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Auslander's Formula for stable ∞-categories

(after Jona Klemenc)

$\mathcal{A}$: (ess. small) abelian cat (e.g $\operatorname{mod}\Lambda$, Λ: fin.-dim algebra)

finite product-preserving

$\operatorname{Mod}(\mathcal{A}) := \operatorname{Fun}^{\pi}(\mathcal{A}^{op}, Ab)$ additive functors $F: \mathcal{A}^{op} \longrightarrow Ab$

$\cup\!\mathsf{I}$

$\operatorname{mod}(\mathcal{A})$: coherent functors $\mathcal{A}(-,X) \xrightarrow{f\circ ?} \mathcal{A}(-,Y) \to F \to 0$

$\mathsf{I}\!\cup$

$\operatorname{eff}(\mathcal{A})$: effaceble functors $\mathcal{A}(-,X) \xrightarrow{f\circ ?} \mathcal{A}(-,Y) \to F \to 0$, $f: X \twoheadrightarrow Y$ epi

$\mathcal{A}$: abelian cat. $\Rightarrow$ $\operatorname{mod}(\mathcal{A})$: abelian cat

$\mathsf{I}\!\cup$

$\operatorname{eff}(\mathcal{A})$: Serre subcategory

(Auslander's Formula 1965) $\mathcal{A} \xrightarrow{\sim} \dfrac{\operatorname{mod}(\mathcal{A})}{\operatorname{eff}(\mathcal{A})}$

$$\delta: X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z \quad \underset{\text{Left exact}}{\overset{\text{Yoneda}}{\longmapsto}} \quad 0 \to \mathcal{A}(-,X) \xrightarrow{i\circ ?} \mathcal{A}(-,Y) \xrightarrow{p\circ ?} \mathcal{A}(-,Z) \longrightarrow \underset{\text{defect}}{\overset{\operatorname{eff}(\mathcal{A}) \ni}{\delta^*}} \longrightarrow 0$$

(Krause 2015)

$$\begin{tikzcd}
Ac^b(\mathcal{A}) \arrow[r, hook] \arrow[d, "\wr"] & K^b(\mathcal{A}) \arrow[r, two heads] \arrow[d, "\wr"] & D^b(\mathcal{A}) \arrow[d, "\wr"] \\
\operatorname{thick}(\operatorname{eff}(\mathcal{A})) \arrow[r, hook] & D^b(\operatorname{mod}(\mathcal{A})) \arrow[r, two heads] & \dfrac{D^b(\operatorname{mod}(\mathcal{A}))}{\operatorname{thick}(\operatorname{eff}(\mathcal{A}))}
\end{tikzcd}$$

Aim Generalise from $\mathcal{D}^b(\mathcal{A})$ to a larger class of triangulated categories.

Thm (Folklore) $\mathcal{A}$: small additive cat. $\Rightarrow$ $\operatorname{Mod}(\mathcal{A}) = \operatorname{Fun}^{\pi}(\mathcal{A}^{op}, Ab) \xrightarrow{\sim} \operatorname{Fun}^{\pi}(\mathcal{A}^{op}, Set)$

Problem $D(\operatorname{Mod}(\mathcal{A})) \not\simeq \operatorname{Fun}^{\pi}(\mathcal{A}^{op}, D(Ab))$ due to the bad categorical properties of triangulated categories.

∞-cat. theory Set $\hookrightarrow \mathrm{Gpd}_\infty$: ∞-groupoids

$X \in \mathrm{Gpd}_\infty \rightsquigarrow \pi_0(X)$: set of "path connected components"
$\rotatebox{90}{$\in$}$
x, $k \geq 1 \rightsquigarrow \pi_k(X, x)$: <u>k-th homotopy group</u> (abelian for $k \geq 2$)

$\mathrm{Set} \simeq \{ X \in \mathrm{Gpd}_\infty \mid \forall x \in X \ \forall k \geq 1 \ \pi_k(X,x) = 0 \}$

$\mathcal{C}$: ∞-cat $\rightsquigarrow \forall X, Y \in \mathcal{C}$ $\mathrm{Map}_{\mathcal{C}}(X,Y)$: <u>∞-groupoid of maps</u> $X \to Y$

$\mathrm{Cat} \simeq \{ \mathcal{C} \in \mathrm{Cat}_\infty \mid \forall X, Y \in \mathcal{C} \ \mathrm{Map}_{\mathcal{C}}(X,Y) \in \mathrm{Set} \}$

$\mathrm{Cat} \hookrightarrow \mathrm{Cat}_\infty$ admits a left adjoint $\mathrm{Ho} : \mathrm{Cat}_\infty \longrightarrow \mathrm{Cat}$

$\therefore \forall \mathcal{C} \in \mathrm{Cat}_\infty \rightsquigarrow \mathrm{Ho}(\mathcal{C}) \in \mathrm{Cat}$ <u>homotopy cat.</u> of $\mathcal{C}$

e.g. $\mathrm{Ho}(\mathrm{Gpd}_\infty) \simeq \mathrm{HTop}$: cat. of top. spaces up to weak htpy. eq.

Def (Lurie) $\mathcal{A}$: ∞-cat is <u>additive</u> if

(0) $\exists\, 0 \in \mathcal{A}$: zero object

(1) $\forall X, Y \in \mathcal{A}$ $\exists\, X \amalg Y$, $X \times Y \in \mathcal{A}$

(2) $\mathrm{Ho}(\mathcal{A})$ is an additive category $\left\{ \begin{array}{lll} \forall X, Y \in \mathcal{A} & X \amalg Y \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} X \times X & \text{iso} \\ \forall X \in X & X \oplus X \xrightarrow{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} X \oplus X & \text{iso} \end{array} \right.$

Thm (Lurie) $\mathcal{A}$: small additive ∞-cat. $\mathrm{Fun}^{\pi}(\mathcal{A}^{op}, ?) \xrightarrow{\sim} \mathrm{Fun}^{\pi}(\mathcal{A}^{op}, \mathrm{Gpd}_\infty)$

$\mathrm{Ab} \hookrightarrow$? What are the analogues of abelian groups in ∞-cat theory?

Def $\mathcal{C}$: ∞-cat is <u>stable</u> if

(0) $\exists\, 0 \in \mathcal{C}$: zero object

(1) $\forall f : X \to Y$ in $\mathcal{C}$ $\exists$ $\begin{array}{ccc} W & \to & X \\ \downarrow & \text{PB} & \downarrow f \\ 0 & \to & Y \end{array}$ & $\exists$ $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \text{PO} & \downarrow \\ 0 & \to & Z \end{array}$ $W := \mathrm{fib}(f)$, $Z := \mathrm{cofib}(f)$

(2) A square $\begin{array}{ccc} X & \to & Y \\ \downarrow & & \downarrow \\ 0 & \to & Z \end{array}$ in $\mathcal{C}$ is PB $\Leftrightarrow$ it is PO $\Big\}$ <u>fibre-cofibre sequences</u>

Thm (Lurie) $\mathcal{C}$: stable ∞-cat $\Rightarrow$ $\mathcal{C}$: additive ∞-cat. & $\mathrm{Ho}(\mathcal{C})$: triangulated cat.

Prop A : small cat & $\mathcal{C}$: stable ∞-cat $\Rightarrow$ $\mathrm{Fun}(A, \mathcal{C})$: stable ∞-cat

Warning $\mathrm{Ho}(\mathrm{Fun}(A, \mathcal{C}))$ is triangulated but $\mathrm{Ho}(\mathrm{Fun}(A,\mathcal{C})) \not\simeq \mathrm{Fun}(A, \mathrm{Ho}(\mathcal{C}))$

Def $\mathcal{C}, \mathcal{D}$: stable ∞-cat's. $F : \mathcal{C} \to \mathcal{D}$ is exact if $F(0) \simeq 0$

and F preserves fibre-cofibre sequences

Rmk $F : \mathcal{C} \to \mathcal{D}$ exact $\Rightarrow$ $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ triangle functor

exact functors, defined in obvious

Prop $\mathcal{C}, \mathcal{D}$: stable ∞-cat's $\Rightarrow$ $\mathrm{Fun}^{ex}(\mathcal{C}, \mathcal{D})$: stable ∞-cat

$(\mathrm{Grp}_\infty)_* :=$ ∞-cat of pointed ∞-groupoids $\quad (X, x) = (* \xrightarrow{x} X)$

Def / Thm (Lurie) The ∞-cat of spectra is $\mathrm{Sp} := \lim \left(\cdots \xrightarrow{\Omega} (\mathrm{Grp}_\infty)_* \xrightarrow{\Omega} (\mathrm{Grp}_\infty)_* \right)$

(1) The ∞-cat Sp is stable and admits (small) colimits

free spectrum $\qquad$ forgetful functor

(2) $\exists\ \Sigma^\infty : \mathrm{Gpd}_\infty \rightleftarrows \mathrm{Sp} : \Omega^\infty$ adjunction

(3) $\mathbb{S} := \Sigma^\infty(S^0)$ is a compact object of Sp

Define $\pi_i := \mathrm{Ho}(\mathrm{Sp})(\Sigma^i(\mathbb{S}), -) : \mathrm{Ho}(\mathrm{Sp}) \longrightarrow \mathrm{Ab}$, $i \in \mathbb{Z}$

(4) The ~~pair~~ following pair $(\mathrm{Sp}_{\geq 0}, \mathrm{Sp}_{\leq 0})$ of full subcategories of Sp

is a non-degenerate t-structure with heart $\mathrm{Sp}^\heartsuit = \mathrm{Sp}_{\geq 0} \cap \mathrm{Sp}_{\leq 0} \simeq \mathrm{Ab}$

$\mathbb{S} \in \mathrm{Sp}_{\geq 0} := \{ X \in \mathrm{Sp} \mid \pi_{<0}(X) = 0 \}$, $\quad \mathrm{Sp}_{\leq 0} := \{ X \in \mathrm{Sp} \mid \pi_{>0}(X) = 0 \}$

Thm (Lurie) $\mathcal{A}$: small additive ∞-cat. $\Longrightarrow$ $\Omega^\infty : \mathrm{Fun}^\pi(\mathcal{A}^{op}, \mathrm{Sp}_{\geq 0}) \xrightarrow{\sim} \mathrm{Fun}^\pi(\mathcal{A}^{op}, \mathrm{Gpd}_\infty)$

$$\begin{tikzcd}
\mathrm{Fun}^\pi(\mathcal{A}^{op}, \mathrm{Ab}) \arrow[r, hook] \arrow[d, phantom, "\wr"] & \mathrm{Fun}^\pi(\mathcal{A}^{op}, \mathrm{Sp}_{\geq 0}) \arrow[r, hook] \arrow[d, phantom, "\parallel"] & \mathrm{Fun}^\pi(\mathcal{A}^{op}, \mathrm{Sp}) \arrow[d, phantom, "\parallel"] \\
\mathrm{Mod}(\mathrm{Ho}(\mathcal{A})) \arrow[r, hook] & \underbrace{\mathrm{Sp}_{\geq 0}\text{-}\mathrm{Mod}(\mathcal{A})}_{\text{Aisle of pointwise t-structure}} \arrow[r, hook] & \mathrm{Sp}\text{-}\mathrm{Mod}(\mathcal{A})
\end{tikzcd}$$

Example $\mathcal{A}$: small additive 1-cat

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{A}) \arrow[r, hook] \arrow[d, phantom, "\parallel"] & \mathrm{Sp}_{\geq 0}\text{-}\mathrm{Mod}(\mathcal{A}) \arrow[r, hook] \arrow[d, leftrightarrow, "\wr"] & \mathrm{Sp}\text{-}\mathrm{Mod}(\mathcal{A}) \arrow[d, leftrightarrow, "\wr"] \\
\mathrm{Mod}(\mathcal{A}) \arrow[r, hook] & \mathcal{D}(\mathrm{Mod}(\mathcal{A}))_{\geq 0} \arrow[r, hook] & \mathcal{D}(\mathrm{Mod}(\mathcal{A}))
\end{tikzcd}$$

$\mathrm{Sp}\text{-}\mathrm{mod}(\mathcal{A}) \subseteq \mathrm{Sp}\text{-}\mathrm{Mod}(\mathcal{A})$: smallest stable subcat. containing $\{\hat{x} \mid x \in \mathcal{A}\}$

Example $\mathcal{A}$: small additive 1-cat $\rightsquigarrow$ $\mathrm{Sp}\text{-}\mathrm{mod}(\mathcal{A}) \simeq \mathcal{K}^b(\mathrm{proj}(\mathcal{A})) \simeq \mathcal{D}^b(\mathcal{A}, \mathcal{S}_\oplus)$

Def (Quillen 1972, Barwick 2015) $\mathcal{E}$: additive ∞-cat.

$\mathcal{S} \subseteq \mathrm{Fun}(\square, \mathcal{A})$ class of fibre-cofibre sequences
$$\begin{tikzcd}
X \arrow[r, tail, "i"', "\text{inflation}"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, two heads, "p"', "\text{deflation}"] \\
0 \arrow[r] & Z
\end{tikzcd}$$
in $\mathcal{E}$ (conflation)

We say that $(\mathcal{E}, \mathcal{S})$ is an <u>exact ∞-cat</u> if $\mathcal{S}$ satisfies the apparent ∞-categorical variant of the axioms of exact cat's.

Gabriel-Quillen Embedding $(\mathcal{E}, \mathcal{S})$: small exact cat.

$\Longrightarrow$ $\exists\, i : \mathcal{E} \hookrightarrow \mathcal{A}$, exact functor with $\mathcal{A}$: abelian s.t.
$i(\mathcal{E}) \subseteq \mathcal{A}$ is closed under extensions & i reflects conflations

Q What is the analogue of the Gabriel-Quillen Embedding for exact ∞-cat's?

$$\begin{array}{ccl} \mathscr{C} & \mathrm{St}_\infty & : \infty\text{-cat of (small) stable } \infty\text{-cat's \& exact functors} \\ \downarrow & \iota \downarrow \uparrow ? & \\ (\mathscr{C}, \mathcal{S}_{max}) & \mathrm{Ex}_\infty & : \infty\text{-cat of (small) exact } \infty\text{-cat's \& exact functors} \end{array}$$

$(\mathcal{E}, \mathcal{S})$: exact ∞-cat

$$\delta : \begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, two heads, "p"] \\ 0 \arrow[r] & Z \end{tikzcd} \longmapsto \begin{tikzcd} \hat{X} \arrow[r, "\hat{i}"] \arrow[d] & \hat{Y} \arrow[d, "\hat{p}"] \\ \hat{0} \arrow[r] & \hat{Z} \end{tikzcd} \in \mathrm{Sp\text{-}mod}(\mathcal{E}), \quad \mathrm{tot}(\delta) := \mathrm{colim}\left(\begin{tikzcd}[row sep=small, column sep=small] \hat{X} \arrow[r, "\hat{i}"] \arrow[d] \arrow[dr] & \hat{Y} \arrow[d] \arrow[dr] & \\ 0 \arrow[r] \arrow[dr] & \delta \arrow[r] \arrow[d] & \hat{Z} \arrow[d] \\ & 0 \arrow[r] & 0 \end{tikzcd} \right)$$

not bicartesian in general

"defect"

Example $(\mathcal{E}, \mathcal{S})$: exact 1-cat $\rightsquigarrow$ $\mathrm{tot}(X \rightarrowtail Y \twoheadrightarrow Z) = (\cdots 0 \to X \rightarrowtail Y \twoheadrightarrow Z \to 0 \to \cdots) \in \mathcal{K}^b(\mathcal{E})$

$\mathcal{H}_{st}(\mathcal{E}, \mathcal{S}) := \mathrm{Sp\text{-}mod}(\mathcal{E}) / \underbrace{\mathrm{thick}\{\mathrm{tot}(\delta) \mid \delta \in \mathcal{S}\}}_{\mathrm{Sp\text{-}eff}(\mathcal{E}, \mathcal{S})}$: stable hull

Thm (Klemenc 2022) $\mathcal{H}_{st} : \mathrm{Ex}_\infty \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{St}_\infty : \iota$ adjunction whose unit η satisfies :

(1) $\forall \mathscr{C}$: stable ∞-cat, $\eta^* : \mathrm{Fun}^{ex}(\mathcal{H}_{st}(\mathcal{E}, \mathcal{S}), \mathscr{C}) \xrightarrow{\sim} \mathrm{Fun}^{ex}((\mathcal{E}, \mathcal{S}), (\mathcal{H}_{st}(\mathcal{E}, \mathcal{S}), \mathcal{S}_{max}))$

(2) $\eta : \mathcal{E} \longrightarrow \mathcal{H}_{st}(\mathcal{E}, \mathcal{S})$ is fully faithful, exact, reflects conflations and $i(\mathcal{E}) \subseteq \mathcal{H}_{st}(\mathcal{E}, \mathcal{S})$ is closed under extensions

Corollary $\mathscr{C}$: stable ∞-cat $\rightsquigarrow$ $\eta : \mathscr{C} \xrightarrow{\sim} \mathcal{H}_{st}(\mathscr{C}, \mathcal{S}_{max}) = \mathrm{Sp\text{-}mod}(\mathscr{C}) / \mathrm{Sp\text{-}eff}(\mathscr{C})$

Krause's Derived Auslander Formula (Bunke - Cisinski - Kasprowski - Winges 2019)

$\mathcal{A}$: small abelian cat .

$$\begin{tikzcd} \mathrm{thick}(\mathrm{eff}(\mathcal{A})) \arrow[r, hook] \arrow[d, leftrightarrow, "\wr"] & \mathcal{D}^b(\mathrm{mod}(\mathcal{A})) \arrow[r, two heads] \arrow[d, leftrightarrow, "\wr"] & \dfrac{\mathcal{D}^b(\mathrm{mod}(\mathcal{A}))}{\mathrm{thick}(\mathrm{eff}(\mathcal{A}))} \arrow[d, leftrightarrow, "\wr"] \\ \mathrm{Ac}^b(\mathcal{A}) \arrow[r, hook] \arrow[d, leftrightarrow, "\wr"] & K^b(\mathcal{A}) \arrow[r, two heads] \arrow[d, leftrightarrow, "\wr"] & \mathcal{D}^b(\mathcal{A}) \arrow[d, leftrightarrow, "\wr"] \\ \mathrm{Sp\text{-}eff}(\mathcal{A}) \arrow[r, hook] & \mathrm{Sp\text{-}mod}(\mathcal{A}) \arrow[r, two heads] & \dfrac{\mathrm{Sp\text{-}mod}(\mathcal{A})}{\mathrm{Sp\text{-}eff}(\mathcal{A})} \end{tikzcd}$$