ON ELLIPTIC SOLUTIONS OF THE ASSOCIATIVE YANG–BAXTER EQUATION

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Abstract. We give a direct proof of the fact that elliptic solutions of the associative Yang–Baxter equation arise from an appropriate spherical order on an elliptic curve.

1. Introduction

Let $\mathfrak{A} = \mathsf{Mat}_n(\mathbb{C})$ be the algebra of square matrices of size $n \in \mathbb{N}$ and $\big(\mathbb{C}^3, 0\big) \xrightarrow{r} \mathfrak{A} \otimes \mathfrak{A}$ be the germ of a meromorphic function. The following version of the associative Yang–Baxter equation (AYBE) with spectral parameters was introduced by Polishchuk in [9]:

$$(1) \quad r(u; x_1, x_2)^{12} r(u+v; x_2, x_3)^{23} = r(u+v; x_1, x_3)^{13} r(-v; x_1, x_2)^{12} + r(v; x_2, x_3)^{23} r(u; x_1, x_3)^{13}.$$

The upper indices in this equation indicate the corresponding embeddings of $\mathfrak{A} \otimes \mathfrak{A}$ into $\mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A}$. For example, the germ r^{13} is defined as

$$r^{13} : \mathbb{C}^3 \xrightarrow{r} \mathfrak{A} \otimes \mathfrak{A} \xrightarrow{\imath_{13}} \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A},$$

where $\imath_{13}(x \otimes y) = x \otimes 1 \otimes y$. Two other germs r^{12} and r^{23} are defined in a similar way. We are interested in those solutions of AYBE, which are non-degenerate, skew-symmetric (meaning that $r(v; x_1, x_2) = -r^{21}(-v; x_2, x_1)$) and which admit a Laurent expansion of the form

$$(2) \qquad r(v; x_1, x_2) = \frac{\mathbb{1} \otimes \mathbb{1}}{v} + r_0(x_1, x_2) + v r_1(x_1, x_2) + v^2 r_2(x_1, x_2) + \dots$$

All elliptic and trigonometric solutions of AYBE satisfying (2) were classified in [9, 10]. Recall the description of elliptic solutions of AYBE.

Let $\varepsilon = \exp\big(\frac{2\pi i d}{n}\big)$, where $0 < d < n$ is such that $\gcd(d, n) = 1$. We put

$$(3) \qquad X = \left(\begin{array}{cccc} 1 & 0 & \dots & 0 \\ 0 & \varepsilon & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varepsilon^{n-1} \end{array}\right) \quad \text{and} \quad Y = \left(\begin{array}{cccc} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{array}\right).$$

For any $(k, l) \in I := \big\{1, \dots, n\big\} \times \big\{1, \dots, n\big\}$ denote $Z_{(k,l)} = Y^k X^{-l}$ and $Z_{(k,l)}^{\vee} = \frac{1}{n} X^l Y^{-k}$. Then the following expression

$$(4) \qquad r_{((n,d),\tau)}(v; x_1, x_2) = \sum_{(k,l) \in I} \exp\Big(\frac{2\pi i d}{n} k x\Big) \sigma\Big(v + \frac{d}{n}\big(k\tau + l\big), x\Big) Z_{(k,l)}^{\vee} \otimes Z_{(k,l)}$$

is a solution of AYBE satisfying (2), where $x = x_2 - x_1$ and

$$\sigma(a, z) = 2\pi i \sum_{n\in\mathbb{Z}} \frac{\exp(-2\pi i n z)}{1 - \exp\bigl(-2\pi i(a - 2\pi i n\tau)\bigr)} \tag{5}$$

is the Kronecker elliptic function [12] for $\tau \in \mathbb{C}$ such that $\mathsf{Im}(\tau) > 0$. See also [8, Section III] for a direct proof of this fact.

In his recent work [11] Polishchuk showed that non-degenerate skew-symmetric solutions of AYBE satisfying (2) can be obtained from appropriate triple Massey products in the perfect derived category of coherent sheaves $\mathsf{Perf}(\mathbb{E})$ on a non-commutative projective curve $\mathbb{E} = (E, \mathcal{A})$, where E is an irreducible projective curve over $\mathbb{C}$ of arithmetic genus one and $\mathcal{A}$ is a symmetric spherical order on E. A simplest example of such an order is given by $\mathcal{A} = End_E(\mathcal{F})$, where $\mathcal{F}$ is a simple vector bundle on E. Let $E = E_\tau := \mathbb{C}/\langle 1, \tau\rangle$ be the elliptic curve determined by $\tau \in \mathbb{C}$ and $\mathcal{F}$ be a simple vector bundle of rank n and degree d on E. It follows from results of Atiyah [1] that such $\mathcal{F}$ exists and the sheaf of algebras $\mathcal{A} = \mathcal{A}_{(n,d)} := End_E(\mathcal{F})$ does not depend on the choice of $\mathcal{F}$. We show that the solution of AYBE arising from the non-commutative projective curve $\bigl(E_\tau, \mathcal{A}_{(n,d)}\bigr)$ is given by the formula (4). In [9, Section 2], the corresponding computations were performed using the homological mirror symmetry and explicit formulae for triple Massey products in the Fukaya category of a torus. The expression for the resulted solution of AYBE (see [9, formula (2.3)]) was different from (4). Our computations are straightforward and based by techniques developed in the articles [6, 5].

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2. Symmetric spherical orders on curves of genus one and AYBE

In this section we make a brief review of Polishchuk's construction [11]. Let E be an irreducible projective curve over $\mathbb{C}$ of arithmetic genus one, $\breve{E}$ its smooth part, $\mathcal{O}$ its structure sheaf, $\mathcal{K}$ the sheaf of rational functions on E and Ω the sheaf of regular differential one-forms on E. There exists a regular differential one-form $\omega \in \Gamma(E, \Omega)$ such that $\Gamma(E, \Omega) = \mathbb{C}\omega$. Such ω also defines an isomorphism $\mathcal{O} \cong \Omega$. If E is singular then it is rational. In this case, let $\mathbb{P}^1 \stackrel{\nu}{\to} E$ be the normalization morphism and $\widetilde{\mathcal{O}} = \nu_*\bigl(\mathcal{O}_{\mathbb{P}^1}\bigr)$.

Let $\mathcal{A}$ be a sheaf of orders on E. By definition, $\mathcal{A}$ is a torsion free coherent sheaf of $\mathcal{O}$-algebras on E such that $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{K} \cong Mat_n(\mathcal{K})$ for some $n \in \mathbb{N}$. For any order $\mathcal{A}$ we have the canonical trace morphism $\mathcal{A} \stackrel{t}{\to} \widetilde{\mathcal{O}}$, which coincides with the restriction of the trace morphism $\mathcal{A} \hookrightarrow \mathcal{A} \otimes_{\mathcal{O}} \mathcal{K} \cong Mat_n(\mathcal{K}) \stackrel{t}{\to} \mathcal{K}$ (if E is smooth then $\widetilde{\mathcal{O}} = \mathcal{O}$). Following [11], the order $\mathcal{A}$ is called *symmetric spherical* if the following conditions are fulfilled:

- The image of the trace morphism t is $\mathcal{O}$ and the induced morphism of coherent sheaves $\mathcal{A} \stackrel{t^\sharp}{\longrightarrow} \mathcal{A}^\vee := Hom_E(\mathcal{A}, \mathcal{O})$ is an isomorphism.
- We have: $\Gamma(E, \mathcal{A}) \cong \mathbb{C}$.

Consider the non-commutative projective curve $\mathbb{E} = (E, \mathcal{A})$. Let $\mathsf{Coh}(\mathbb{E})$ be the category of coherent sheaves on $\mathbb{E}$ (these are sheaves of $\mathcal{A}$-modules which are coherent as $\mathcal{O}$-modules) and $\mathsf{Perf}(\mathbb{E})$ be the corresponding perfect derived category. Recall that $\Omega_{\mathbb{E}} := \mathit{Hom}_E(\mathcal{A}, \Omega)$ is a dualising bimodule of $\mathbb{E}$. If $\mathcal{A}$ is symmetric then $\Omega_{\mathbb{E}} \cong \mathcal{A}$ as $\mathcal{A}$-bimodules and $\mathsf{Perf}(\mathbb{E})$ is a triangulated 1-Calabi–Yau category. The last assertion means that for any pair of objects $\mathcal{G}^\bullet, \mathcal{H}^\bullet$ in $\mathsf{Perf}(\mathbb{E})$ there is an isomorphism of vector spaces

$$\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{G}^\bullet, \mathcal{H}^\bullet\big) \cong \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{H}^\bullet, \mathcal{G}^\bullet[1]\big)^* \tag{6}$$

which is functorial in both arguments.

Let $P = \underline{\mathsf{Pic}}^0(E)$ be the Jacobian of E and $\mathcal{L} \in \mathsf{Pic}(P \times E)$ be a universal line bundle. For any $v \in P$, let $\mathcal{L}^v := \mathcal{L}\big|_{\{v\}\times E} \in \mathsf{Pic}^0(E)$ and $\mathcal{A}^v := \mathcal{A} \otimes_{\mathcal{O}} \mathcal{L}^v \in \mathsf{Coh}(\mathbb{E})$.

Lemma 2.1. *The coherent sheaf $\mathcal{A}$ is semi-stable of slope zero. Moreover,*

$$\Gamma(E, \mathcal{A}^v) = 0 = H^1(E, \mathcal{A}^v) \tag{7}$$

for all but finitely many points $v \in P$.

Proof. Let $\mathcal{B}$ be the kernel of the trace morphism $\mathcal{A} \xrightarrow{t} \mathcal{O}$. It follows from the long exact cohomology sequence of $0 \to \mathcal{B} \to \mathcal{A} \xrightarrow{t} \mathcal{O} \to 0$ that $H^0(E, \mathcal{B}) = 0 = H^1(E, \mathcal{B})$. Hence, $\mathcal{B}$ is a semi-stable coherent sheaf on E of slope zero and $\mathcal{A} \cong \mathcal{B} \oplus \mathcal{O}$. It follows that $\mathcal{A}$ is semi-stable, too. The latter fact also implies the vanishing $\Gamma(E, \mathcal{A}^v) = 0 = H^1(E, \mathcal{A}^v)$ for all but finitely many $v \in P$. □

Corollary 2.2. *There exists a proper closed subset $D \subset P \times P$ such that*

$$\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^{v_1}, \mathcal{A}^{v_2}\big) = 0 = \mathsf{Ext}^1_{\mathbb{E}}\big(\mathcal{A}^{v_1}, \mathcal{A}^{v_2}\big) \tag{8}$$

for all $v_1, v_2 \in (P \times P) \setminus D$.

Proof. This statement follows from the isomorphisms

$$\mathsf{Ext}^i_{\mathbb{E}}\big(\mathcal{A}^{v_1}, \mathcal{A}^{v_2}\big) \cong H^i\big(E, \mathit{End}_{\mathbb{E}}(\mathcal{A}^{v_1}, \mathcal{A}^{v_2})\big) \cong H^i\big(E, \mathcal{A}^{v_2 - v_1}\big), \quad i = 0, 1 \tag{9}$$

and the vanishing (7). □

Recall that for any $x, y \in \breve{E}$ we have the following standard short exact sequences

$$0 \longrightarrow \Omega \longrightarrow \Omega(x) \xrightarrow{\mathsf{res}_x} \mathbb{C}_x \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{O}(-y) \longrightarrow \mathcal{O} \xrightarrow{\mathsf{ev}_y} \mathbb{C}_y \longrightarrow 0, \tag{10}$$

where $\underline{\mathsf{res}}_x$ and $\underline{\mathsf{ev}}_y$ are the residue and evaluation morphisms, respectively. Using the isomorphism $\mathcal{O} \xrightarrow{\omega} \Omega$, we can rewrite the first short exact sequence as

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(x) \xrightarrow{\mathsf{res}^\omega_x} \mathbb{C}_x \longrightarrow 0. \tag{11}$$

For any $\mathcal{H} \in \mathsf{Coh}(\mathbb{E})$ denote $\mathcal{H}\big|_x := \mathcal{H} \otimes_{\mathcal{O}} \mathbb{C}_x \in \mathsf{Coh}(\mathbb{E})$. Tensoring (11) by $\mathcal{A}^v$ (where $v \in P$ is an arbitrary point), we get the following short exact sequence in $\mathsf{Coh}(\mathbb{E})$:

$$0 \longrightarrow \mathcal{A}^v \longrightarrow \mathcal{A}^v(x) \longrightarrow \mathcal{A}^v\big|_x \longrightarrow 0.$$

Next, for any $(u, v) \in P \times P$ we have the induced long exact sequence of vector spaces

$$0 \longrightarrow \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{A}^v\big) \longrightarrow \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{A}^v(x)\big) \longrightarrow \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{A}^v\big|_x\big) \longrightarrow \mathsf{Ext}^1_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{A}^v\big).$$

It follows from (7) that the linear map

$$\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{A}^v(x)\big) \xrightarrow{\mathsf{res}^{\mathcal{A}}(u,v;x)} \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^v\big|_x, \mathcal{A}^v\big|_x\big) \tag{12}$$

is an isomorphism if $(u, v) \in (P \times P) \setminus D$.

Similarly, for any $v \in P$ and $x \neq y \in \breve{E}$ we have the following short exact sequence

$$0 \longrightarrow \mathcal{A}^v(x - y) \longrightarrow \mathcal{A}^v(x) \longrightarrow \mathcal{A}^v(x)\big|_y \longrightarrow 0$$

in $\mathsf{Coh}(\mathbb{E})$, which is obtained by tensoring the evaluation sequence (10) by $\mathcal{A}^v(x)$. After applying to it the functor $\mathsf{Hom}_{\mathbb{E}}(\mathcal{A}^u, -)$ and using a canonical isomorphism $\mathcal{A}^v\big|_y \cong \mathcal{A}^v(x)\big|_y$, we obtain a linear map

$$\mathsf{Hom}_{\mathbb{E}}(\mathcal{A}^u, \mathcal{A}^v(x)) \xrightarrow{\underline{\mathsf{ev}}^{\mathcal{A}}(u,v;x,y)} \mathsf{Hom}_{\mathbb{E}}(\mathcal{A}^u\big|_y, \mathcal{A}^v\big|_y). \tag{13}$$

Let $\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_x, \mathcal{A}^v\big|_x\big) \xrightarrow{\alpha(u,v;x,y)} \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_y, \mathcal{A}^v\big|_y\big)$ be (the unique) linear map making the following diagram of vector spaces

$$\begin{array}{ccc} & \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{A}^v(x)\big) & \\ {\scriptstyle \underline{\mathsf{res}}^{\mathcal{A}}(u,v;x)} \swarrow & & \searrow {\scriptstyle \underline{\mathsf{ev}}^{\mathcal{A}}(u,v;x,y)} \\ \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_x, \mathcal{A}^v\big|_x\big) & \xrightarrow{\alpha(u,v;x,y)} & \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_y, \mathcal{A}^v\big|_y\big) \end{array} \tag{14}$$

commutative. Let $\gamma(u, v; x, y) \in \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^v\big|_x, \mathcal{A}^u\big|_x\big) \otimes \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_y, \mathcal{A}^v\big|_y\big)$ be the image of $\alpha(u, v; x, y)$ under the composition of the following canonical isomorphisms of vector spaces:

$$\begin{aligned} \mathsf{Lin}\Big(\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_x, \mathcal{A}^v\big|_x\big), \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_y, \mathcal{A}^v\big|_y\big)\Big) &\cong \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_x, \mathcal{A}^v\big|_x\big)^* \otimes \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_y, \mathcal{A}^v\big|_y\big) \\ &\cong \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^v\big|_x, \mathcal{A}^u\big|_x\big) \otimes \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_y, \mathcal{A}^v\big|_y\big), \end{aligned}$$

where the last isomorphism is induced by the trace morphism t.

Let $P \times P \xrightarrow{\eta} P$ be the group operation on P and $o \in P$ be the corresponding neutral element (i.e. $\mathcal{O} \cong \mathcal{L}^o$). Consider the canonical projections $P \times P \times E \xrightarrow{\pi_i} P \times E$, $(x_1, x_2; x) \mapsto (x_i, x)$ for $i = 1, 2$ and $P \times P \times E \xrightarrow{\pi_\circ} P \times E$, $(x_1, x_2; x) \mapsto (x_1, x_2)$. Then there exists $\mathcal{S} \in \mathsf{Pic}(P \times P)$ such that

$$(\eta \times \mathbb{1})^*\mathcal{L} \cong \pi_1^*\mathcal{L} \otimes \pi_2^*\mathcal{L} \otimes \pi_\circ^*\mathcal{S}. \tag{15}$$

In particular, $\mathcal{L}^{v_1} \otimes \mathcal{L}^{v_2} \cong \mathcal{L}^{v_1+v_2}$, where $v_1 + v_2 = \eta(v_1, v_2)$.

For any type of E (elliptic, nodal or cuspidal) there exists a complex analytic covering map $(\mathbb{C}, +) \xrightarrow{\chi} (P, \eta)$, which is also a group homomorphism. In this way we get a local coordinate on P in a neighbourhood of o. Next, we put: $\overline{\mathcal{L}} := (\chi \times \mathbb{1})^*\mathcal{L}$. Since any line bundle on $\mathbb{C} \times \mathbb{C}$ is trivial, we get from (15) an induced isomorphism

$$(\bar{\eta} \times \mathbb{1})^*\overline{\mathcal{L}} \cong \bar{\pi}_1^*\overline{\mathcal{L}} \otimes \bar{\pi}_2^*\overline{\mathcal{L}}, \tag{16}$$

where $\bar{\eta}$ (respectively, $\bar{\pi}_i$) is the composition of η (respectively, π_i) with $\chi \times \chi$. It follows that we have isomorphisms

$$\mathcal{O}_E \xrightarrow{\alpha} \overline{\mathcal{L}}\Big|_{0 \times E} \quad \text{and} \quad \mathcal{O}_{\mathbb{C} \times \mathbb{C} \times E} \xrightarrow{\beta} \bar{\eta}^*\overline{\mathcal{L}}^\vee \otimes \bar{\pi}_1^*\overline{\mathcal{L}} \otimes \bar{\pi}_2^*\overline{\mathcal{L}}. \tag{17}$$

Let $U \subset \breve{E}$ be an open subset for which there exists an isomorphism of $\Gamma(U, \mathcal{O}_E)$-algebras

$$\Gamma(U, \mathcal{A}) \xrightarrow{\xi} \mathfrak{A} \otimes_{\mathbb{C}} \Gamma(U, \mathcal{O}_E) \tag{18}$$

as well as a trivialization

$$\Gamma(\mathbb{C} \times U, \overline{\mathcal{L}}) \xrightarrow{\bar{\zeta}} \Gamma(\mathbb{C} \times U, \mathcal{O}_{\mathbb{C}\times E}), \tag{19}$$

which identify the sections α and β from (17) with the identity section. Since η is a complex analytic covering map, we get from $\bar{\zeta}$ a local trivialization ζ of the universal family $\mathcal{L}$. Then such trivializations ξ and ζ allow to identify $\gamma(u, v; x, y)$ with a tensor $\rho(u, v; x, y) \in \mathfrak{A} \otimes \mathfrak{A}$. Note that by the construction the tensor $\rho(u, v; x, y)$ depends only the difference $w := u - v \in P$ with respect to the group law on the Jacobian P.

Theorem 2.3 (Polishchuk [11])**.** *The constructed tensor $\varrho(w; x, y) = \rho(u, v; x, y)$ is a non-degenerate skew-symmetric solution of the associatiove Yang–Baxter equation (1).*

Recall the key steps of the proof of this result. For any $x \in \breve{E}$, let $\mathcal{S}^x \in \mathsf{Coh}(\mathbb{E})$ be a simple object of finite length supported at x (which is unique, up to an isomorphism). For any $(u, v) \in (P \times P) \setminus D$ and $(x, y) \in (\breve{E} \times \breve{E}), x \neq y$ consider the triple Massey product

$$\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{S}^x\big) \otimes \mathsf{Ext}^1_{\mathbb{E}}(\mathcal{S}^x, \mathcal{A}^v) \otimes \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^v, \mathcal{S}^y\big) \xrightarrow{m_3(u,v;x,y)} \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{S}^y\big)$$

in the triangulated category $\mathsf{Perf}(\mathbb{E})$. Since $\mathsf{Ext}^1_{\mathbb{E}}(\mathcal{S}^x, \mathcal{A}^v)^* \cong \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^v, \mathcal{S}^x\big)$ (see (6)), we get from $m_3(u, v; x, y)$ a linear map

$$\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{S}^x\big) \otimes \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^v, \mathcal{S}^y\big) \xrightarrow{m^{u,v}_{x,y}} \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^v, \mathcal{S}^x\big) \otimes \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{S}^y\big). \tag{20}$$

The constructed family of maps $m^{u,v}_{x,y}$ satisfies the identity

$$(m^{v_3,v_2}_{x_1,x_2})^{12}(m^{v_1,v_3}_{x_1,x_3})^{13} - (m^{v_1,v_3}_{x_2,x_3})^{23}(m^{v_1,v_2}_{x_1,x_2})^{12} + (m^{v_1,v_2}_{x_1,x_3})^{13}(m^{v_2,v_3}_{y_2,y_3})^{23} = 0, \tag{21}$$

both sides of which are viewed as linear maps

$$\mathsf{Hom}_{\mathbb{E}}(\mathcal{A}^{v_1}, \mathcal{S}^{x_1}) \otimes \mathsf{Hom}_{\mathbb{E}}(\mathcal{A}^{v_2}, \mathcal{S}^{x_2}) \otimes \mathsf{Hom}_{\mathbb{E}}(\mathcal{A}^{v_3}, \mathcal{S}^{x_3}) \longrightarrow$$

$$\longrightarrow \mathsf{Hom}_{\mathbb{E}}(\mathcal{A}^{v_2}, \mathcal{S}^{x_1}) \otimes \mathsf{Hom}_{\mathbb{E}}(\mathcal{A}^{v_3}, \mathcal{S}^{x_2}) \otimes \mathsf{Hom}_{\mathbb{E}}(\mathcal{A}^{v_1}, \mathcal{S}^{x_3}).$$

Moreover, $m^{u,v}_{x,y}$ is non-degenerate and skew-symmetric:

$$\iota(m^{u,v}_{x,y}) = -m^{v,u}_{y,x}, \tag{22}$$

where ι is the isomorphism

$$\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{S}^x\big) \otimes \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^v, \mathcal{S}^y\big) \longrightarrow \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^v, \mathcal{S}^y\big) \otimes \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{S}^x\big)$$

given by $\iota(f \otimes g) = g \otimes f$. Both identities (21) and (22) are consequences of existence of an A_∞-structure on $\mathsf{Perf}(\mathbb{E})$ which is cyclic with respect to the Serre duality (6). Applying appropriate canonical isomorphisms, one can identify $m^{u,v}_{x,y}$ with the linear map $\alpha(u, v; x, y)$ from the commutative diagram (14). See also [6, Theorem 2.2.17] for a detailed exposition in a similar setting. □

3. Solutions of AYBE as a section of a vector bundle

Following the work [6], we provide a global version of the commutative diagram (14). Let

$$B := P \times P \times \breve{E} \times \breve{E} \setminus \big(D \times \breve{E} \times \breve{E}\big) \cup \big(P \times P \times \Xi\big),$$

where $D \subset P \times P$ is the locus defined by (8) and $\Xi \subset \breve{E} \times \breve{E}$ is the diagonal. Let $X := B \times E$. Then the canonical projection $X \stackrel{\pi}{\longrightarrow} B$ admits two canonical sections $B \stackrel{\sigma_i}{\longrightarrow} X$ given by $\sigma_i(v_1, v_2; x_1, x_2) := (v_1, v_2; x_1, x_2; x_i)$ for $i = 1, 2$. Let $\Sigma_i := \sigma_i(B) \subset X$ be the corresponding Cartier divisor. Note that $\Sigma_1 \cap \Sigma_2 = \emptyset$.

Similarly to (11), we have the following short exact sequence in the category $\mathsf{Coh}(X)$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(\Sigma_1) \xrightarrow{\underline{\mathsf{res}}^{\omega}_{\Sigma_1}} \mathcal{O}_{\Sigma_1} \longrightarrow 0. \tag{23}$$

Here, for a local section $g(v_1, v_2; x_1, x_2; x) = \dfrac{f(v_1, v_2; x_1, x_2; x)}{x - x_1}$ of the line bundle $\mathcal{O}_X(\Sigma_1)$ we put: $\underline{\mathsf{res}}^{\omega}_{\Sigma_1}(g) = \mathsf{res}_{x=x_1}\big(g\omega_x\big)$, where ω_x is the pull-back of ω under the canonical projection $X \stackrel{\pi_5}{\longrightarrow} E$.

Consider the non-commutative scheme $\mathbb{X} = \big(X, \pi_5^*(\mathcal{A})\big)$ as well as coherent sheaves $\mathcal{A}^{(i)} := \pi_5^*(\mathcal{A}) \otimes \pi_{i,5}^*(\mathcal{L}) \in \mathsf{Coh}(\mathbb{X})$, where $X \stackrel{\pi_{i,5}}{\longrightarrow} P \times E$ is the canonical projection for $i = 1, 2$. Tensoring (23) by $\mathcal{A}^{(2)}$, we get a short exact sequence

$$0 \longrightarrow \mathcal{A}^{(2)} \longrightarrow \mathcal{A}^{(2)}(\Sigma_1) \longrightarrow \mathcal{A}^{(2)}\Big|_{\Sigma_1} \longrightarrow 0 \tag{24}$$

in the category $\mathsf{Coh}(\mathbb{X})$. Since $\mathcal{A}^{(1)}$ is a locally projective $\mathcal{O}_{\mathbb{X}}$-module, applying the functor $\mathit{Hom}_{\mathbb{X}}(\mathcal{A}^{(1)}, -)$ to (24), we get an induced short exact sequence

$$0 \to \mathit{Hom}_{\mathbb{X}}\big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}\big) \longrightarrow \mathit{Hom}_{\mathbb{X}}\big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}(\Sigma_1)\big) \longrightarrow \mathit{Hom}_{\mathbb{X}}\Big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}\Big|_{\Sigma_1}\Big) \to 0 \tag{25}$$

in the category $\mathsf{Coh}(X)$. Base-change isomorphism combined with the vanishing (8) imply that $R\pi_*\big(\mathit{Hom}_{\mathbb{X}}\big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}\big)\big) = 0$, where $R\pi_* : D^b\big(\mathsf{Coh}(X)\big) \longrightarrow D^b\big(\mathsf{Coh}(X)\big)$ is the derived direct image functor. Applying the functor π_* to the short exact sequence (25), we get the following isomorphism

$$\pi_*\Big(\mathit{Hom}_{\mathbb{X}}\big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}(\Sigma_1)\big)\Big) \stackrel{\cong}{\longrightarrow} \pi_* \mathit{Hom}_{\mathbb{X}}\Big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}\Big|_{\Sigma_1}\Big)$$

of coherent sheaves on B. Since $\mathit{Hom}_{\mathbb{X}}\Big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}\Big|_{\Sigma_1}\Big) \cong \mathit{Hom}_{\mathbb{X}}\Big(\mathcal{A}^{(1)}\Big|_{\Sigma_1}, \mathcal{A}^{(2)}\Big|_{\Sigma_1}\Big)$, we get an isomorphism $\underline{\mathsf{res}}^{\mathcal{A}}_{\Sigma_1}$ of coherent sheaves on B (which are even locally free) given as the composition

$$\pi_*\Big(\mathit{Hom}_{\mathbb{X}}\big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}(\Sigma_1)\big)\Big) \stackrel{\cong}{\longrightarrow} \pi_* \mathit{Hom}_{\mathbb{X}}\Big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}\Big|_{\Sigma_1}\Big) \stackrel{\cong}{\longrightarrow} \pi_* \mathit{Hom}_{\mathbb{X}}\Big(\mathcal{A}^{(1)}\Big|_{\Sigma_1}, \mathcal{A}^{(2)}\Big|_{\Sigma_1}\Big).$$

Next, we have the following short exact sequence of coherent sheaves on X:

$$0 \longrightarrow \mathcal{O}_X(-\Sigma_2) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\Sigma_2} \longrightarrow 0. \tag{26}$$

Since $\Sigma_1 \cap \Sigma_2 = \emptyset$, the canonical morphism $\mathcal{O}_{\Sigma_2} \to \mathcal{O}(\Sigma_1)\big|_{\Sigma_2}$ is an isomorphism. Tensoring (26) by $\mathcal{A}^{(2)}(\Sigma_1)$, we get a short exact sequence

$$0 \longrightarrow \mathcal{A}^{(2)}(\Sigma_1 - \Sigma_2) \longrightarrow \mathcal{A}^{(2)}(\Sigma_1) \longrightarrow \mathcal{A}^{(2)}\Big|_{\Sigma_2} \longrightarrow 0 \tag{27}$$

in the category $\mathsf{Coh}(\mathbb{X})$. Applying to (27) the functor $\mathit{Hom}_{\mathbb{X}}(\mathcal{A}^{(1)}, -)$, we get an induced short exact sequence

$$0 \to \mathit{Hom}_{\mathbb{X}}\big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}(\Sigma_1 - \Sigma_2)\big) \longrightarrow \mathit{Hom}_{\mathbb{X}}\big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}(\Sigma_1)\big) \longrightarrow \mathit{Hom}_{\mathbb{X}}\Big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}\Big|_{\Sigma_1}\Big) \to 0$$

in the category $\mathsf{Coh}(X)$. Applying the functor π_*, we get a morphism of locally free sheaves $\underline{\mathsf{ev}}^{\mathcal{A}}_{\Sigma_2}$ on B given as the composition

$$\pi_*\Big(\mathit{Hom}_{\mathbb{X}}\big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}(\Sigma_1)\big)\Big) \longrightarrow \pi_*\mathit{Hom}_{\mathbb{X}}\Big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}\Big|_{\Sigma_2}\Big) \xrightarrow{\cong} \pi_*\mathit{Hom}_{\mathbb{X}}\Big(\mathcal{A}^{(1)}\Big|_{\Sigma_2}, \mathcal{A}^{(2)}\Big|_{\Sigma_2}\Big).$$

In other words, we get the following global version

$$\begin{array}{ccc} & \pi_*\Big(\mathit{Hom}_{\mathbb{X}}\big(\mathcal{A}^{(1)}, \mathcal{A}^{(2)}(\Sigma_1)\big)\Big) & \\ \swarrow^{\underline{\mathsf{res}}^{\mathcal{A}}_{\Sigma_1}} & & \searrow^{\underline{\mathsf{ev}}^{\mathcal{A}}_{\Sigma_2}} \\ \pi_*\mathit{Hom}_{\mathbb{X}}\Big(\mathcal{A}^{(1)}\Big|_{\Sigma_1}, \mathcal{A}^{(2)}\Big|_{\Sigma_1}\Big) & \xrightarrow{\qquad\alpha^{\mathcal{A}}\qquad} & \pi_*\mathit{Hom}_{\mathbb{X}}\Big(\mathcal{A}^{(1)}\Big|_{\Sigma_2}, \mathcal{A}^{(2)}\Big|_{\Sigma_2}\Big) \end{array}$$

of the the commutative diagram (14), where $\alpha^{\mathcal{A}} := \underline{\mathsf{ev}}^{\mathcal{A}}_{\Sigma_2} \circ \big(\underline{\mathsf{res}}^{\mathcal{A}}_{\Sigma_1}\big)^{-1}$.

For any $1 \le i, j \le 2$, consider the canonical projection

$$P \times P \times \breve{E} \times \breve{E} \xrightarrow{\kappa_{ij}} P \times E, \ (v_1, v_2; x_1, x_2) \mapsto (v_j, x_i).$$

Then we have the following canonical isomorphism of coherent sheaves on B:

$$\pi_*\mathit{Hom}_{\mathbb{X}}\Big(\mathcal{A}^{(1)}\Big|_{\Sigma_i}, \mathcal{A}^{(2)}\Big|_{\Sigma_i}\Big) \cong \mathcal{A}^{\langle i \rangle} \otimes \mathit{Hom}_B\big(\kappa^*_{1i}(\mathcal{L}), \kappa^*_{2i}(\mathcal{L})\big),$$

where $\mathcal{A}^{\langle i \rangle}$ is the pull-back of $\mathcal{A}$ on B via the projection morphism

$$P \times P \times \breve{E} \times \breve{E} \longrightarrow E, \ (v_1, v_2; x_1, x_2) \mapsto x_i$$

for $i = 1, 2$. The morphism of locally free $\mathcal{O}_B$-modules

$$\mathcal{A}^{\langle 1 \rangle} \otimes \mathit{Hom}_B\big(\kappa^*_{11}(\mathcal{L}), \kappa^*_{21}(\mathcal{L})\big) \xrightarrow{\alpha^{\mathcal{A}}} \mathcal{A}^{\langle 2 \rangle} \otimes \mathit{Hom}_B\big(\kappa^*_{12}(\mathcal{L}), \kappa^*_{22}(\mathcal{L})\big)$$

determines a distinguished section

$$\gamma^{\mathcal{A}} \in \Gamma\Big(B, \mathcal{A}^{\langle 1 \rangle} \otimes \mathcal{A}^{\langle 2 \rangle} \otimes \kappa^*_{11}(\mathcal{L}) \otimes \kappa^*_{21}(\mathcal{L}^\vee) \otimes \kappa^*_{22}(\mathcal{L}) \otimes \kappa^*_{12}(\mathcal{L}^\vee)\Big). \tag{28}$$

For $i = 1, 2$ consider the canonical projections $P \times P \times E \xrightarrow{\psi_i} P \times E, (v_1, v_2; x) \mapsto (v_i; x)$ as well as $P \times P \times E \xrightarrow{\psi} P \times P, (v_1, v_2; x) \mapsto (v_1, v_2)$. Then there exists $\mathcal{S} \in \mathsf{Pic}(P \times P)$ such that

$$\psi_1^*(\mathcal{L}) \otimes \psi_2^*(\mathcal{L}^\vee) \cong (\mu \times \mathbb{1})^* \otimes \psi^*(\mathcal{S}),$$

where $P \times P \xrightarrow{\mu} P, (v_1, v_2) \mapsto v_1 - v_2$. Finally, for $i = 1, 2$ consider the morphism

$$P \times P \times \breve{E} \times \breve{E} \xrightarrow{\mu_i} P \times \breve{E}, (v_1, v_2; x_1, x_2) \mapsto (v_1 - v_2; x_i).$$

Then we have an isomorphism of locally free sheaves

$$\kappa_{11}^*(\mathcal{L}) \otimes \kappa_{21}^*(\mathcal{L}^\vee) \otimes \kappa_{22}^*(\mathcal{L}) \otimes \kappa_{12}^*(\mathcal{L}^\vee) \cong \mu_1^*(\mathcal{L}) \otimes \mu_2^*(\mathcal{L}^\vee).$$

In these terms, we can regard $\gamma^{\mathcal{A}}$ from (28) as a section

$$\gamma^{\mathcal{A}} \in \Gamma\Big(B, \mathcal{A}^{\langle 1 \rangle} \otimes \mathcal{A}^{\langle 2 \rangle} \otimes \mu_1^*(\mathcal{L}) \otimes \mu_2^*(\mathcal{L}^\vee)\Big). \tag{29}$$

Applying trivialisations ξ of $\mathcal{A}$ (see (18)) and ζ of $\mathcal{L}$ (see (19)), we obtain from $\gamma^{\mathcal{A}}$ a tensor-valued function

$$\rho^{\mathcal{A}}_{\xi,\zeta} : V \times V \times U \times U \longrightarrow \mathfrak{A} \otimes \mathfrak{A},$$

which satisfies the translation property

$$\rho^{\mathcal{A}}_{\xi,\zeta}(v_1 + u, v_2 + u; x_1, x_2) = \varrho^{\mathcal{A}}_{\xi,\zeta}(v_1, v_2; x_1, x_2).$$

Recall that for all types of the genus one curve E (smooth, nodal or cuspidal) we have a group homomorphism $(\mathbb{C}, +) \longrightarrow (P, +)$, which is locally a biholomorphic map.

After making these identifications, we get the germ of a meromorphic function

$$(\mathbb{C}^3, 0) \xrightarrow{\varrho} \mathfrak{A} \otimes \mathfrak{A}, \quad \text{where} \quad \varrho(v_1 - v_2; x_1, x_2) := \rho^{\mathcal{A}}_{\xi,\zeta}(v_1, v_2; x_1, x_2). \tag{30}$$

This function is a non-degenerate skew-symmetric solution of AYBE.

Summary. Let $\mathbb{E} = (E, \mathcal{A})$ be a non-commutative projective curve, where E is an irreducible projective curve of arithmetic genus one and $\mathcal{A}$ be a symmetric spherical order on E. Let P be the Jacobian of E and $\mathcal{L}$ be a universal family of degree zero line bundles on E. Then we have a distinguished section $\gamma^{\mathcal{A}} \in \Gamma\Big(B, \mathcal{A}^{\langle 1 \rangle} \otimes \mathcal{A}^{\langle 2 \rangle} \otimes \mu_1^*(\mathcal{L}) \otimes \mu_2^*(\mathcal{L}^\vee)\Big)$. Choosing trivializations ξ of $\mathcal{A}$ (see (18)) and ζ of $\mathcal{L}$ (see (19)), we get the germ of a meromorphic function $(\mathbb{C}^3, 0) \xrightarrow{\varrho} \mathfrak{A} \otimes \mathfrak{A}$, which is a non-degenerate skew-symmetric solution of AYBE. A different trivialization $\tilde{\xi}$ of $\mathcal{A}$ leads to a gauge-equivalent solution $\big(\varphi(x_1) \otimes \varphi(x_1)\big)\varrho(v; x_1, x_2)$, where $(\mathbb{C}, 0) \xrightarrow{\varphi} \mathsf{Aut}_{\mathbb{C}}(\mathfrak{A})$ is the germ of $\tilde{\xi}\xi^{-1}$. Analogously, another choice of a trivialization ζ leads to an equivalent solution $\exp\big(v(\beta(x_1) - \beta(x_2))\big)\varrho(v; x_1, x_2)$ for some holomorphic $(\mathbb{C}, 0) \xrightarrow{\beta} \mathbb{C}$.

Remark 3.1. The simplest example of a symmetric spherical order is $\mathcal{A} = \mathit{End}_E(\mathcal{F})$, where $\mathcal{F}$ is a simple vector bundle on E of rank n and degree d. It follows from [1, 4, 3] that such $\mathcal{F}$ exists if and only if n and d are coprime and the sheaf of algebras $\mathcal{A} = \mathcal{A}_{(n,d)} = \mathit{End}_E(\mathcal{F})$ does not depend on the choice of $\mathcal{F}$. Moreover, according to [11, Proposition 1.8.1], any symmetric spherical order on an elliptic curve E is isomorphic to $\mathcal{A}_{(n,d)}$ for some $0 < d < n$ mutually prime.

Remark 3.2. Let $(u, v) \in P \times P \setminus D$ and $(x, y) \in \breve{E} \times \breve{E} \setminus \Xi$. Then we have canonical isomorphisms

$$\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_x, \mathcal{A}^v\big|_x\big) \cong H^0(E, \mathit{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_x, \mathcal{A}^v\big|_x\big) \cong H^0\big(E, \mathcal{A}^{v-u}([x])\big).$$

Analogously, we have canonical isomorphisms

$$\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_x, \mathcal{A}^v\big|_x\big) \cong \mathcal{A}^{v-u}\big|_x \quad \text{and} \quad \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_y, \mathcal{A}^v\big|_y\big) \cong \mathcal{A}^{v-u}\big|_y$$

such that the following diagram

$$\begin{array}{ccccc}
\mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_x, \mathcal{A}^v\big|_x\big) & \xleftarrow{\mathsf{res}^{\mathcal{A}}(u,v;x)} & \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u, \mathcal{A}^v(x)\big) & \xrightarrow{\mathsf{ev}^{\mathcal{A}}(u,v;x,y)} & \mathsf{Hom}_{\mathbb{E}}\big(\mathcal{A}^u\big|_y, \mathcal{A}^v\big|_y\big) \\
\downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} \\
\mathcal{A}^{v-u}\big|_x & \xleftarrow{\mathsf{res}_x^{v-u}} & H^0\big(E, \mathcal{A}^{v-u}([x])\big) & \xrightarrow{\mathsf{ev}_y^{v-u}} & \mathcal{A}^{v-u}\big|_y
\end{array} \tag{31}$$

is commutative. Here, the linear maps res_x^{v-u} and ev_y^{v-u} are induced by the standard short exact sequences (10).

4. Elliptic solutions of AYBE

Let $\tau \in \mathbb{C}$ be such that $\mathsf{Im}(\tau) > 0$, $\mathbb{C} \supset \Lambda = \langle 1, \tau\rangle \cong \mathbb{Z}^2$ and $E = E_\tau = \mathbb{C}/\Lambda$. Recall some standard techniques to deal with holomorphic vector bundles on complex tori. An *automorphy factor* is a pair (A, V), where V is a finite dimensional vector space over $\mathbb{C}$ and $A : \Lambda \times \mathbb{C} \longrightarrow \mathsf{GL}(V)$ is a holomorphic function such that $A(\lambda+\mu, z) = A(\lambda, z+\mu)A(\mu, z)$ for all $\lambda, \mu \in \Lambda$ and $z \in \mathbb{C}$. Such (A, V) defines the following holomorphic vector bundle on the torus E:

$$\mathcal{E}(A, V) := \mathbb{C} \times V/\sim, \text{ where } \quad (z, v) \sim \big(z+\lambda, A(\lambda, z)v\big) \quad \text{for all} \quad (\lambda, z, v) \in \Lambda \times \mathbb{C} \times V.$$

Given two automorphy factors (A, V) and (B, V), the corresponding vector bundles $\mathcal{E}(A, V)$ and $\mathcal{E}(B, V)$ are isomorphic if and only if there exists a holomorphic function $H : \mathbb{C} \to \mathsf{GL}(V)$ such that

$$B(\lambda, z) = H(z+\lambda)A(\lambda, z)H(z)^{-1} \quad \text{for all} \quad (\lambda, z) \in \Lambda \times \mathbb{C}.$$

Let $\Phi : \mathbb{C} \longrightarrow \mathsf{GL}_n(\mathbb{C})$ be a holomorphic function such that $\Phi(z+1) = \Phi(z)$ for all $z \in \mathbb{C}$. Then one can define the automorphy factor $(A, \mathbb{C}^n)$ in the following way.

– $A(0, z) = I_n$ (the identity $n \times n$ matrix).

– For any $k \in \mathbb{N}_0$ we set:

$$A(k\tau, z) = \Phi\big(z + (k-1)\tau\big) \dots \Phi(z) \text{ and } A(-k\tau, z) = A(k\tau, z - k\tau)^{-1}.$$

For a proof of the following result, see [5, Proposition 5.1].

Proposition 4.1. *Let $0 < d < n$ be coprime. Then the sheaf of orders $\mathcal{A} = \mathcal{A}_{(n,d)}$ has the following description:*

$$\mathcal{A} \cong \mathbb{C} \times \mathfrak{A} / \sim, \quad \textit{where} \quad (z, Z) \sim (z+1, \mathsf{Ad}_X(Z)) \sim (z+\tau, \mathsf{Ad}_Y(Z)), \tag{32}$$

X and Y are matrices given by (3) and $\mathsf{Ad}_T(Z) = TZT^{-1}$ for $T \in \big\{X, Y\big\}$ and $Z \in \mathfrak{A}$.

For any $(k,l) \in I := \{1,\dots,n\} \times \{1,\dots,n\}$ denote $Z_{(k,l)} = Y^k X^{-l}$ and $Z^\vee_{(k,l)} = \frac{1}{n} X^l Y^{-k}$. Note that the operators $\mathsf{Ad}_X, \mathsf{Ad}_Y \in \mathsf{End}_{\mathbb{C}}(\mathfrak{A})$ commute. Moreover,

$$\mathsf{Ad}_X(Z_{(k,l)}) = \varepsilon^k Z_{(k,l)} \quad \text{and} \quad \mathsf{Ad}_Y(Z_{(k,l)}) = \varepsilon^l Z_{(k,l)}$$

for any $(k,l) \in I$. As a consequence $\big(Z_{(k,l)}\big)_{(k,l)\in I}$ is a basis of $\mathfrak{A}$.

Let $\mathsf{can} : \mathfrak{A} \otimes \mathfrak{A} \longrightarrow \mathsf{End}_{\mathbb{C}}(\mathfrak{A})$ be the canonical isomorphism sending a simple tensor $Z' \otimes Z''$ to the linear map $Z \mapsto \mathsf{tr}(Z' \cdot Z) \cdot Z''$. Then we have:

$$\mathsf{can}(Z^\vee_{(k,l)} \otimes Z_{(k,l)})(Z_{(k',l')}) = \begin{cases} Z_{(k,l)} & \text{if} \quad (k',l') = (k,l) \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

Recall the expressions for the first and third Jacobian theta-functions (see e.g. [7]):

$$\begin{cases} \bar{\theta}(z) = \theta_1(z|\tau) = 2q^{\frac{1}{4}} \sum\limits_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin\big((2n+1)\pi z\big), \\ \theta(z) = \theta_3(z|\tau) = 1 + 2 \sum\limits_{n=1}^{\infty} q^{n^2} \cos(2\pi n z), \end{cases} \tag{34}$$

where $q = \exp(\pi i \tau)$. They are related by the following identity:

$$\theta\Big(z + \frac{1+\tau}{2}\Big) = i \exp\Big(-\pi i \big(z + \frac{\tau}{4}\big)\Big) \bar{\theta}(z). \tag{35}$$

Lemma 4.2. *For any $x \in \mathbb{C}$ consider the function $\varphi_x(w) = -\exp\big(-2\pi i (w + \tau - x)\big)$. Then the following results are true.*

- *The vector space*

$$\left\{ \mathbb{C} \xrightarrow{f} \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is holomorphic} \\ f(w+1) = f(w) \\ f(w+\tau) = \varphi_x(w) f(w) \end{array} \right\} \tag{36}$$

  is one-dimensional and generated by $\theta_x(w) := \theta\big(w + \frac{1+\tau}{2} - x\big)$.
- *We have: $\mathcal{E}(\varphi_x) \cong \mathcal{O}_E\big([x]\big)$.*
- *For $a, b \in \mathbb{R}$ let $v = a\tau + b \in \mathbb{C}$ and $[v] = v(v) \in E$. Then we have:*

$$\mathcal{E}(\exp(-2\pi i v)) \cong \mathcal{O}_E\big([0] - [v]\big). \tag{37}$$

  In these terms we also get a description of a universal family $\mathcal{L}$ of degree zero line bundles on E.

A proof of these statements can be for instance found in [7] or [6, Section 4.1].

Let $U \subset \mathbb{C}$ be a small open neighborhood of 0 and $O = \Gamma(U, \mathcal{O}_{\mathbb{C}})$ be the ring of holomorphic functions on U. Let z be a coordinate on U, $\mathbb{C} \xrightarrow{\eta} E$ be the canonical covering map, $\omega = dz \in H^0(E, \Omega)$ and $\Gamma(U, \mathcal{A}) \xrightarrow{\xi} \mathfrak{A} \otimes_{\mathbb{C}} O$ be the standard trivialization induced by the automorphy data $(\mathsf{Ad}_X, \mathsf{Ad}_Y)$. One can also define a trivialization ζ of the universal family $\mathcal{L}$ of degree zero line bundles on E compatible with the isomorphisms (37).

Consider the following vector space

$$\mathsf{Sol}\big((n,d), v, x\big) = \left\{ \mathbb{C} \xrightarrow{F} \mathfrak{A} \;\middle|\; \begin{array}{l} F \quad \text{is holomorphic} \\ F(w+1) = \mathsf{Ad}_X\big(F(w)\big) \\ F(w+\tau) = \varphi_{x-v}(w) \mathsf{Ad}_Y\big(F(w)\big) \end{array} \right\}.$$

Proposition 4.3. *The following diagram*

$$
\begin{array}{ccccc}
\mathcal{A}^v\big|_x & \xleftarrow{\mathsf{res}^v_x} & H^0\big(\mathcal{A}^v(x)\big) & \xrightarrow{\mathsf{ev}^v_y} & \mathcal{A}^v\big|_y \\
\Big\downarrow \jmath^v_x & & \Big\downarrow \jmath & & \Big\downarrow \jmath^v_y \\
\mathfrak{A} & \xleftarrow{\overline{\mathsf{res}}_x} & \mathsf{Sol}\big((n,d),v,x\big) & \xrightarrow{\overline{\mathsf{ev}}_y} & \mathfrak{A}
\end{array} \tag{38}
$$

is commutative, where for $F \in \mathsf{Sol}\big((n,d),v,x\big)$ we have:

$$\overline{\mathsf{res}}_x(F) = \frac{F(x)}{\theta'\big(\frac{1+\tau}{2}\big)} \quad \text{and} \quad \overline{\mathsf{ev}}_y(F) = \frac{F(y)}{\theta\big(y-x+\frac{1+\tau}{2}\big)}.$$

The isomorphisms of vector spaces $\jmath^v_x, \jmath^v_y$ and $\jmath$ are induced by the trivializations ξ and ζ as well as the pull-back functor η^*.

Comment on the proof. Since an analogous result is proven in [6, Corollary 4.2.1], we omit details here. □

Now we are prepared to prove the main result of this work.

Theorem 4.4. *Let $r_{((n,d),\tau)}(v;x,y)$ be the solution of AYBE corresponding to the datum $\big(E_\tau, \mathcal{A}_{(n,d)}\big)$ with respect to the trivializations ξ (respectively, ζ) of $\mathcal{A}$ (respectively, $\mathcal{L}$) introduced above. Then it is given by the expression (4).*

Proof. We first compute an explicit basis of the vector space $\mathsf{Sol}\big((n,d),v,x\big)$. Let

$$F(w) = \sum_{(k,l)\in I} f_{(k,l)}(w) Z_{(k,l)}.$$

The condition $F \in \mathsf{Sol}\big((n,d),v,x\big)$ yields the following constraints on the coefficients $f_{(k,l)}$:

$$
\left\{
\begin{array}{lcl}
f_{(k,l)}(w+1) & = & \varepsilon^k f_{(k,l)}(w) \\
f_{(k,l)}(w+\tau) & = & \varepsilon^l \varphi_{x-v}(w) f_{(k,l)}(w).
\end{array}
\right. \tag{39}
$$

It follows from Lemma 4.2 that the vector space of holomorphic solutions of the system (39) is one-dimensional and generated by the function

$$f_{(k,l)}(w) = \exp\Big(-\frac{2\pi i d}{n} kw\Big)\theta\Big(w + \frac{1+\tau}{2} + v - x - \frac{d}{n}(k\tau - l)\Big).$$

From Proposition 4.3 and formula (33) it follows that $r_{((n,d),\tau)}(v;x,y)$ is given by the following expression:

$$r_{((n,d),\tau)}(v;x,y) = \sum_{(k,l)\in I} r_{(k,l)}(v;z)\, Z^\vee_{(k,l)} \otimes Z_{(k,l)},$$

where $z = y - x$ and

$$r_{(k,l)}(v;z) = \exp\Big(-\frac{2\pi i d}{n} kz\Big) \frac{\theta'\Big(\dfrac{1+\tau}{2}\Big)\theta\Big(z + v + \dfrac{1+\tau}{2} - \dfrac{d}{n}(k\tau - l)\Big)}{\theta\Big(v + \dfrac{1+\tau}{2} - \dfrac{d}{n}(k\tau - l)\Big)\theta\Big(z + \dfrac{1+\tau}{2}\Big)}.$$

Relation (35) implies that

$$\frac{\theta'\Big(\frac{1+\tau}{2}\Big)\theta\Big(z+v+\frac{1+\tau}{2}-\frac{d}{n}(k\tau-l)\Big)}{\theta\Big(v+\frac{1+\tau}{2}-\frac{d}{n}(k\tau-l)\Big)\theta\Big(z+\frac{1+\tau}{2}\Big)} = \frac{\theta'\Big(\frac{1+\tau}{2}\Big)\bar\theta\Big(z+v-\frac{d}{n}(k\tau-l)\Big)}{i\exp\Big(-\pi i\frac{\tau}{4}\Big)\bar\theta\Big(v-\frac{d}{n}(k\tau-l)\Big)\bar\theta(z)}$$

Moreover, it follows from (35) that $\theta'\Big(\frac{1+\tau}{2}\Big) = i\exp\Big(-\pi i\frac{\tau}{4}\Big)\bar\theta'(0)$. Hence, we get:

$$\begin{aligned} r_{(k,l)}(v;z) = \exp\Big(-\frac{2\pi i d}{n}kz\Big)\frac{\bar\theta'(0)\bar\theta\Big(z+v-\frac{d}{n}(k\tau-l)\Big)}{\bar\theta\Big(v-\frac{d}{n}(k\tau-l)\Big)\bar\theta(z)} \\ = \exp\Big(-\frac{2\pi i d}{n}kz\Big)\sigma\Big(v-\frac{d}{n}(k\tau-l),z\Big). \end{aligned}$$

Here we use the fact that the Kronecker elliptic function $\sigma(u,z)$ defined by (5) satisfies the formula: $\sigma(u,z) = \dfrac{\bar\theta'(0)\bar\theta_1(u+z)}{\bar\theta(u)\bar\theta(z)}$ (see for instance [13, Section 3]). We have a bijection $\{1,\dots,n\} \longrightarrow \{0,\dots,n-1\}, k \mapsto (n-k)$. Using this substitution as well as the identity $\sigma(u-d\tau,x) = \exp(2\pi i dz)\sigma(u,z)$, we end up with the expression (4), as asserted. □

Remark 4.5. Let $r(u;x_1,x_2)$ be a non-degenerate skew-symmetric solution of AYBE (1) satisfying (2). Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{A} \xrightarrow{\pi} \mathfrak{g}, Z \mapsto Z - \frac{1}{n}\mathsf{tr}(Z)I_n$. Then

$$\bar r(x_1,x_2) = (\pi\otimes\pi)(r_1(x_1,x_2))$$

is a solution of the classical Yang–Baxter equation

$$\begin{cases} \big[\bar r^{12}(x_1,x_2),\bar r^{13}(x_1,x_3)\big] + \big[\bar r^{13}(x_1,x_3),\bar r^{23}(x_2,x_3)\big] + \big[\bar r^{12}(x_1,x_2),\bar r^{23}(x_2,x_3)\big] = 0 \\ \bar r^{12}(x_1,x_2) = -\bar r^{21}(x_2,x_1), \end{cases}$$

see [9, Lemma 1.2]. Under certain additional assumptions (which are fulfilled provided $\bar r(x_1,x_2)$ is elliptic or trigonometric), the function $R(x_1,x_2) = r(u_\circ;x_1,x_2)$ (where $u=u_\circ$ from the domain of definition of r is fixed) satisfies the quantum Yang–Baxter equation

$$R(x_1,x_2)^{12}R(x_1,x_3)^{13}R(x_2,x_3)^{23} = R(x_2,x_3)^{23}R(x_1,x_3)^{13}R(x_1,x_2)^{12},$$

see [10, Theorem 1.5]. In fact, the expression (4) is a well-known elliptic solution of Belavin of the quantum Yang–Baxter equation; see [2].

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