $l \le \frac{1}{p}(m-1)$ *and* $V \in L^p(\mathbb{R})$

or

(ii) $l = 0$ *and* $V \in L^p(\mathbb{R}) + L^\infty(\mathbb{R})$.

Let $\gamma \in \mathbb{C}$ and $a \in L^1([-1,0])$. Then there is $n \in \mathbb{N}$ such that (10) *has a unique solution for all $\binom{x}{f} \in D(\mathcal{A}^n)$, where $\mathcal{A}$ is the matrix operator associated to* (10).

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