HMS symmetries (and their decategorification) for toric boundary divisors

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(joint with Špela Špenko)

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Plan

- The lecture is centered around the idea of actions of fundamental groups on derived categories.
- Most of the lecture will be a survey of existing results.
- Towards the end I will mention some new results.

Crepant resolutions

Everything is linear over $\mathbb{C}$.

Definition

If X is a normal algebraic variety with Gorenstein singularities (i.e. X is Cohen-Macaulay and the dualizing sheaf $\omega_X = (\det \Omega_X)^{**}$ is locally free) then a resolution of singularities $\pi : Y \to X$ is said to be crepant if $\pi^* \omega_X = \omega_Y$.

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- It need not exist.
- If it exists then it is generally not unique.

Crepant resolutions II

Example

The conifold, i.e. the quadratic singularity $xy - zw = 0$ has two distinct crepant resolutions given by blowing up (x, z) and (x, w). This is the so-called "Atiyah flop".

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Example

The three-dimensional hypersurface singularity

$$x^2 + y^2 + z^2 + w^n = 0 \qquad (n \geq 2)$$

has a crepant resolution if and only if n is even.

Nonetheless different crepant resolutions appear to be strongly related.

The Bondal-Orlov, Kawamata conjecture

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Bondal-Orlov and independently Kawamata conjectured a categorification of this result. Write $\mathcal{D}(Y) := D^b(\mathrm{coh}(Y))$.

Conjecture (Bondal-Orlov, Kawamata)

Assume $Y_i \to X$ for $i = 1, 2$ are two crepant resolutions of X. Then there is an equivalence of triangulated categories $\mathcal{D}(Y_1) \cong \mathcal{D}(Y_2)$ (linear over X).

Despite overwhelming evidence, this conjecture is still wide open!

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- In the case of the Atiyah flop (connecting the two crepant resolutions of the conifold) one possible equivalence is given by the Fourier-Mukai functor for the "fiber product kernel" $\mathcal{O}_{Y_1 \times_X Y_2}$ but this is far from the only possibility.

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- The fiber product kernel does not always work.

Example (Cautis)

The cotangent bundles $T^* \operatorname{Gr}(d, n)$ and $T^* \operatorname{Gr}(n-d, n)$, for complementary Grassmannians with $d \leq n/2$ are crepant resolutions of $\{X \in M_n(k) \mid X^2 = 0, \operatorname{rk} X \leq d\}$. There is an equivalence $\mathcal{D}(T^* \operatorname{Gr}(d, n)) \to \mathcal{D}(T^* \operatorname{Gr}(n-d, n))$ but it is not given by the fiber product kernel.

The Bondal-Orlov, Kawamata conjecture III

Known cases:

- Dimension 3 (Bridgeland).
- Toric varieties (Kawamata).
- Symplectic singularities (Kaledin).
- Many crepant resolutions obtained by variation of GIT (Halpern-Leistner-Sam, Ballard-Favero-Katzarkov).

The stringy Kähler moduli space

- It is now understood, thanks to intuition from physics, that the equivalences $\mathcal{D}(Y_1) \cong \mathcal{D}(Y_2)$ should be canonically associated to paths connecting two points in a topological space called the "stringy Kähler moduli space" (SKMS).

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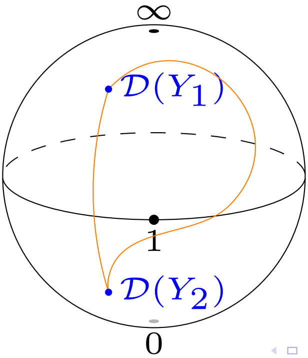
- In the case of the conifold the SKMS is given by $\mathbb{P}^1 - \{0, 1, \infty\}$.



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- To understand the conifold it is convenient to write it as a quotient singularity.
- Assume that $T = \mathbb{C}^*$ acts on $Z = \mathbb{C}^4$ with weights $(1, 1, -1, -1)$. I.e. via $t \cdot (x_1, x_2, x_3, x_4) = (tx_1, tx_2, t^{-1}x_3, t^{-1}x_4)$.

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- Then $Z /\!\!/ T = \{(u, v, w, x) \in \mathbb{C}^4 \mid ux = vw\}$ with $u = x_1x_3$, $v = x_1x_4$, $w = x_2x_3$, $x = x_2x_4$. Thus $Z /\!\!/ T$ is the conifold!

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- Then $Z/\!\!/T = \{(u, v, w, x) \in \mathbb{C}^4 \mid ux = vw\}$ with $u = x_1x_3$, $v = x_1x_4$, $w = x_2x_3$, $x = x_2x_4$. Thus $Z/\!\!/T$ is the conifold!
- The two crepant resolutions are given by
$$Z^{ss,\pm}/T \to Z/\!\!/T$$
with $Z^{ss,\pm} = Z - N^{\pm}$, $N^+ = \{x_1 = 0, x_2 = 0\}$, $N^- = \{x_3 = 0, x_4 = 0\}$.

Windows (Donovan-Segal, Halpern-Leistner)

- For a reductive group G acting on an algebraic variety we may consider the quotient stack $[Z/G]$. We have

$$\mathrm{coh}([Z/G]) := \mathrm{coh}_G(Z)$$

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- Note: if Z is affine then $\mathcal{W}(U) \cong \mathcal{D}(\Lambda(U))$ where $\Lambda(U) := \mathrm{End}_Z(U \otimes \mathcal{O}_Z)^G$. This is a non-commutative ring.

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- By composing these equivalences and their inverses we get many autoequivalences of $\mathcal{D}(Z^{ss,\pm}/T)$ (the crepant resolutions of the conifold)!
- We can organize these in a $\mathbb{Z}$-equivariant local system of triangulated categories.

Local systems

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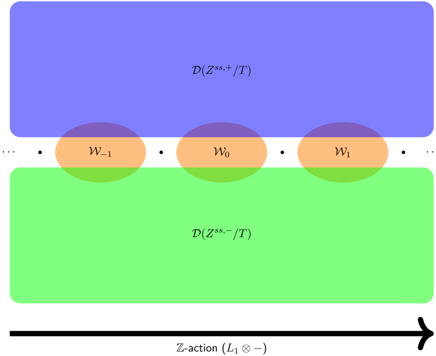
- One may also specify a local system L by specifying $L(U_i)$ for an open cover $\cup_i U_i = M$ with U_i simply connected, together with gluing data.

Local systems of categories

Put $\mathcal{W}_n = \mathcal{W}(L_n \oplus L_{n+1})$. We obtain a $\mathbb{Z}$-equivariant local system of triangulated categories on $\mathbb{C} - \mathbb{Z}$.



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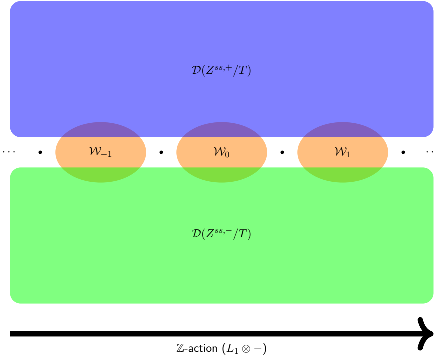

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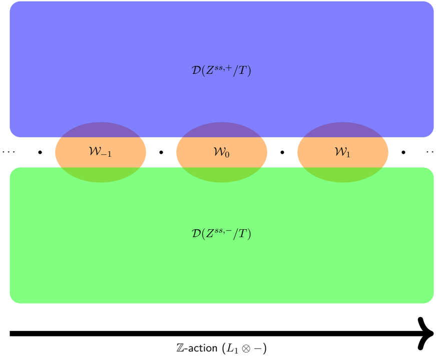


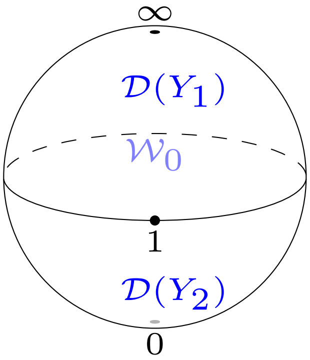
- Choosing a base point in the blue area we get an action of $\pi_1(\mathbb{C} - \mathbb{Z})$ on $\mathcal{D}(Z^{ss,+}/T)$.
- Using the $\mathbb{Z}$-action we get a $\pi_1((\mathbb{C} - \mathbb{Z})/\mathbb{Z})$-action.

Local systems of categories II

There is a homeomorphism of topological spaces

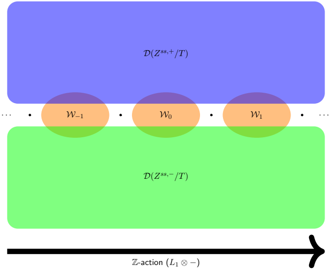
$$(\mathbb{C} - \mathbb{Z})/\mathbb{Z} \cong \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\} : \bar{z} \mapsto e^{2\pi i z}.$$

Note that $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ is a sphere minus three points!



- Near the poles we have "commutative resolutions" $\mathcal{D}(Z^{ss,+}/T)$.
- Near the equator we have a "noncommutative resolution" such as $\mathcal{W}_0 \cong \mathcal{D}(\Lambda(L_0 \oplus L_1))$.
- The $\mathbb{Z}$-action corresponds to loops around the poles.

The standard pattern



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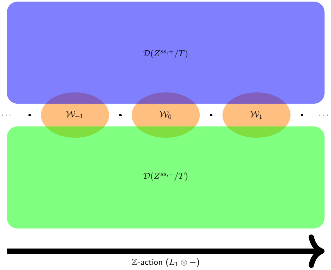
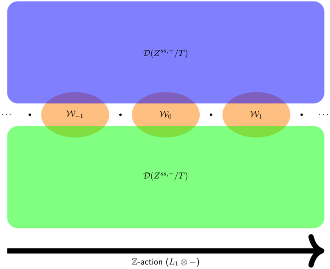


- The SKMS $(\mathbb{C} - \mathbb{Z})/\mathbb{Z}$ of the conifold is a special case of a common (but not universal) pattern.
- It is of the form $(\mathbb{C}^n - \mathcal{H}_\mathbb{C})/L$ where $\mathcal{H}_\mathbb{C}$ is the complexification of a real affine hyperplane arrangement $\mathcal{H}$ and L is a real lattice leaving $\mathcal{H}$ invariant (in the conifold case we have $n = 1$, $\mathcal{H} = \mathbb{Z}$, $L = \mathbb{Z}$).

The standard pattern II



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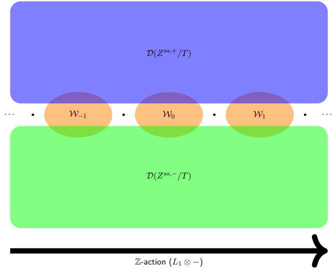
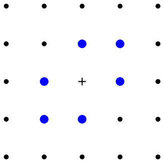


- The commutative crepant resolutions are given by the connected components of $i\mathbb{R}^n - i\mathcal{H}_c$ where $\mathcal{H}_c$ is the central hyperplane arrangement corresponding to $\mathcal{H}$.
- The noncommutative (crepant) resolutions are given by the connected components of $(\mathbb{R}^n - \mathcal{H})/L$.

A four-dimensional example

Consider $(\mathbb{C}^*)^2$ acting on $\mathbb{C}^6$ with weights $(0,1)$, $(1,1)$, $(1,0)$, $(0,-1)$, $(-1,-1)$, $(-1,0)$.

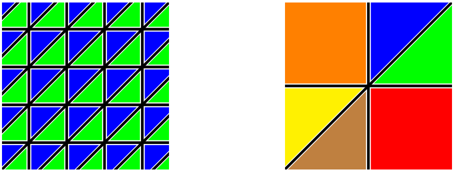


Quotient

$$V(uvw - pq) \subset \mathbb{C}^5$$

Hyperplane arrangement

$\mathbb{R}^2$ $\qquad$ $i\mathbb{R}^2$



Resolutions which are partially commutative and partially non-commutative also appear in this setting.

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- GIT quotients of "quasi-symmetric" representations (Špenko-VdB, Halpern-Leistner-Sam). Note: a G-representation W is said to be symmetric if $W \cong W^*$. "Quasi-symmetry" is a weaker version of this (see below).

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- Non-quasi-symmetric representations do not satisfy the pattern.

Decategorification

If $\mathcal{D}$ is a local system of categories on a topological space M then $U \mapsto K_0(\mathcal{D}(U))_{\mathbb{C}}$ defines a local system of vector spaces which we call the decategorification of $\mathcal{D}$. It is often given by the solutions of an interesting differential equation.

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$$z(1-z)\frac{d^2 f}{dz^2} + (c - (a+b+1)z)\frac{df}{dz} - abf = 0$$

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Fact

The local system corresponding to the conifold is the rank two local system given by the solutions of the hypergeometric equation for $a = b = c = 0$.

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- By using H-equivariant derived categories in the construction of the local system of triangulated categories on $(\mathbb{C} - \mathbb{Z})/\mathbb{Z} \cong \mathbb{P}^1 - \{0, 1, \infty\}$ we get, after decategorification, a local system of modules of rank two over $R(H)$. Specializing we get indeed a local system depending on 3 parameters!

Gorenstein affine toric varieties

- Let $P \subset \mathbb{R}^{k-1} \times \{1\}$ be a lattice polygon (for the lattice $\mathbb{Z}^{k-1} \times \{1\} \subset \mathbb{R}^{k-1} \times \{1\}$) and let σ be the cone over P. Then $X_P := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^k]$ for

$$\sigma^\vee = \{x \in \mathbb{R}^k \mid \forall y \in \mathbb{R}^k : \langle x, y \rangle \geq 0\}.$$

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Example

If $P \subset \mathbb{R}^2 \times \{1\}$ is the square with corners $\{(0,0,1),(0,1,1),(1,1,1),(1,0,1)\}$ then X_P is the conifold.

Gorenstein affine toric varieties II

- Another way to construct Gorenstein affine toric varieties is as GIT quotients $\mathbb{C}^d /\!\!/ T$ where $T \cong (\mathbb{C}^*)^l$ acts linearly on $\mathbb{C}^d$ with weights $\beta_1, \ldots, \beta_d \in X(T) := \mathrm{Hom}(T, \mathbb{C}^*)$ such that $\sum_i \beta_i = 0$.

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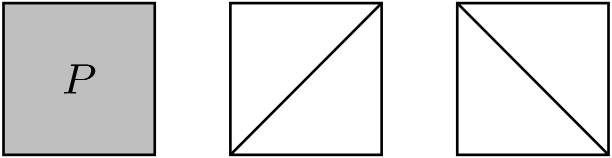
- We say that the weights are quasi-symmetric is $\sum_{\beta_i \in \ell} \beta_i = 0$ for every line $\ell \subset X(T)_{\mathbb{R}}$ through the origin.

Gorenstein affine toric varieties III

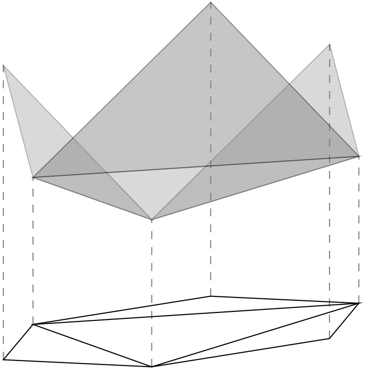
Fact: the crepant resolutions of X_P (by Deligne-Mumford stacks) correspond to lattice triangulations of P. If Σ is the fan-associated to a lattice triangulation, i.e. the collection of cones spanned by the triangles then the resolution is the corresponding toric stack X_Σ.



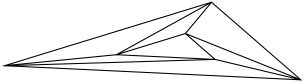
The two crepant resolutions of the conifold

Gorenstein affine toric varieties IV

- The projective crepant resolutions are given by so-called "regular triangulations" (linear loci of piecewise linear convex functions)

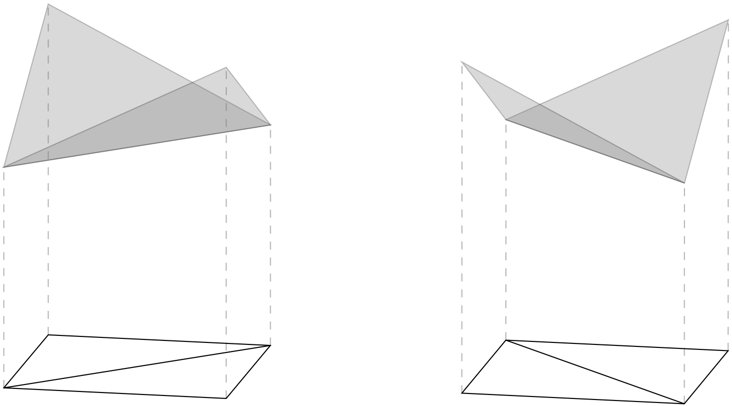


Regular



Non-regular

Gorenstein affine toric varieties V



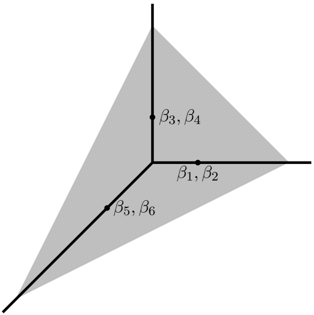
The two triangulations of the square are regular

Gorenstein affine toric varieties VI

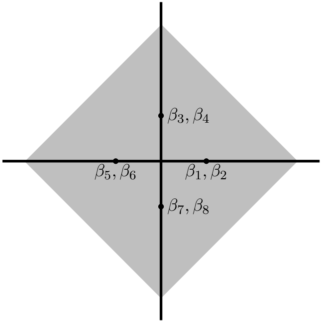
- Fact: the regular triangulations (= projective crepant resolutions) with vertices in a fixed $A \subset P \cap \mathbb{Z}^k$ with $\mathrm{conv}(A) = P$ correspond to the maximal cones in the so-called "secondary fan" (Gelfand-Kapranov-Zelevinsky). In general the secondary fan does not correspond to the complement of a hyperplane arrangement.

Examples of secondary fans

- We assume $T = (\mathbb{C}^*)^2$ so that $X(T) \cong \mathbb{Z}^2$. In that case the secondary fan can be deduced from the weights $(\beta_i)_i$.



Non-quasi-symmetric case



Quasi-symmetric case

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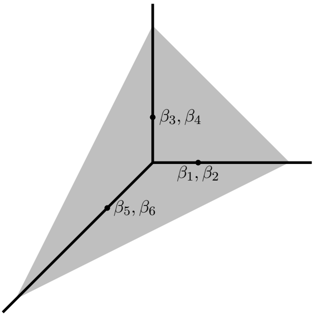

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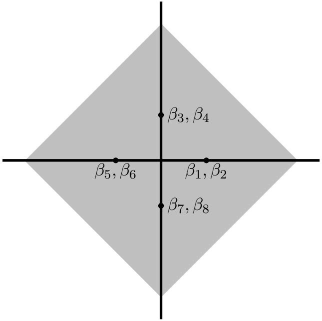

Quasi-symmetric case

- Fact: in the quasi-symmetric case the secondary fan corresponds to the complement of a central hyperplane arrangement.

The SKMS for Gorenstein affine toric varieties

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- For F a face of P (e.g. $F = P$) let $\nabla_F \subset \mathbb{C}^{F \cap A}$ be the set of $(\alpha_a)_a \in \mathbb{C}^{F \cap A}$ such that the variety

$$\{x \in (\mathbb{C}^*)^k \mid \sum_{a \in F \cap A} \alpha_a x^a = 0\}$$

is singular.

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- Set $V(A) := \bigcup_F p_F^{-1}(\nabla_F)$, where $p_F : \mathbb{C}^A \to \mathbb{C}^{F \cap A}$ is the projection.

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- We let $(\mathbb{C}^*)^k$ act on $\mathbb{C}^A$ with weights given by the elements of A. $V(A)$ is invariant under this action. We set

$$\mathcal{K}_A := [(\mathbb{C}^A \setminus V(A))/(\mathbb{C}^*)^k].$$

  This is a Deligne-Mumford stack.

The SKMS for the conifold

- In the conifold case the elements of A are the columns of

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Consider the case $F = P$. The Laurent polynomial

$$az + bxz + cyz + dxyz$$

is singular when

$$\begin{aligned} bz + dyz &= 0 \\ cz + dxz &= 0 \\ a + bx + cy + dxy &= 0 \end{aligned}$$

which has a solution in $(\mathbb{C}^*)^3$ if and only if $ad - bc \neq 0$.

The SKMS of the conifold

- Taking the other faces of $P = \mathrm{conv}(A)$ (a square) into account one gets that the SKMS of the conifold is

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- Sending (a, b, c, d) to ad/bc defines an isomorphism $(\mathbb{C}^*)^4/(\mathbb{C}^*)^3 \cong \mathbb{C}^*$. It identifies the SKMS with

$$\mathbb{C}^* - \{1\} = \mathbb{P}^1 - \{0, 1, \infty\}$$

(like before).

The SKMS for Gorenstein affine toric varieties II

Theorem (Kite)

Assume that that X_P is given by a quasi-symmetric GIT quotient $\mathbb{C}^d /\!\!/ T$. Then $\mathcal{K}_A$ is of the form $(X(T)_{\mathbb{C}} - \mathcal{H}_{\mathbb{C}})/X(T)$ where $\mathcal{H}_{\mathbb{C}}$ is the complexification of real affine hyperplane arrangement $\mathcal{H}$ in $X(T)_{\mathbb{R}}$.

Standard pattern!!

The SKMS for Gorenstein affine toric varieties III

Conjecture (SKMS conjecture)

$\pi_1(\mathcal{K}_A)$ acts on $\mathcal{D}(Y)$ for any crepant resolution $Y \to X_P$ corresponding to a triangulation of P with vertices in A.

The SKMS for Gorenstein affine toric varieties III

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Using earlier work of Halpern-Leistner and Špenko-VdB it follows

Theorem

The SKMS conjecture is true in the quasi-symmetric case.

The SKMS for Gorenstein affine toric varieties III

- The SKMS conjecture has been checked for some non-quasi-symmetric examples by Kite and Segal.

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- Various “wall-crossing” results “near infinity” are known (Balard-Favero-Katzarkov, Halpern-Leistner, Halpern-Leistner-Shipman, Segal-Kite,...) which are in particular sufficient to prove the Bondal-Orlov-Kawamata conjecture in the toric case.

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- The SKMS is related to the Bridgeland moduli space of stability conditions although they are not the same.

Decategorification: the Gelfand-Kaparanov-Zelevinsky system

- Let $A \subset P \cap \mathbb{Z}^k$ with $P = \mathrm{conv}(A)$. Put $d = |A|$. We think of A as a $k \times d$-matrix (i.e. the elements of A correspond to the columns, and the last row of A consists of 1's).

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- The GKZ system is the system of differential equations given by $(E_i - b_i)(f) = 0$ for $i = 1, \ldots, k$ and $\Box_l(f) = 0$ for all l such that $Al = 0$.

The GKZ system for the conifold

In the conifold case we have

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and one find that the GKZ system is given by

$$\begin{aligned} (x_2\partial_2 + x_4\partial_4)f &= b_1 \\ (x_3\partial_3 + x_4\partial_4)f &= b_2 \\ (x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4)f &= b_3 \\ (\partial_1\partial_4 - \partial_2\partial_3)f &= 0 \end{aligned}$$

One may show that this is equivalent to the hypergeometric system we have introduced above.

Decategorification: the GKZ system II

Assuming the SKMS conjecture, we can add:

Conjecture

The $\pi_1(\mathcal{K}_A)$ action after decategorification corresponds to GKZ system for a suitable parameter b.

To avoid having to choose a parameter one can work with equivariant derived categories as explained above for the conifold.

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In the general case "wall-crossing" results "near infinity" are known (Borisov-Horja, Borisov-Han).

The toric boundary

Definition

If X is a smooth toric stack for a torus H then X has a unique dense H-orbit. The complement of this orbit, denoted by ∂X, is called the the toric boundary. It is a normal crossing divisor.

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If X is a smooth toric stack for a torus H then X has a unique dense H-orbit. The complement of this orbit, denoted by ∂X, is called the the toric boundary. It is a normal crossing divisor.

It turns out that a version of the SKMS conjecture holds for the toric boundary!

Theorem (Špenko-VdB)

$\pi_1(\mathcal{K}_A)$ acts on $\mathcal{D}(\partial Y)$ for any crepant resolution $Y \to X_P$ corresponding to a triangulation of P with vertices in $A \subset P \cap \mathbb{Z}^k$ (with $\mathrm{conv}(A) = P$).

The toric boundary II

We can also describe the decategorification for this action.

Theorem (Špenko-VdB)

Let L be the local system which is the decategorification of the $\pi(\mathcal{K}_A)$-action on $D^b(\mathrm{coh}(\partial Y))$. Then we have an exact sequence of local systems

$$0 \to \underline{\mathbb{C}}^k \to L \to G \to \underline{\mathbb{C}} \to 0$$

where the first and the last local system are constant and where G is a GKZ system for suitable parameters.

In other words: up to constant local system the decategorification is given by a GKZ system!

Idea of proof: homological mirror symmetry

Let $f_\alpha(x)$ be the function on $(\mathbb{C}^*)^k$ given by $f_\alpha(x) = \sum_{a \in A} \alpha_a x^a$.
Let $r \in \mathbb{C}$ be generic and put $F_\alpha := f_\alpha^{-1}(r)$.
Note that $(\mathbb{C}^*)^k$ is a symplectic manifold with symplectic form $\sum_i dx_i \overline{dx_i}/(x_i \bar{x}_i)$. One shows that F_α is a symplectic submanifold (a Liouville manifold to be more precise)

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We have $\mathcal{D}(\partial Y) \cong \mathrm{DFuk}(F_\alpha)$ for a suitable choice of $\alpha \in \mathbb{C}^A$ where $\mathrm{DFuk}(F_\alpha)$ is the thick closure of the (wrapped) Fukaya category of F_α.

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The following result yields the existence of the $\pi_1(\mathcal{K}_A)$-action.

Theorem (Špenko-VdB)

$\alpha \to \mathrm{DFuk}(F_\alpha)$ defines a local system of categories on $\mathbb{C}^A - V(A)$.

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- Gammache and Shende show that $\mathcal{D}(Y) \cong \mathrm{DFuk}(((\mathbb{C}^*)^k, f_\alpha)$.
- However it is not known to how to define $\mathrm{DFuk}((\mathbb{C}^*)^k, f_\alpha)$ for every α, in such a way that it defines a local system! This breaks the approach ☹

Alternative approach

- An alternative approach is to try to manipulate the stopped Liouville manifold $((\mathbb{C}^*)^k, f_\alpha^{-1}(r))$ directly by “mutating the stop”.

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- An alternative approach is to try to manipulate the stopped Liouville manifold $((\mathbb{C}^*)^k, f_\alpha^{-1}(r))$ directly by "mutating the stop".
- So far this does not work in general. However it has been carried out by Huang and Zhou in the quasi-symmetric case using similar combinatorics as before.