

Auslander's Formula for stable ∞ -categories

(after Jona Klemenc)

\mathcal{A} : (ess. small) abelian cat (e.g. $\text{mod } \Lambda$, Λ : fin.-dim algebra)

finite product-preserving

$\text{Mod}(\mathcal{A}) := \text{Fun}^{\text{tr}}(\mathcal{A}^{\text{op}}, \text{Ab})$ additive functors $F: \mathcal{A}^{\text{op}} \rightarrow \text{Ab}$

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$\text{mod}(\mathcal{A})$: coherent functors $\mathcal{A}(-, X) \xrightarrow{f \circ ?} \mathcal{A}(-, Y) \rightarrow F \rightarrow 0$

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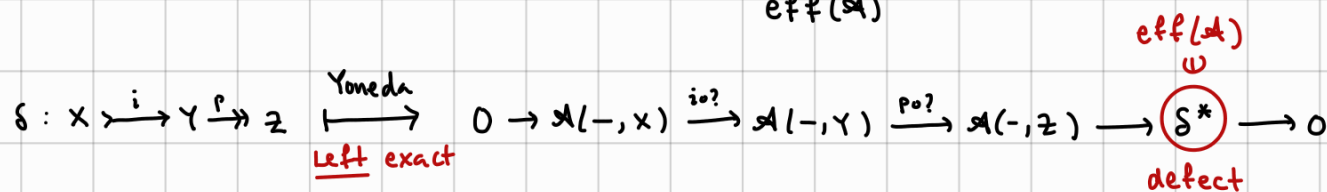
$\text{eff}(\mathcal{A})$: effaceable functors $\mathcal{A}(-, X) \xrightarrow{f \circ ?} \mathcal{A}(-, Y) \rightarrow F \rightarrow 0$, $f: X \twoheadrightarrow Y$ epi

\mathcal{A} : abelian cat. $\Rightarrow \text{mod}(\mathcal{A})$: abelian cat

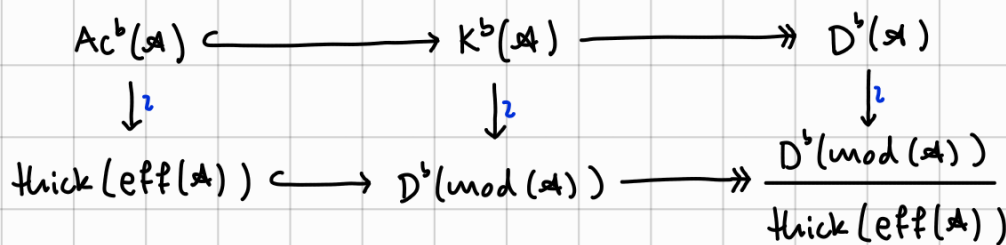
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$\text{eff}(\mathcal{A})$: Serre subcategory

(Auslander's Formula 1965) $\mathcal{A} \xrightarrow{\sim} \frac{\text{mod}(\mathcal{A})}{\text{eff}(\mathcal{A})}$



(Krause 2015)



Aim Generalise from $\text{D}^b(\mathcal{A})$ to a larger class of triangulated categories.

Thm (Folklore) \mathcal{A} : small additive cat. $\Rightarrow \text{Mod}(\mathcal{A}) = \text{Fun}^{\text{tr}}(\mathcal{A}^{\text{op}}, \text{Ab}) \xrightarrow{\sim} \text{Fun}^{\text{tr}}(\mathcal{A}^{\text{op}}, \text{Set})$

Problem $\text{D}(\text{Mod}(\mathcal{A})) \neq \text{Fun}^{\text{tr}}(\mathcal{A}^{\text{op}}, \text{D}(\text{Ab}))$ due to the bad categorical properties of triangulated categories.

∞ -cat. theory $\text{Set} \hookrightarrow \text{Gpd}_\infty$: ∞ -groupoids

$X \in \text{Gpd}_\infty \rightsquigarrow \pi_0(X)$: set of "path connected components"

$\pi_k, k \geq 1 \rightsquigarrow \pi_k(X, x)$: k -th homotopy group (abelian for $k \geq 2$)

$$\text{Set} \simeq \{ X \in \text{Gpd}_\infty \mid \forall x \in X \forall k \geq 1 \pi_k(X, x) = 0 \}$$

$\mathcal{C} : \infty\text{-cat} \rightsquigarrow \forall X, Y \in \mathcal{C} \text{ Map}_{\mathcal{C}}(X, Y)$: ∞ -groupoid of maps $X \rightarrow Y$

$$\text{Cat} \simeq \{ \mathcal{C} \in \text{Cat}_\infty \mid \forall X, Y \in \mathcal{C} \text{ Map}_{\mathcal{C}}(X, Y) \in \text{Set} \}$$

$\text{Cat} \hookrightarrow \text{Cat}_\infty$ admits a left adjoint $\text{Ho} : \text{Cat}_\infty \rightarrow \text{Cat}$

$\therefore \forall \mathcal{C} \in \text{Cat}_\infty \rightsquigarrow \text{Ho}(\mathcal{C}) \in \text{cat}$ homotopy cat. of \mathcal{C}

e.g. $\text{Ho}(\text{Gpd}_\infty) \simeq \text{HTop}$: cat. of top. spaces up to weak htpy. eq.

Def (Lurie) $\mathcal{A} : \infty\text{-cat}$ is additive if

(0) $\exists 0 \in \mathcal{A}$: zero object

(1) $\forall X, Y \in \mathcal{A} \exists X \amalg Y, X \times Y \in \mathcal{A}$

(2) $\text{Ho}(\mathcal{A})$ is an additive category

$$\left\{ \begin{array}{l} \forall X, Y \in \mathcal{A} \quad X \amalg Y \xrightarrow{(\text{id}, \text{id})} X \times Y \quad \text{iso} \\ \forall X \in \mathcal{A} \quad X \oplus X \xrightarrow{(\text{id}, \text{id})} X \oplus X \quad \text{iso} \end{array} \right.$$

Thm (Lurie) \mathcal{A} : small additive ∞ -cat. $\text{Fun}^{\text{tr}}(\mathcal{A}^{\text{op}}, ?) \rightsquigarrow \text{Fun}^{\text{tr}}(\mathcal{A}^{\text{op}}, \text{Gpd}_\infty)$

$\text{Ab} \hookrightarrow ?$ What are the analogues of abelian groups in ∞ -cat theory?

Def $\mathcal{C} : \infty\text{-cat}$ is stable if

(0) $\exists 0 \in \mathcal{C}$: zero object

(1) $\forall f: X \rightarrow Y$ in $\mathcal{C} \exists \text{fib}(f) \rightarrow X \rightarrow Y \rightarrow \text{cofib}(f)$

(2) A square

$$\begin{array}{ccc} X & \rightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \rightarrow & Z \end{array}$$

in \mathcal{C} is PB

\Leftrightarrow it is PO

fibre-cofibre sequences

$W := \text{fib}(f), Z := \text{cofib}(f)$

$$\begin{array}{ccc} W & \rightarrow & X \\ \downarrow \text{PB} & & \downarrow f \\ 0 & \rightarrow & Y \end{array} \quad \& \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \text{PO} & & \downarrow \\ 0 & \rightarrow & Z \end{array}$$

Thm (Lurie) \mathcal{C} : stable ω -cat $\Rightarrow \mathcal{C}$: additive ω -cat. & $\text{Ho}(\mathcal{C})$: triangulated cat.

Prop A : small cat & \mathcal{C} : stable ω -cat $\Rightarrow \text{Fun}(A, \mathcal{C})$: stable ω -cat

Warning $\text{Ho}(\text{Fun}(A, \mathcal{C}))$ is triangulated but $\text{Ho}(\text{Fun}(A, \mathcal{C})) \neq \text{Fun}(A, \text{Ho}(\mathcal{C}))$

Def \mathcal{C}, \mathcal{D} : stable ω -cat's. $F: \mathcal{C} \rightarrow \mathcal{D}$ is exact if $F(0) \simeq 0$
and F preserves fibre-wfibre sequences

Prop $F: \mathcal{C} \rightarrow \mathcal{D}$ exact $\Rightarrow \text{Ho}(F): \text{Ho}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{D})$ triangle functor

Prop \mathcal{C}, \mathcal{D} : stable ω -cat's $\Rightarrow \text{Fun}^{\text{ex}}(\mathcal{C}, \mathcal{D})$: stable ω -cat
exact functors, defined in obvious

$(\text{Grp}_{\infty})_*$:= ω -cat of pointed ω -groupoids $(X, x) = (* \xrightarrow{x} X)$

Def / Thm (Lurie) The ω -cat of spectra is $\text{Sp} := \text{lim}(\dots \xrightarrow{\Sigma} (\text{Grp}_{\infty})_* \xrightarrow{\Sigma} (\text{Grp}_{\infty})_*)$

(1) The ω -cat Sp is stable and admits (small) colimits

(2) $\exists \Sigma^{\infty}: \text{Gpd}_{\infty} \xleftrightarrow{\text{free spectrum}} \text{Sp} \xleftrightarrow{\text{forgetful functor}} \Omega^{\infty}$ adjunction

(3) $\mathbb{S} := \Sigma^{\infty}(S^0)$ is a compact object of Sp

Define $\pi_i := \text{Ho}(\text{Sp})(\Sigma^i(\mathbb{S}), -): \text{Ho}(\text{Sp}) \rightarrow \text{Ab}$, $i \in \mathbb{Z}$

(4) The pair following pair $(\text{Sp}_{>0}, \text{Sp}_{\leq 0})$ of full subcategories of Sp
is a non-degenerate t -structure with heart $\text{Sp}^{\heartsuit} = \text{Sp}_{>0} \cap \text{Sp}_{\leq 0} \simeq \text{Ab}$

$\mathbb{S} \in \text{Sp}_{>0} := \{X \in \text{Sp} \mid \pi_{\leq 0}(X) = 0\}$, $\text{Sp}_{\leq 0} := \{X \in \text{Sp} \mid \pi_{>0}(X) = 0\}$

Thm (Lurie) \mathcal{A} : small additive ∞ -cat. $\Rightarrow \Omega^\infty: \text{Fun}^\pi(\mathcal{A}^{\text{op}}, \text{Sp}_{\geq 0}) \xrightarrow{\sim} \text{Fun}^\pi(\mathcal{A}^{\text{op}}, \text{Sp}_{\text{dual}})$

$$\begin{array}{ccccc} \text{Fun}^\pi(\mathcal{A}^{\text{op}}, \text{Ab}) & \hookrightarrow & \text{Fun}^\pi(\mathcal{A}^{\text{op}}, \text{Sp}_{\geq 0}) & \hookrightarrow & \text{Fun}^\pi(\mathcal{A}^{\text{op}}, \text{Sp}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \text{Mod}(\text{Ho}(\mathcal{A})) & \hookrightarrow & \text{Sp}_{\geq 0}\text{-Mod}(\mathcal{A}) & \hookrightarrow & \text{Sp-Mod}(\mathcal{A}) \end{array}$$

Aisle of pointwise t-structure

Example \mathcal{A} : small additive 1-cat

$$\begin{array}{ccccc} \text{Mod}(\mathcal{A}) & \hookrightarrow & \text{Sp}_{\geq 0}\text{-Mod}(\mathcal{A}) & \hookrightarrow & \text{Sp-Mod}(\mathcal{A}) \\ \parallel & & \downarrow \cong & & \downarrow \cong \\ \text{Mod}(\mathcal{A}) & \hookrightarrow & \mathcal{D}(\text{Mod}(\mathcal{A}))_{\geq 0} & \hookrightarrow & \mathcal{D}(\text{Mod}(\mathcal{A})) \end{array}$$

$\text{Sp-mod}(\mathcal{A}) \subseteq \text{Sp-Mod}(\mathcal{A})$: smallest stable subcat. containing $\{\hat{x} \mid x \in \mathcal{A}\}$

Example \mathcal{A} : small additive 1-cat $\rightsquigarrow \text{Sp-mod}(\mathcal{A}) \simeq \mathcal{K}^b(\text{proj}(\mathcal{A})) \simeq \mathcal{D}^b(\mathcal{A}, \mathcal{S}_\oplus)$

Def (Quillen 1972, Barwick 2015) \mathcal{E} : additive ∞ -cat.

$\mathcal{S} \subseteq \text{Fun}(\text{fib-cofib}, \mathcal{A})$ class of fibre-cofibre sequences

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow & \square & \downarrow P \\ 0 & \longrightarrow & Z \end{array} \text{ in } \mathcal{E}$$

inflation (pointing to i), deflation (pointing to P), conflation (under $0 \rightarrow Z$)

We say that $(\mathcal{E}, \mathcal{S})$ is an exact ∞ -cat if \mathcal{S} satisfies the apparent ∞ -categorical variant of the axioms of exact cat's.

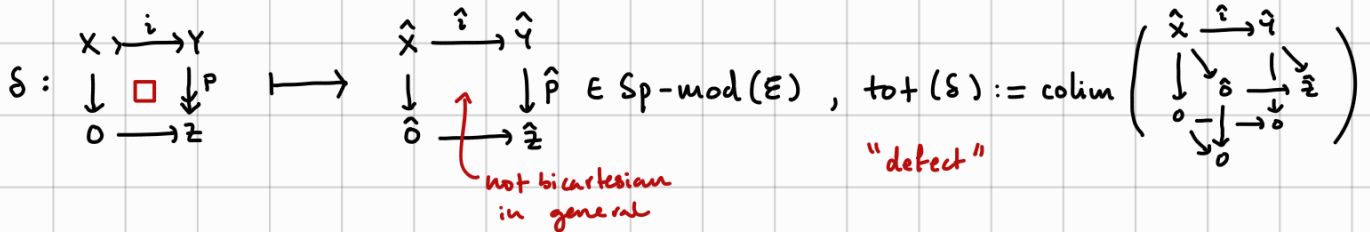
Gabriel-Quillen Embedding $(\mathcal{E}, \mathcal{S})$: small exact cat.

$\Rightarrow \exists i: \mathcal{E} \hookrightarrow \mathcal{A}$, exact functor with \mathcal{A} : abelian s.t.
 $i(\mathcal{E}) \subseteq \mathcal{A}$ is closed under extensions & i reflects conflations

Q What is the analogue of the Gabriel-Quillen Embedding for exact ∞ -cat's?

\mathcal{C} St_{∞} : ∞ -cat of (small) stable ∞ -cat's & exact functors
 \downarrow $\downarrow \uparrow$?
 $(\mathcal{C}, \mathcal{S}_{\max})$ Ex_{∞} : ∞ -cat of (small) exact ∞ -cat's & exact functors

$(\mathcal{E}, \mathcal{S})$: exact ∞ -cat



Example $(\mathcal{E}, \mathcal{S})$: exact 1-cat $\rightsquigarrow \text{tot}(X \rightarrow Y \rightarrow Z) = (\dots \rightarrow 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \rightarrow \dots) \in \mathcal{K}^b(\mathcal{E})$

$\mathcal{H}_{st}(\mathcal{E}, \mathcal{S}) := Sp\text{-mod}(\mathcal{E}) / \underbrace{\text{thick} \{ \text{tot}(\mathcal{S}) \mid \mathcal{S} \in \mathcal{S} \}}_{Sp\text{-eff}(\mathcal{E}, \mathcal{S})}$: stable hull

Thm (Klemenc 2022) $\mathcal{H}_{st}: Ex_{\infty} \xrightleftharpoons{\perp} St_{\infty} : \perp$ adjunction whose unit η satisfies:

- (1) $\forall \mathcal{C}$: stable ∞ -cat, $\eta^*: \text{Fun}^{ex}(\mathcal{H}_{st}(\mathcal{E}, \mathcal{S}), \mathcal{C}) \xrightarrow{\sim} \text{Fun}^{ex}(\mathcal{E}, \mathcal{S}, (\mathcal{H}_{st}(\mathcal{E}, \mathcal{S}), \mathcal{S}_{\max}))$
- (2) $\eta: \mathcal{E} \rightarrow \mathcal{H}_{st}(\mathcal{E}, \mathcal{S})$ is fully faithful, exact, reflects conflation and $i(\mathcal{E}) \subseteq \mathcal{H}_{st}(\mathcal{E}, \mathcal{S})$ is closed under extensions

Corollary \mathcal{C} : stable ∞ -cat $\rightsquigarrow \eta: \mathcal{C} \xrightarrow{\sim} \mathcal{H}_{st}(\mathcal{C}, \mathcal{S}_{\max}) = Sp\text{-mod}(\mathcal{C}) / Sp\text{-eff}(\mathcal{C})$

Krause's Derived Auslander Formula (Bunke-Cisinski-Kasprowski-Winges 2019)

\mathcal{A} : small abelian cat.

