A classification of polyharmonic Maaß forms via quiver representations

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Abstract: We give a classification of the Harish-Chandra modules generated by the pullback to $\mathrm{SL}_2(\mathbb{R})$ of *poly*harmonic Maaß forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ with exponential growth allowed at the cusps. This extends results of Bringmann–Kudla in the harmonic case. While in the harmonic setting there are nine cases, our classification comprises ten; A new case arises in weights $k > 1$. To obtain the classification we introduce quiver representations into the topic and show that those associated with polyharmonic Maaß forms are cyclic, indecomposable representations of the two-cyclic or the Gelfand quiver. A classification of these transfers to a classification of polyharmonic weak Maaß forms. To realize all possible cases of Harish-Chandra modules we develop a theory of weight shifts for Taylor coefficients of vector-valued spectral families. We provide a comprehensive computer implementation of this theory, which allows us to provide explicit examples.

polyharmonic Maaß forms ▪ Harish-Chandra modules ▪ Gelfand quiver ▪ mock modular forms ▪ Kronecker limit formula

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HARMONIC weak Maaß forms have proved to be immensely useful in number theory and related areas such as arithmetic geometry, combinatorics, and mathematical physics. Numerous applications were showcased by Bringmann–Folsom–Ono–Rolen [6], often rooted in generating series that are "completed" to produce harmonic weak Maaß forms. Special highlights are the confirmation of the Andrews–Dragonette Conjecture [8], incoherent Eisenstein series in the Kudla Program [27], and Mathieu Moonshine [21, 23] associated with $K3$–surfaces. As work by Duke, İmamoḡlu, and Tóth [20] shows *poly*harmonic weak Maaß forms play an evenly important role in number theory. Moreover, these forms appear in string theory, especially via the theory of modular graph functions [26], and in the correspond mathematical theory of iterated period integrals on the moduli space of marked genus-1 curves [9, 10].

The study of harmonic weak Maaß forms from a representation theoretic perspective was initiated by Bringmann and Kudla [7]. They provided a classification of the Harish-Chandra modules generated by the pullback to $\mathrm{SL}_2(\mathbb{R})$ of harmonic weak Maaß forms of integral weight.

In the present work we study polyharmonic (weak) Maaß forms, i.e. forms that vanish under a power of the Laplace operator, transform like modular forms, and have at most exponential growth at the cusps. Here, the weight k Laplace operator is defined as

$$\Delta_k = -\mathrm{R}_{k-2}\mathrm{L}_k, \quad \text{where } \mathrm{R}_k = 2i\partial_\tau + ky^{-1},\, \mathrm{L}_k = -2iy^2\partial_{\overline{\tau}}.$$

We call a polyharmonic Maaß form f of weight k satisfying $\Delta_k^{d+1} f = 0$ and $\Delta_k^d f \neq 0$ polyharmonic of *exact depth d* (compare Definition 3.1). A classical example of such a form, which appears already in the Kronecker limit formula, is the function

$$-\tfrac{1}{6}\log\bigl(y^{12}\bigl|\eta(\tau)\bigr|^{48}\bigr),$$

where $\eta(\tau) = \exp(\pi i\tau/12)\prod_{n=1}^{\infty}(1-\exp(2\pi i n\tau))$.

The classification Our first main result is the classification of Harish-Chandra modules corresponding to polyharmonic Maaß forms of integral weight. As a special case we recover the classification for harmonic Maaß forms obtained by Bringmann and Kudla. Harish-Chandra modules associated with polyharmonic Maaß forms are formally described in Section 3.2. Informally, they capture all $\mathrm{SL}_2(\mathbb{R})$-covariant differential equations satisfied by a given form.

Classification A. *In Sections 3.4 and 3.5 we give a classification comprising* 10 *cases of Harish-Chandra modules associated with weight-k polyharmonic weak Maaß forms of exact depth d: Cases Ia–d for $k < 1$, Cases IIa–b for $k = 1$, and Cases IIIa–d for $k > 1$. The classification is given in terms of the vanishing and non-vanishing of the following functions:*

$k < 1$	$k = 1$	$k > 1$
$\mathrm{L}_k \Delta_k^d f \overset{?}{=} 0$, $\mathrm{R}_k^{1-k} \Delta_k^d f \overset{?}{=} 0$	$\mathrm{L}_k \Delta_k^d f \overset{?}{=} 0$	$\mathrm{L}^k \Delta_k^{d-1} f \overset{?}{=} 0$, $\mathrm{L}_k \Delta_k^d f \overset{?}{=} 0$, $\mathrm{L}_k^k \Delta_k^d f \overset{?}{=} 0$.

Remark. (1) The three vanishing conditions that appear for $k > 1$ imply one another from left to right, so that they yield 4 as opposed to 8 cases.

(2) The classification of harmonic weak Maaß forms, i.e. the case $d = 0$, encompasses only 9 cases. Each of them generalizes in a suitable way, but there is an additional Case IIId if $k > 1$, which occurs only in positive depth d.

Example B. In our classification, the previously mentioned form $-\frac{1}{6}\log(y^{12}|\eta(\tau)|^{48})$ falls into Case Ia. Bringmann–Kudla highlighted s-derivatives of Eisenstein series as a further class of interesting functions that are not covered by their classification. These are polyharmonic Maaß forms of type Id if $k < 0$, of type Ia if $k = 0$, of type IIIb if $k = 2$, and of type IIIa if $k > 2$. The case $k = 1$ gives rise to incoherent Eisenstein series that fall under type IIa. See Section 5 for details.

Elementary Lie algebra considerations that serve well in the harmonic setup cease to work in ours. In particular, the method of classification employed by Bringmann–Kudla relies heavily on the harmonicity condition $\Delta_k f = 0$, which implies that K-types of the Harish-Chandra module associated with f occur with multiplicity at most 1. The transition between K-types can therefore be adequately described by the vanishing or non-vanishing of scalars. This holds no longer true for general polyharmonic Maaß forms, which is the major obstacle to the classification that we achieved. It turns out that the theory of quiver representations beautifully enables us to circumvent this obstacle. In Proposition 1.5 and Theorem 1.8, we provide equivalences between categories of specific Harish-Chandra modules and quiver representations following [4, 5, 15, 17, 24, 25]. This naturally leads us to representations of the Gelfand quiver, which are well-known to be intricate. A key step in our classification is Theorem 3.6, in which we show that quiver representations that arise from polyharmonic Maaß forms are cyclic and indecomposable. There are only few cyclic representations of the Gelfand quiver, which we give in Theorem 2.5 and which mirror the

Cases Ia–d and IIIa–d in our classification. Cases IIa and IIb arise from the two-cyclic quiver as discussed in Remark 2.8.

The explicit realization Our second main result is the explicit realization of all cases of the classification. Classification A constrains the possible Harish-Chandra modules associated with polyharmonic weak Maaß forms, however a priori it is not clear whether each case appears.

Theorem C. *For any $k \in \mathbb{Z}$ and any case of Classification A associated with this k, there is a (vector-valued) polyharmonic weak Maaß form of weight k that realizes this case.*

Remark. In Theorem 4.14 we provide a comprehensive theory that allows us to alter the weight of a modular realization of Cases Ia–d or IIIa–d if it is obtained from a spectral family. This theory involves a detailed analysis of the action of several differential operators on spectral families and is interesting in its own right.

To prove Theorem C we construct preimages under Δ_k^d of the realizations provided by Bringmann and Kudla. For scalar-valued realizations we can employ a classical approach and extract polyharmonic Maaß forms from the Taylor coefficients of a spectral family. This idea was already used by Lagarias and Rhoades [28] for Eisenstein series and by Duke–İmamoḡlu–Tóth [20] for Poincaré series. For vector-valued forms we vastly generalize this method. We show that the weight of a modular realization can be adjusted when taking products with specific vector-valued Maaß forms. In special cases this was also used by Bringmann and Kudla. Our Theorem 4.14 extends this approach to the broadest context compatible with our classification.

The proof of Theorem 4.14 depends on a delicate analysis of the action of the Laplace operator on specific products of Maaß forms. It can be viewed as an explicit instance of a theory by Bernstein–Gelfand [3], who studied tensor products of Harish-Chandra modules with finite dimensional representation. Our treatment is sufficiently explicit to accommodate a computer implementation, which is available on the third author's homepage and allows for the precise calculation of coefficients in every single case.

Structure of the paper The paper is structured as follows. In Section 1 we provide preliminaries on Harish-Chandra modules and investigate the structure of the relevant path algebra. In Section 2 we classify the corresponding cyclic modules. The bridge between these results and the classification of the Harish-Chandra modules arising from polyharmonic weak Maaß forms is provided in Section 3. In Section 4 we develop our results that enable us to alter the weight of spectral families. We use these results to show that there exist polyharmonic weak Maaß forms for each of the cases of the classification in Section 5.

1 Blocks of the category of Harish–Chandra–modules for $\mathrm{SL}_2(\mathbb{R})$

1.1 Preliminaries on Harish-Chandra–modules Let $G = \mathrm{SL}_2(\mathbb{R})$ and $K = \mathrm{SO}_2(\mathbb{R})$ be a maximal compact subgroup. Let

$$\mathfrak{g} := \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C}) = \left\langle H = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right), X = \frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right), Y = \left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right) \right\rangle$$

be the complexified Lie algebra of G and $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. The following relations in $\mathfrak{g}$ are satisfied:

$$[H, X] = 2X,\ [H, Y] = -2Y \text{ and } [X, Y] = H.$$

Let

$$C = H^2 - 2H + 4XY = H^2 + 2H + 4YX \in U(\mathfrak{g}) \tag{1.1}$$

be a Casimir element. It is well-known that the center $\mathfrak{z}$ of $U(\mathfrak{g})$ is $\mathbb{C}[C]$ (the algebra of polynomials in C). A calculation shows that

$$\begin{aligned} XY &= \frac{1}{4}\big(C - H^2 + 2H\big) = \frac{1}{4}\big(C - (p^2 - 1)\big), \\ YX &= \frac{1}{4}\big(C - H^2 - 2H\big) = \frac{1}{4}\big(C - (p^2 - 1)\big). \end{aligned} \tag{1.2}$$

Definition 1.1. A complex vector space M is a $(\mathfrak{g}, K)$–module if it has a structure $(M, \circ)$ of a representation of $\mathfrak{g}$ as well as a structure $(M, \cdot)$ of a representation of K such that

$$\big(\mathrm{Ad}_g(Z)\big) \circ v = g^{-1} \cdot \big(Z \circ (g \cdot v)\big)$$

for any $v \in M$, $g \in G$ and $Z \in \mathfrak{g}$, where Ad is the adjoint action of G on $\mathfrak{g}$. In what follows, we shall omit $\circ$ and $\cdot$ from the notation for the action on M. For $n \in \mathbb{Z}$ we put:

$$M_n = \big\{ v \in M : H \circ v = nv \big\}.$$

A $(\mathfrak{g}, K)$–module M is a *Harish–Chandra* module if it is finitely generated over $U(\mathfrak{g})$ and

$$M \cong \bigoplus_{n \in \mathbb{Z}} M_n \text{ with } \dim_{\mathbb{C}}(M_n) < \infty \text{ for all } n \in \mathbb{Z}.$$

In what follows, $\mathsf{HC}(\mathfrak{g}, K)$ denotes the category of Harish–Chandra $(\mathfrak{g}, K)$–modules.

Definition 1.2. Let $l \in \mathbb{N}_0$ and $\gamma = l^2 - 1$. Let $\mathsf{HC}_l(\mathfrak{g}, K)$ be the full subcategory of the category of $(\mathfrak{g}, K)$-modules consisting of such modules M which are

(i) finitely generated as $U(\mathfrak{g})$-modules;

(ii) there exists $m \in \mathbb{N}$ (depending on M) such that $(C - \gamma I)^m \cdot M = 0$.

We have the following standard result.

Lemma 1.3. *The category* $\mathsf{HC}_l(\mathfrak{g}, K)$ *is a full subcategory of* $\mathsf{HC}(\mathfrak{g}, K)$. *For any object M of* $\mathsf{HC}_l(\mathfrak{g}, K)$ *we have a direct sum decomposition*

$$M = \bigoplus_{i \in \mathbb{Z}} M_{-l-1+2i}$$

and its $\mathfrak{g}$–module structure can be visualized by the following (infinite) diagram of finite dimensional vector spaces and linear maps:

$$\cdots \underset{Y}{\overset{X}{\rightleftarrows}} M_{-l-1} \underset{Y}{\overset{X}{\rightleftarrows}} M_{-l+1} \underset{Y}{\overset{X}{\rightleftarrows}} \cdots \underset{Y}{\overset{X}{\rightleftarrows}} M_{l-1} \underset{Y}{\overset{X}{\rightleftarrows}} M_{l+1} \underset{Y}{\overset{X}{\rightleftarrows}} \cdots. \tag{1.3}$$

We call (1.3) a *diagram description* of a Harish–Chandra module $M \in \mathsf{HC}_l(\mathfrak{g}, K)$. For any $p \in \mathbb{Z}$, consider the following fragment of (1.3):

$$M_{p-1} \underset{Y}{\overset{X}{\rightleftarrows}} M_{p+1}.$$

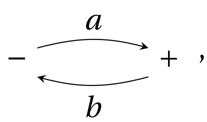
As a consequence of (1.2), we obtain the following statements about the diagram description (1.3) of $M \in \mathsf{HC}_l(\mathfrak{g}, K)$:

- $X|_{M_{p-1}}$ is an isomorphism for $p \neq \pm l$.
- $Y|_{M_{p+1}}$ is an isomorphism for $p \neq \pm l$.
- For $p = \pm l$, the corresponding endomorphisms XY and YX are nilpotent.

1.2 Path algebras Let $\mathfrak{D} = \mathbb{C}[\![t]\!]$, $\mathfrak{m} = (t)$ and

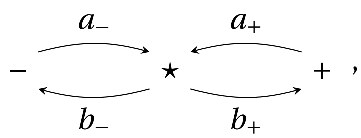
$$\mathfrak{A} = \begin{pmatrix} \mathfrak{D} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{D} & \mathfrak{D} & \mathfrak{m} \\ \mathfrak{D} & \mathfrak{m} & \mathfrak{D} \end{pmatrix} \subset \mathrm{Mat}_{3\times 3}(\mathfrak{D}) \quad \text{and} \quad \mathfrak{B} = \begin{pmatrix} \mathfrak{D} & \mathfrak{m} \\ \mathfrak{D} & \mathfrak{D} \end{pmatrix} \subset \mathrm{Mat}_{2\times 2}(\mathfrak{D}). \tag{1.4}$$

Note that the algebra $\mathfrak{B}$ is isomorphic to the arrow ideal completion of the path algebra of the cyclic quiver



(1.5)

whereas $\mathfrak{A}$ is isomorphic to the arrow ideal completion of the path algebra of the so-called Gelfand quiver

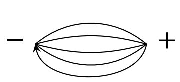


$$a_- b_- = a_+ b_+. \tag{1.6}$$

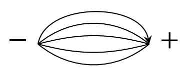
Example 1.4. The isomorphisms between $\mathfrak{B}$ and $\mathfrak{A}$ and the completed path algebras of (1.5) and (1.6) can be implemented as follows:

$$a \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\, b \longmapsto \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}; \quad a_- \longmapsto \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\, a_+ \longmapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & t & 0 \end{pmatrix},\, b_- \longmapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\, b_+ \longmapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

More graphically, a matrix with a single nonzero, monomial entry corresponds to a path, whose source and target are recorded by its position. The column position records the source and the row position records the target of a path. In the case of the cyclic quiver we assign the labels "−" and "+" to the first and second position, respectively. The exponent of t in a monomial records the number of times a path arrives at "−". For instance, we next depict the path starting at "+" and ending at "−" with 2 loops and the reverse path together with the corresponding elements of $\mathfrak{B}$:



$\begin{pmatrix} 0 & t^3 \\ 0 & 0 \end{pmatrix}$,



$\begin{pmatrix} 0 & 0 \\ t^2 & 0 \end{pmatrix}$.

The composition of the left with the right path yields five loops starting at "+", corresponding to $\begin{pmatrix} 0 & t^3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ t^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & t^5 \end{pmatrix} \in \mathfrak{B}$.

In the case of the Gelfand quiver, the source and target "$*$", "−", and "+" in that order correspond to the positions in the matrix. The exponent of t records the number of times a path arrives at "$*$". It is crucial for this correspondence that the two loops starting at "$*$" are considered equivalent by the relation $a_- b_- = a_+ b_+$ in (1.6).

In what follows, $\mathsf{Rep}(\mathfrak{A})$ and $\mathsf{Rep}(\mathfrak{B})$ denote the categories of finite dimensional left $\mathfrak{A}$–modules and $\mathfrak{B}$–modules. They are equivalent to the categories of finite dimensional *nilpotent* representations of (1.6) and (1.5).

The cyclic quiver Let $M \in \mathsf{HC}_0(\mathfrak{g}, K)$, whose diagram description is (1.3). We use the following notation:

$$Z_- = X\big|_{M_{-1}} \quad \text{and} \quad Z_+ = Y\big|_{M_{+1}}. \tag{1.7}$$

Proposition 1.5. *There is an equivalence of categories*

$$\mathsf{HC}_0(\mathfrak{g}, K) \xrightarrow{\mathbb{E}} \mathsf{Rep}(\mathfrak{B}) \tag{1.8}$$

given on the level of objects by the assignment

$$M \overset{\mathbb{E}}{\longmapsto} \left[\; V_- \underset{B}{\overset{A}{\rightleftarrows}} V_+ \;\right] = \left[\; M_{-1} \underset{Z_+}{\overset{Z_-}{\rightleftarrows}} M_{+1} \;\right].$$

Proof. Let M and N be a pair of objects of $\mathsf{HC}_0(\mathfrak{g}, K)$ and $M \xrightarrow{\psi} N$ be any morphism. It follows that $\psi(M_i) \subseteq N_i$ for any $i \in \mathbb{Z}$ implying that $\mathbb{E}$ is a functor. Proofs of the statements that $\mathbb{E}$ is fully faithful and dense are straightforward. ∎

The Gelfand quiver Let $l \in \mathbb{N}$ and $M \in \mathsf{HC}_l(\mathfrak{g}, K)$, whose diagram description is (1.3). We use the following notation:

$$\begin{array}{llll} X_- = X\big|_{M_{-l-1}}, & X_+ = X\big|_{M_{l-1}} & \text{and} & X_i = X\big|_{M_{-l-1+2i}} \text{ for } 1 \le i \le l-1; \\ Y_- = Y\big|_{M_{-l+1}}, & Y_+ = Y\big|_{M_{l+1}} & \text{and} & Y_i = Y\big|_{M_{-l+1+2i}} \text{ for } 1 \le i \le l-1; \\ X_* = X_{l-1} \cdots X_1 & \text{and} & Y_* = Y_1 \cdots Y_{l-1}. & \end{array} \tag{1.9}$$

Lemma 1.6. *In the above notation* (1.9)*, the following identities are true:*

$$X_* X_- Y_- = Y_+ X_+ X_* \quad \textit{and} \quad X_- Y_- Y_* = Y_* Y_+ X_+.$$

Proof. We only show the first statement since a proof of the second one is completely analogous. First choose bases in all vector spaces M_{l+1+2i}. Assume first that $X_1 =$

$\cdots = X_{l-1} = I$ with respect to this choice. For any $1 \le i \le l$, let $C_i := C|_{M_{-l-1+2i}}$. It follows from (1.2) that $C_1 = \cdots = C_l$ and

$$X_- Y_- = (C_1 - \gamma I) = (C_l - \gamma I) = Y_+ X_+.$$

Since $X_* = I$, we get the result in this special case.

To prove the statement in general, recall that all linear maps $X_1, \dots, X_{l-1}$ are isomorphisms. Moreover, the following diagram is commutative:

$$\begin{array}{ccccccccccc}
M_{-l+1} & \xrightarrow{X_1} & M_{-l+3} & \xrightarrow{X_2} & M_{-l+5} & \xrightarrow{X_3} & \dots & \xrightarrow{X_{l-2}} & M_{l-3} & \xrightarrow{X_{l-1}} & M_{l-1} \\
{\scriptstyle I}\big\downarrow & & {\scriptstyle X_1^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle X_1^{-1}X_2^{-1}} & & & & \big\downarrow & & \big\downarrow{\scriptstyle X_*^{-1}} \\
M_{-l+1} & \xrightarrow[I]{} & M_{-l+3} & \xrightarrow[I]{} & M_{-l+5} & \xrightarrow[I]{} & \dots & \xrightarrow[I]{} & M_{l-3} & \xrightarrow[I]{} & M_{l-1}.
\end{array}$$

Observe that when applying the vertical errors we identify spaces of different weight via their chosen bases. It follows that any object M of $\mathsf{HC}_l(\mathfrak{g}, K)$ whose diagram description is

$$\cdots \rightleftarrows M_{-l-1} \underset{Y_-}{\overset{X_-}{\rightleftarrows}} M_{-l+1} \underset{Y_1}{\overset{X_1}{\rightleftarrows}} \cdots \underset{Y_{l-1}}{\overset{X_{l-1}}{\rightleftarrows}} M_{l-1} \underset{Y_+}{\overset{X_+}{\rightleftarrows}} M_{l+1} \rightleftarrows \cdots \tag{1.10}$$

is isomorphic to an object given by

$$\cdots \rightleftarrows M_{-l-1} \underset{Y_-}{\overset{X_-}{\rightleftarrows}} M_{-l+1} \underset{\widetilde{Y}_1}{\overset{I}{\rightleftarrows}} \cdots \underset{\widetilde{Y}_{l-1}}{\overset{I}{\rightleftarrows}} M_{l-1} \underset{X_*^{-1}Y_+}{\overset{X_+X_*}{\rightleftarrows}} M_{l+1} \rightleftarrows \cdots. \tag{1.11}$$

where $\widetilde{Y}_1, \dots, \widetilde{Y}_{l-1}$ are appropriate invertible matrices. Finally, it was shown above that $X_- Y_- = X_*^{-1} Y_+ X_+ X_*$, which implies that $X_* X_- Y_- = Y_+ X_+ X_*$. ∎

Lemma 1.7. *Let $l \in \mathbb{N}$ and M be an object of $\mathsf{HC}_l(\mathfrak{g}, K)$ whose diagram description is* (1.10)*. Then*

$$M_{-l-1} \underset{Y_-}{\overset{X_-}{\rightleftarrows}} M_{-l+1} \underset{X_*^{-1}Y_+}{\overset{X_+X_*}{\rightleftarrows}} M_{l+1} \quad \textit{and} \quad M_{-l-1} \underset{Y_-Y_*}{\overset{Y_*^{-1}X_-}{\rightleftarrows}} M_{l-1} \underset{Y_+}{\overset{X_+}{\rightleftarrows}} M_{l+1} \tag{1.12}$$

are nilpotent representations of the Gelfand quiver (1.6) *which are moreover isomorphic.*

Proof. The fact that (1.12) are representations of the Gelfand quiver follows from Lemma 1.6. The requested isomorphism of representations is constructed as follows:

$$\begin{array}{ccccc} M_{-l-1} & \underset{Y_-}{\overset{X_-}{\rightleftarrows}} & M_{-l+1} & \underset{X_*^{-1}Y_+}{\overset{X_+X_*}{\rightleftarrows}} & M_{l+1} \\ \Big\downarrow T & & \Big\downarrow X_* & & \Big\downarrow I \\ M_{-l-1} & \underset{Y_-Y_*}{\overset{Y_*^{-1}X_-}{\rightleftarrows}} & M_{l-1} & \underset{Y_+}{\overset{X_+}{\rightleftarrows}} & M_{l+1} \end{array} \tag{1.13}$$

First note that both squares in the right part of (1.13) are commutative. To define T, note that

$$Y_*X_* = Y_1 \dots Y_{l-1}X_{l-1} \dots X_1 = p(C_1),$$

where $p(t) = \frac{1}{4^{l-1}}(t-\gamma_1)\dots(t-\gamma_{l-1}) \in \mathbb{C}[t]$ for appropriate $\gamma_1, \dots, \gamma_{l-1} \in \mathbb{Z}$, which are all different from γ. We put $T = p(C_0)$, where $C_0 = C|_{M_{-l-1}}$. It follows that T is an isomorphism of vector spaces. The commutativity of both left squares of the diagram (1.13) follows in particular from the fact that C is a central element of $U(\mathfrak{g})$. ■

Theorem 1.8. *For any $l \in \mathbb{N}$ there is an equivalence of categories*

$$\mathsf{HC}_l(\mathfrak{g}, K) \xrightarrow{\mathbb{E}} \mathsf{Rep}(\mathfrak{A}) \tag{1.14}$$

given on the level of objects by the assignment

$$M \overset{\mathbb{E}}{\longmapsto} \left[V_- \underset{B_-}{\overset{A_-}{\rightleftarrows}} V_\star \underset{B_+}{\overset{A_+}{\leftrightarrows}} V_+ \right] = \left[M_{-l-1} \underset{Y_-}{\overset{X_-}{\rightleftarrows}} M_{-l+1} \underset{X_*^{-1}Y_+}{\overset{X_+X_*}{\rightleftarrows}} M_{l+1} \right].$$

Proof. We have to define $\mathbb{E}$ on morphisms. To do it, it suffices to define $\mathbb{E}$ on the full subcategory $\mathsf{HC}'_l(\mathfrak{g}, K)$ of $\mathsf{HC}_l(\mathfrak{g}, K)$ consisting of those objects M whose diagram description (1.10) has the property: $X_1 = \dots = X_{l-1} = I$ (note that $\mathsf{HC}'_l(\mathfrak{g}, K)$ is equivalent to $\mathsf{HC}_l(\mathfrak{g}, K)$). Given any morphism $M \xrightarrow{\psi} N$ in $\mathsf{HC}'_l(\mathfrak{g}, K)$, we have: $\psi(M_i) \subseteq N_i$ for any $i \in \mathbb{Z}$. We define $\mathbb{E}(M) \xrightarrow{\mathbb{E}(\psi)} \mathbb{E}(N)$ by putting $\psi_- := \psi|_{M_{-l-1}}$, $\psi_* := \psi|_{M_{-l+1}}$ and $\psi_+ := \psi|_{M_{l+1}}$. It is easy to see that $\mathsf{HC}'_l(\mathfrak{g}, K) \xrightarrow{\mathbb{E}} \mathsf{Rep}(\mathfrak{A})$ is a functor. The fact that $\mathbb{E}$ is fully faithful and dense can be verified by straightforward computation. Hence, we can extend $\mathbb{E}$ to an equivalence of categories $\mathsf{HC}_l(\mathfrak{g}, K) \xrightarrow{\mathbb{E}} \mathsf{Rep}(\mathfrak{A})$. ■

Remark 1.9. In what follows we shall use the fact in the spirit of Lemma 1.7 that

$$\mathbb{E}(M) \cong \quad M_{-l-1} \underset{Y_- Y_*}{\overset{Y_*^{-1} X_-}{\rightleftarrows}} M_{l-1} \underset{Y_+}{\overset{X_+}{\rightleftarrows}} M_{l+1}. \tag{1.15}$$

2 Cyclic modules over $\mathfrak{A}$ and $\mathfrak{B}$

We refer to [18] for basic notions and results from the representation theory of associative algebras. Let

$$e_* = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), e_+ = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), \text{ and } e_- = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$$

be three primitive idempotents of the algebra $\mathfrak{A}$ corresponding to three vertices of the Gelfand quiver (1.6). Then

$$P_* = \mathfrak{A}e_* \cong \left(\begin{smallmatrix} \mathfrak{D} \\ \mathfrak{D} \\ \mathfrak{D} \end{smallmatrix}\right), P_+ = \mathfrak{A}e_+ \cong \left(\begin{smallmatrix} \mathfrak{m} \\ \mathfrak{D} \\ \mathfrak{m} \end{smallmatrix}\right), \text{ and } P_- = \mathfrak{A}e_- \cong \left(\begin{smallmatrix} \mathfrak{m} \\ \mathfrak{m} \\ \mathfrak{D} \end{smallmatrix}\right)$$

are indecomposable projective $\mathfrak{A}$–modules. Conversely, any indecomposable projective $\mathfrak{A}$–module is isomorphic to P_* or $P_\pm$.

Definition 2.1. A finite dimensional $\mathfrak{A}$–module V is called *cyclic* if there exists an epimorphism $P \twoheadrightarrow V$ for an indecomposable projective $\mathfrak{A}$–module P.

Lemma 2.2. *Any cyclic $\mathfrak{A}$–module is indecomposable.*

Proof. Assume that $V \cong V' \oplus V''$ is a direct sum decomposition with $V' \neq 0 \neq V''$. Take projective coverings $P' \stackrel{f'}{\twoheadrightarrow} V'$ and $P'' \stackrel{f''}{\twoheadrightarrow} V''$. Then $P' \oplus P'' \xrightarrow{(f'\, f'')} V$ is a projective covering of V. It follows that $P \cong P' \oplus P''$ is decomposable. Contradiction. ∎

Remark 2.3. The above result is true for arbitrary semi-perfect rings (i.e. those rings for which any finitely generated left module has a projective cover).

Our next goal is to give a classification of cyclic $\mathfrak{A}$–modules.

Definition 2.4. A finitely generated $\mathfrak{A}$–module Q is called a *lattice* if it is free as $\mathfrak{D}$–module.

The algebra $\mathfrak{A}$ belongs to the class of the so-called nodal orders; see [16] and in particular [16, Example 3.18]. The study of lattices over orders is a classical subject of representation theory of associative algebras [18]. Consider the so-called hereditary envelope

$$\mathfrak{H} = \begin{pmatrix} \mathfrak{D} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{D} & \mathfrak{D} & \mathfrak{D} \\ \mathfrak{D} & \mathfrak{D} & \mathfrak{D} \end{pmatrix} \tag{2.1}$$

of the nodal order $\mathfrak{A}$ (see [16] for the definition). Then any indecomposable $\mathfrak{A}$–lattice is a direct summand of $\mathfrak{A} \oplus \mathfrak{H}$ (this result is true for arbitrary nodal orders; see [16]). As a consequence, there are only four isomorphism classes of indecomposable $\mathfrak{A}$–lattices: indecomposable projective modules P_*, P_+ and P_- and as well as

$$Q = \begin{pmatrix} \mathfrak{m} \\ \mathfrak{D} \\ \mathfrak{D} \end{pmatrix}.$$

This fact is essential for the classification of cyclic $\mathfrak{A}$–modules given below.

For a finite dimensional $\mathfrak{A}$–module

$$V = \left[\begin{array}{ccccc} V_- & \underset{B_-}{\overset{A_-}{\rightleftarrows}} & V_\star & \underset{B_+}{\overset{A_+}{\leftrightarrows}} & V_+ \end{array} \right] \tag{2.2}$$

we put: $C_\pm := B_\pm A_\pm$ and $C_* = A_+B_+ = A_-B_-$. Let $m_\pm$ and m_* be the nilpotency degrees of the endomorphisms $C_\pm$ and C_*, respectively, whereas

$$\underline{\dim}(V) = \big(\dim(V_-), \dim(V_*), \dim(V_+)\big) \in \mathbb{N}_0^3$$

is the dimension vector of V.

A cyclic $\mathfrak{A}$–module V has *type* $i \in \{+,-,*\}$ if there exists an epimorphism $P_i \twoheadrightarrow V$.

Theorem 2.5. *Let V be a cyclic $\mathfrak{A}$–module of type $i \in \{+,-,*\}$. Then its isomorphism class is uniquely determined by the corresponding dimension vector $\underline{\dim}(V)$. For $d \in \mathbb{N}_0$, the possibilities are as follows:*

(I) Cyclic modules of type "$*$". Possible dimension vectors are:

(a) $(d, d+1, d)$,
(b) $(d+1, d+1, d+1)$,
(c) $(d, d+1, d+1)$,
(d) $(d+1, d+1, d)$.

For all these representations we have: $m_* = d+1$.

(II) *Cyclic modules of type "+". Possible dimension vectors are:*

(a) $(d, d, d+1)$.
(b) $(d, d+1, d+1)$.
(c) $(d+1, d+1, d+1)$.
(d) $(d-1, d, d+1)$ *(it exists only for* $d \geq 1$*).*

For these cyclic modules we have: $m_+ = d+1$.

(III) *A description of the cyclic modules of type "−" is symmetric to those given in (II).*

Proof. Let V be a cyclic $\mathfrak{A}$–module and $P \stackrel{\pi}{\twoheadrightarrow} V$ be its projective cover. Consider the short exact sequence

$$0 \longrightarrow N \stackrel{\imath}{\longrightarrow} P \stackrel{\pi}{\longrightarrow} V \longrightarrow 0.$$

Since N is a free $\mathfrak{D}$–module, it is an $\mathfrak{A}$–lattice. As V is a $\mathfrak{D}$–module of finite length, we have:

$$\mathfrak{K} \otimes N \cong \mathfrak{K} \otimes P \cong U := \begin{pmatrix} \mathfrak{K} \\ \mathfrak{K} \\ \mathfrak{K} \end{pmatrix},$$

where $\mathfrak{K} = \mathbb{C}((t))$. Hence, N is an indecomposable $\mathfrak{A}$–lattice, i.e. $N \in \{P_*, P_\pm, Q\}$. Next, we can view P and N as *subsets* of U. Then we get:

$$\mathrm{Hom}_{\mathfrak{A}}(N, P) = \{a \in \mathfrak{K} : aN \subseteq P\} = \{a \in \mathfrak{D} : aN \subseteq P\}.$$

It follows that any non-zero $\mathfrak{A}$–module map $N \longrightarrow P$ is injective and its cokernel is a cyclic $\mathfrak{A}$–module. We hence have the following cases, corresponding to the enumeration in the Theorem 2.5.

(I) Cyclic modules of type "$*$".

(a) The dimension vector of

$$\mathrm{cok}\Big(Q \stackrel{t^d}{\longrightarrow} P_*\Big) = \mathrm{cok}\left(\begin{pmatrix} \mathfrak{m} \\ \mathfrak{D} \\ \mathfrak{D} \end{pmatrix} \stackrel{t^d}{\longrightarrow} \begin{pmatrix} \mathfrak{D} \\ \mathfrak{D} \\ \mathfrak{D} \end{pmatrix}\right)$$

is $(d, d+1, d)$.

(b) The dimension vector of

$$\mathrm{cok}\Big(P_* \stackrel{t^{d+1}}{\longrightarrow} P_*\Big) = \mathrm{cok}\left(\begin{pmatrix} \mathfrak{D} \\ \mathfrak{D} \\ \mathfrak{D} \end{pmatrix} \stackrel{t^{d+1}}{\longrightarrow} \begin{pmatrix} \mathfrak{D} \\ \mathfrak{D} \\ \mathfrak{D} \end{pmatrix}\right)$$

is $(d+1, d+1, d+1)$.

(c) The dimension vector of

$$\operatorname{cok}\Big(P_- \xrightarrow{t^d} P_*\Big) = \operatorname{cok}\left(\begin{pmatrix}\mathfrak{m}\\ \mathfrak{m}\\ \mathfrak{D}\end{pmatrix} \xrightarrow{t^d} \begin{pmatrix}\mathfrak{D}\\ \mathfrak{D}\\ \mathfrak{D}\end{pmatrix}\right)$$

is $(d, d+1, d+1)$.

(d) The dimension vector of

$$\operatorname{cok}\Big(P_+ \xrightarrow{t^d} P_*\Big) = \operatorname{cok}\left(\begin{pmatrix}\mathfrak{m}\\ \mathfrak{D}\\ \mathfrak{m}\end{pmatrix} \xrightarrow{t^d} \begin{pmatrix}\mathfrak{D}\\ \mathfrak{D}\\ \mathfrak{D}\end{pmatrix}\right)$$

is $(d+1, d+1, d)$.

(II) (a) The dimension vector of

$$\operatorname{cok}\Big(P_* \xrightarrow{t^{d+1}} P_+\Big) = \operatorname{cok}\left(\begin{pmatrix}\mathfrak{D}\\ \mathfrak{D}\\ \mathfrak{D}\end{pmatrix} \xrightarrow{t^{d+1}} \begin{pmatrix}\mathfrak{m}\\ \mathfrak{D}\\ \mathfrak{m}\end{pmatrix}\right)$$

is $(d, d, d+1)$.

(b) The dimension vector of

$$\operatorname{cok}\Big(Q \xrightarrow{t^{d+1}} P_+\Big) = \operatorname{cok}\left(\begin{pmatrix}\mathfrak{m}\\ \mathfrak{D}\\ \mathfrak{D}\end{pmatrix} \xrightarrow{t^{d+1}} \begin{pmatrix}\mathfrak{m}\\ \mathfrak{D}\\ \mathfrak{m}\end{pmatrix}\right)$$

is $(d, d+1, d+1)$.

(c) The dimension vector of

$$\operatorname{cok}\Big(P_+ \xrightarrow{t^{d+1}} P_+\Big) = \operatorname{cok}\left(\begin{pmatrix}\mathfrak{m}\\ \mathfrak{D}\\ \mathfrak{D}\end{pmatrix} \xrightarrow{t^{d+1}} \begin{pmatrix}\mathfrak{m}\\ \mathfrak{D}\\ \mathfrak{m}\end{pmatrix}\right)$$

is $(d+1, d+1, d+1)$.

(d) The dimension vector of

$$\operatorname{cok}\Big(P_- \xrightarrow{t^d} P_+\Big) = \operatorname{cok}\left(\begin{pmatrix}\mathfrak{m}\\ \mathfrak{m}\\ \mathfrak{D}\end{pmatrix} \xrightarrow{t^d} \begin{pmatrix}\mathfrak{m}\\ \mathfrak{D}\\ \mathfrak{m}\end{pmatrix}\right)$$

is $(d-1, d, d+1)$ with $d \geq 1$.

The assertion about the nilpotency degrees $m_\pm$ and m_* follows from the observation that C_* (respectively, $C_\pm$) is a nilpotent Jordan block of size $\dim(V_*)$ (respectively, $\dim(V_\pm)$). ■

Remark 2.6. Theorem 2.5 asserts that cyclic $\mathfrak{A}$–modules are determined by purely "discrete data": their types and dimension vectors. It is not true in general, even in the case of arbitrary nodal orders. For example, let $\mathfrak{N} = \mathbb{C}[\![x, y]\!]/\!/(xy)$.Then for $n, m \in \mathbb{N}$ and $\lambda \in \mathbb{C}^*$ the module $V_{(m,n,\lambda)} = \mathfrak{N}/(x^m - \lambda y^n)$ is cyclic. However, $V_{(m,n,\lambda)} \cong V_{(m',n',\lambda')}$ if and only if $(m, n, \lambda) = (m', n', \lambda')$.

Remark 2.7. The problem of classification of *all* indecomposable finite dimensional $\mathfrak{A}$–modules was posed by I. Gelfand in [24]. In 1973 Nazarova and Roiter proved that $\mathsf{Rep}(\mathfrak{A})$ is representation tame, reducing the problem of description of its indecomposable objects to a certain problem of linear algebra (matrix problem). However, the correct combinatorics of indecomposables was obtained only in 1988 by Bondarenko [4, 5]. Independently and about the same time, Crawley-Boevey gave another solution of Gelfand's problem using a completely different approach [17]. In 2004 Burban and Drozd proved that also the derived category $\mathsf{D}^b\big(\mathsf{Rep}(\mathfrak{A})\big)$ is representation tame and gave an explicit description of the corresponding indecomposable complexes of projective modules [15]. See also [25] for further elaborations of this approach.

Remark 2.8. Since $\mathfrak{B}$ is a hereditary order, any indecomposable finite dimensional $\mathfrak{B}$–module is cyclic; see for instance [19]. Let

$$V = \left[\; V_- \underset{A_-}{\overset{A_+}{\rightleftarrows}} V_+ \;\right]$$

be a nilpotent representation of (1.5). We put: $C_\pm := A_\mp A_\pm$. Let $m_\pm$ be the nilpotency degree of $C_\pm$ and $\underline{\dim}(V) = \big(\dim(V_-), \dim(V_+)\big)$ the dimension vector of V. Let

$$e_+ = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{B} \quad \text{and} \quad e_- = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{B}.$$

We put:

$$P_+ = \mathfrak{B}e_+ \cong \left(\begin{smallmatrix} \mathfrak{D} \\ \mathfrak{D} \end{smallmatrix}\right) \quad \text{and} \quad P_- = \mathfrak{B}e_- \cong \left(\begin{smallmatrix} \mathfrak{m} \\ \mathfrak{D} \end{smallmatrix}\right).$$

Cyclic $\mathfrak{B}$–modules are up to isomorphism characterized by their types and dimension vectors. Let $d \in \mathbb{N}_0$.

(I) Cyclic modules of type "+" are the following.

(a) The dimension vector of

$$\operatorname{cok}\Big(P_- \xrightarrow{t^d} P_+\Big) = \operatorname{cok}\Big(\left(\begin{smallmatrix} \mathfrak{m} \\ \mathfrak{D} \end{smallmatrix}\right) \xrightarrow{t^d} \left(\begin{smallmatrix} \mathfrak{D} \\ \mathfrak{D} \end{smallmatrix}\right)\Big)$$

is $(d, d+1)$.

(b) The dimension vector of

$$\mathrm{cok}\Big(P_+ \xrightarrow{t^d} P_+\Big) = \mathrm{cok}\Big(\big(\begin{smallmatrix}\mathfrak{m}\\ \mathfrak{D}\end{smallmatrix}\big) \xrightarrow{t^{d+1}} \big(\begin{smallmatrix}\mathfrak{D}\\ \mathfrak{D}\end{smallmatrix}\big)\Big)$$

is $(d+1, d+1)$.

For both types of cyclic modules we have: $m_+ = d+1$.

(II) The classification of cyclic $\mathfrak{B}$–modules of type "$-$" is analogous.

3 From polyharmonic Maaß forms to quiver representations

We describe and classify quiver representations associated to polyharmonic weak Maaß forms and provide in each case of the classification polyharmonic weak Maaß forms in all possible weights that yield the relevant quiver representation. The classification is done in Section 3.3 by linking it to the classification of cyclic quiver representations from Section 2. Our modular realizations are given in Section 5.

The representation theoretic labels that arise in Section 2 do not match the ones for harmonic Maaß forms that appeared in work of Bringmann–Kudla [7]. We provide a translation in Section 3.5.

3.1 Polyharmonic Maaß forms The group $G = \mathrm{SL}_2(\mathbb{R})$ acts on the upper half-plane $\mathbb{H} = \{\tau = x + iy \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ by Möbius transformations:

$$\big(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\big)\tau = \frac{a\tau + b}{c\tau + d}.$$

For $k \in \mathbb{Z}$, we obtain the weight-k slash action on functions $f : \mathbb{H} \longrightarrow \mathbb{C}$ defined by

$$\big(f\big|_k g\big)(\tau) = (c\tau + d)^{-k} f(g\tau), \quad g = \big(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\big) \in \mathrm{SL}_2(\mathbb{R}).$$

Given $k \in \mathbb{Z}$, we consider the weight-k *Maaß lowering and raising operators*

$$\mathrm{L}_k = -2iy^2\partial_{\overline{\tau}} \quad \text{and} \quad \mathrm{R}_k = 2i\partial_\tau + ky^{-1},$$

and the *Maaß–Laplace operator*

$$\Delta_k = -y^2(\partial_x^2 + \partial_y^2) + iky(\partial_x + i\partial_y) = -\mathrm{R}_{k-2}\,\mathrm{L}_k = -(\mathrm{L}_{k+2}\mathrm{R}_k + k), \tag{3.1}$$

acting on the space of smooth functions on $\mathbb{H}$. We also define iterated versions of the lowering and raising operators by

$$\mathrm{L}_k^j = \mathrm{L}_{k-2(j-1)} \circ \cdots \circ \mathrm{L}_{k-2} \circ \mathrm{L}_k \quad \text{and} \quad \mathrm{R}_k^j = \mathrm{R}_{k+2(j-1)} \circ \cdots \circ \mathrm{R}_{k+2} \circ \mathrm{R}_k.$$

For $k \in \mathbb{Z}_{\le 0}$, we define the *flipping operator*

$$\mathrm{F}_k f = \frac{y^{-k}}{(-k)!} \overline{\mathrm{R}_k^{-k} f}. \tag{3.2}$$

Closely related to this, we record that for all functions f on $\mathbb{H}$ and all $k \in \mathbb{Z}$, $\gamma \in \mathrm{SL}_2(\mathbb{R})$, we have

$$y^k \overline{(f|_k\gamma)} = \big(y^k \overline{f}\big)\big|_{-k}\gamma. \tag{3.3}$$

Definition 3.1. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian subgroup and $\rho : \Gamma \longrightarrow \mathrm{GL}(V(\rho))$ a representation on a finite dimensional, complex vector space $V(\rho)$. We call a smooth function $f : \mathbb{C} \longrightarrow V(\rho)$ a polyharmonic weak Maaß form for ρ of weight $k \in \mathbb{Z}$ and depth $d \in \mathbb{Z}_{\ge 0}$, if

(i) $f|_k\gamma = \rho(\gamma) f$ for all $\gamma \in \Gamma$,

(ii) $\Delta_k^{d+1} f = 0$,

(iii) $\|f(\tau)\| \ll \exp(ay)$ as $y \longrightarrow \infty$ for some $a \in \mathbb{R}$ and some norm $\|\cdot\|$ on $V(\rho)$.

The space of such functions will be denoted by $\mathrm{H}_k^{(d)}(\rho)$. If ρ is the trivial representation, we may instead write $\mathrm{H}_k^{(d)}(\Gamma)$.

We say that a nonzero $f \in \mathrm{H}_k^{(d)}(\rho)$ has exact depth d if $d = 0$ or if $f \not\in \mathrm{H}_k^{(d-1)}(\rho)$.

Remark 3.2. In their seminal work [13] Bruinier and Funke introduced the notion of harmonic weak Maaß forms for subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and its metaplectic cover (in our notation these are the forms of exact depth $d = 0$).

Remark 3.3. Lagarias–Rhoades [28] investigated the space of polyharmonic Maaß forms and showed that the Taylor coefficients of certain Eisenstein series can be used to build bases for these spaces. Their results were refined by Matsusaka [29] to apply to spaces of polyharmonic weak Maaß forms (i.e. forms that satisfy a growth condition as in (iii) as opposed to a polynomial growth condition that Maaß forms satisfy).

Example 3.4. Following Verdier [31] Bringmann and Kudla introduce certain vector-valued harmonic weak Maaß forms in [7]. We let $k \in \mathbb{N}_0$ and set $m = -k$. By Pol_m we denote the space of polynomials of degree at most m in the variable X. The group $\mathrm{SL}_2(\mathbb{R})$ acts on Pol_m via

$$\big(\rho_m(\gamma)p\big)(X) = (-cX+a)^m p\Big(\frac{dX-b}{-cX+a}\Big),\, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

Let $r \in \mathbb{Z}$ with $0 \le r \le m$ and define

$$\mathfrak{e}_{r,m-r}(\tau)(X) = \frac{(-1)^{m-r}}{r!} y^{r-m} (X-\tau)^r (X-\overline{\tau})^{m-r}.$$

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we then have

$$\mathfrak{e}_{r,m-r}(\gamma\tau) = (c\tau+d)^{m-2r} \rho_m(\gamma) \mathfrak{e}_{r,m-r}(\tau),$$

that is $\mathfrak{e}_{r,m-r}$ has weight $m-2r$. The holomorphic function $\mathfrak{e}_{m,0}$ is a harmonic weak Maaß form of weight $-m$ and type ρ_m.

The functions $\mathfrak{e}_{r,m-r}$ in Example 3.4 behave as follows under the lowering and raising operators:

$$\mathrm{L}_{m-2r} \mathfrak{e}_{r,m-r} = (r+1)(m-r)\, \mathfrak{e}_{r+1,m-r-1}, \quad \mathrm{R}_{m-2r} \mathfrak{e}_{r,m-r} = \mathfrak{e}_{r-1,m-r+1}. \tag{3.4}$$

In particular, $\mathrm{L}_{-m}\mathfrak{e}_{m,0} = 0$ and $\mathrm{R}_m \mathfrak{e}_{0,m} = 0$. For later reference we also note that

$$\Delta_{m-2r}\, \mathfrak{e}_{r,m-r} = -(r+1)(m-r)\, \mathfrak{e}_{r,m-r} \tag{3.5}$$

and

$$y^{m-2r}\, \overline{\mathfrak{e}_{r,m-r}} = \frac{(-1)^m (m-r)!}{r!}\, \mathfrak{e}_{m-r,r}. \tag{3.6}$$

Finally, we note that the behavior of the functions in Example 3.4 under the flipping operator is

$$\mathrm{F}_{-m} \mathfrak{e}_{m,0} = (-1)^m\, \mathfrak{e}_{m,0}. \tag{3.7}$$

3.2 Automorphic forms The map $G/K \longrightarrow \mathbb{H}, [g] \longmapsto gi$ is an isomorphism of real manifolds and K is the stabilizer of the point $i \in \mathbb{H}$. This allows us to lift polyharmonic weak Maaß forms to "weak" automorphic forms.

Recall the setting of Definition 3.1. We associate to $f \in \mathrm{H}^{(d)}_k(\rho)$ a smooth function

$$\varphi_f : G \longrightarrow V(\rho),\ g \longmapsto \big(f|_k g\big)(i). \tag{3.8}$$

We see that φ_f satisfies the following properties:

$$\begin{aligned} \varphi_f(hg) &= \rho(h)\big(\varphi_f(g)\big) \quad \text{for all } h \in \Gamma \text{ and } g \in G; \\ \varphi_f(gk_\theta) &= \exp(ik\theta)\,\varphi_f(g) \text{ for all } g \in G \text{ and } \theta \in \mathbb{R}, \end{aligned} \tag{3.9}$$

where k_θ is given by

$$k_\theta = \exp(i\theta H) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in K = \mathrm{SO}_2(\mathbb{R}). \tag{3.10}$$

The second relation in (3.9) implies that we have $H\varphi_f = k\varphi_f$. Further, the condition $\Delta_k^{d+1} f = 0$ in Definition 3.1 translates to

$$\big(C - (k^2 - 2k)\big)^{d+1} \varphi_f = 0, \tag{3.11}$$

where C is the Casimir element in (1.1), which generates $\mathfrak{z} \subset U(\mathfrak{g})$.

Consider the vector space

$$\mathscr{A}\big(G, \Gamma, \rho\big) := \left\{ \mathbb{H} \xrightarrow{\varphi} W \,:\, \begin{array}{l} \varphi(hg) = \rho(h)\big(\varphi(g)\big) \text{ for all } h \in \Gamma, g \in G, \\ \varphi \text{ is } K\text{-finite and } \mathfrak{z}\text{-finite} \end{array} \right\}. \tag{3.12}$$

The space $\mathscr{A}\big(G, \Gamma, (W, \rho)\big)$ is naturally a $(\mathfrak{g}, K)$–module and for any $f \in \mathrm{H}^{(d)}_k(\rho)$ we have: $\varphi_f \in \mathscr{A}(G, \Gamma, \rho)$.

Remark 3.5. Compared to the definition of the space of automorphic forms the space in (3.12) lacks an (exponential) growth condition mirroring the one in Definition 3.1. Since we are merely interested in $(\mathfrak{g}, K)$–submodules that are generated by φ_f for $f \in \mathrm{H}^{(d)}_k(\rho)$, this does not affect our further discussion.

3.3 Harish-Chandra modules and quiver representations We continue to work in the setting of Definition 3.1. We attach to a function $f \in \mathrm{H}_k(\rho)$ the Harish–Chandra module $M(\varphi_f) \subset \mathscr{A}(G, \Gamma, \rho)$ generated by the function φ_f.

Let

$$\gamma = k^2 - 2k = (k-1)^2 - 1.$$

We define $l \in \mathbb{Z}_{\geq 0}$ according to the following cases:

$$\begin{aligned} l &= 1-k, && \text{if } k < 1; \\ l &= 0 = 1-k = k-1, && \text{if } k = 1; \\ l &= k-1, && \text{if } k > 1. \end{aligned} \tag{3.13}$$

With this notion, we have $\gamma = l^2 - 1$ and $M(\varphi_f) \in \mathsf{HC}_l(\mathfrak{g}, K)$. We assume that $f \neq 0$ has exact depth d. Then $d+1 \in \mathbb{N}$ is the nilpotency degree of f with respect to Δ_k, i.e. $\Delta_k^{d+1}(f) = 0$ but $\Delta_k^d(f) \neq 0$.

Our next goal is to characterize the quiver representation $\mathbb{E}\big(M(\varphi_f)\big)$.

Theorem 3.6. *For any nonzero $f \in \mathrm{H}_k^{(d)}(\rho)$ of exact depth d, the corresponding quiver representation $V_f := \mathbb{E}\big(M(\varphi_f)\big)$ is cyclic. In particular, the Harish–Chandra module $M(\varphi_f)$ is indecomposable.*

Proof. First note that the Harish–Chandra module $M(\varphi_f)$ is spanned as $\mathbb{C}$-vector space by the elements of the form $X^{a_1} Y^{b_1} \ldots X^{a_s} Y^{b_s} \varphi_f$ for $a_1, \ldots, a_s, b_1, \ldots, b_s \in \mathbb{N}_0$. For $k \neq 1$ we have an equivalence of categories $\mathsf{HC}_l(\mathfrak{g}, K) \xrightarrow{\mathbb{E}} \mathsf{Rep}(\mathfrak{A})$.

Consider the case $k < 1$. Then $l = 1-k$. We claim that V_f is a cyclic $\mathfrak{A}$–module of type $*$. To show this, we use the description of the functor $\mathbb{E}$ given in Theorem 1.8. Let

$$V_f = \left[\; V_- \underset{B_-}{\overset{A_-}{\rightleftarrows}} V_\star \underset{B_+}{\overset{A_+}{\leftrightarrows}} V_+ \;\right] = \left[\; M_{-l-1} \underset{Y_-}{\overset{X_-}{\rightleftarrows}} M_{-l+1} \underset{X_*^{-1} Y_+}{\overset{X_+ X_*}{\rightleftarrows}} M_{l+1} \;\right].$$

Then we have: $A_- B_- = C_* = A_+ B_+$ and the following statements are true:

$$V_* = \big\{ C_*^n \varphi_f \,\big|\, n \in \mathbb{N}_0 \big\} \quad \text{and} \quad V_\pm = \big\{ B_\pm C_*^n \varphi_f \,\big|\, n \in \mathbb{N}_0 \big\}.$$

It follows that $P_* = \mathfrak{A} e_* \longrightarrow V_f$, $e_* \longmapsto \varphi_f$ is an epimorphism. As a consequence, V_f is cyclic of type $*$, as asserted. Moreover, the nilpotency degree $d+1$ of f is equal to m_* (which is the nilpotency degree of C_*).

The case $k > 1$ is analogous. We have: $k = l + 1$ and V_f is a cyclic $\mathfrak{A}$–module of type $+$. We have: $\varphi_f \in V_+$ and $d + 1 = m_+$ is the nilpotency degree of the endomorphism $C_+ = B_+ A_+$.

The case $k = 1$ is exceptional since in this case we have an equivalence of categories $\mathsf{HC}_0(\mathfrak{g}, K) \xrightarrow{\mathbb{E}} \mathsf{Rep}(\mathfrak{B})$. In this case

$$V_f = \left[\; V_- \underset{Z_-}{\overset{Z_+}{\rightleftarrows}} V_+ \; \right] = \left[\; M_{-1} \underset{Y}{\overset{X}{\rightleftarrows}} M_1 \; \right]$$

is a cyclic representation of $\mathfrak{B}$ of type $+$ and $d + 1 = m_+$ is the nilpotency degree of $C_+ := Z_+ Z_-$. ■

Remark 3.7. It is not true in general that a $\mathfrak{g}$-module generated by a cyclic vector is automatically indecomposable.[1] For example, let $V_\circ = \mathbb{C}$ be the trivial representation and $V_\diamond = \mathbb{C}^2$ be the fundamental representation of $\mathfrak{g}$. Then the decomposable representation $V = V_\circ \oplus V_\diamond$ is generated by a cyclic element $v = \left(1, \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) \in V$.

3.4 Classification by representation theoretic labels

Let $f \in \mathrm{H}^{(d)}(\rho)$ be a polyharmonic form of weight $k \in \mathbb{Z}$ and exact depth d. Recall the operators $C_* = A_- B_-$ and $C_+ = Z_+ Z_-$ from the proof of Theorem 3.6. The correspondence between the Harish–Chandra module $M(\varphi_f)$ and the cyclic quiver representation $V_f = \mathbb{E}\big(M(\varphi_f)\big)$ classified in Theorem 2.5 and Remark 2.8 is as follows. Labels prefixed with G stand for representations of the Gelfand quiver, and those prefixed with C stand for representations of the cyclic quiver.

(GI) Let $k < 1$. Then V_f be a cyclic representation of $\mathfrak{A}$ of type "$*$" and $m_* = d + 1$. Writing $\psi_f := C_*^d(\varphi_f)$ we have the following cases:

(a) V_f has type GIa if $Y_- \psi_f = 0$ and $X_+ X_* \psi_f = 0$.

(b) V_f has type GIb if $Y_- \psi_f \neq 0$ and $X_+ X_* \psi_f \neq 0$.

(c) V_f has type GIc if $Y_- \psi_f = 0$ but $X_+ X_* \psi_f \neq 0$.

(d) V_f has type GId if $Y_- \psi_f \neq 0$ and $X_+ X_* \psi_f = 0$.

(GII) Let $k > 1$. Then V_f be a cyclic representation of $\mathfrak{A}$ of type "$+$" and $m_+ = d + 1$.

[1]The second-named author is grateful to Volodymyr Mazorchuk for drawing his attention to this fact.

(1) Assume that $d = 0$. We have the following cases:

(a) V_f has type GIIa if $Y_+ \varphi_f = 0$.
(b) V_f has type GIIb if $Y_+ \varphi_f \neq 0$ but $Y_- Y_* Y_+ \varphi_f = 0$.
(c) V_f has type GIIc if $Y_- Y_* Y_+ \varphi_f \neq 0$.
(d) Type GIId does not occur for $d = 0$.

(2) Assume that $d \geq 1$. Writing $\psi_f := C_*^{d-1} Y_+ \varphi_f$, where we note that $X_+ \psi_f = C_+^d \varphi_f \neq 0$, we have the following cases:

(a) V_f has type GIIa if $Y_- Y_* \psi_f \neq 0$ and $C_* \psi_f = 0$.
(b) V_f has type GIIb if $Y_- Y_* \psi_f \neq 0$, $C_* \psi_f \neq 0$, and $Y_- Y_* C_* \psi_f = 0$.
(c) V_f has type GIIc if $Y_- Y_* \psi_f \neq 0$ and $Y_- Y_* C_* \psi_f \neq 0$.
(d) V_f has type GIId if $Y_- Y_* \psi_f = 0$.

(CI) Let $k = 1$. Then V_f is a cyclic representation of $\mathfrak{B}$ of type "+" and $d = m_+ + 1$. Writing $\psi_f := C_+^d \varphi_f$ we have the following cases:

(a) V_f has type CIa if $Y \psi_f = 0$.
(b) V_f has type CIb if $Y \psi_f \neq 0$.

3.5 Classification by BK-style labels The classification in Section 3.4 matches with the one obtained by Bringmann and Kudla in Theorem 5.2 of [7] in the case $d = 0$. However, the labels, which are natural from the point of view of representation theory, do not agree with the ones of Bringmann–Kudla, which are natural from the perspective of modular forms. Table 3.5 provides a translation between them. In higher depth we encounter an additional Case IIId, which corresponds to the representation theoretic Case GIId.

Let $f \in \mathrm{H}^{(d)}(\rho)$ be a polyharmonic form of weight $k \in \mathbb{Z}$ and exact depth d. The Harish-Chandra module $M(\varphi_f)$ as in Section 3.4 is classified using the labels of Bringmann–Kudla as follows:

(I) $k < 1$.

(Ia) $\mathrm{L}\Delta^d f = 0$ and $\mathrm{R}^{1-k} \Delta^d f = 0$.
(Ib) $\mathrm{L}\Delta^d f = 0$ and $\mathrm{R}^{1-k} \Delta^d f \neq 0$.
(Ic) $\mathrm{L}\Delta^d f \neq 0$ and $\mathrm{R}^{1-k} \Delta^d f = 0$.
(Id) $\mathrm{L}\Delta^d f \neq 0$ and $\mathrm{R}^{1-k} \Delta^d f \neq 0$.

(II) $k = 1$.

(IIa) $\mathrm{L}\Delta^d f = 0$.
(IIb) $\mathrm{L}\Delta^d f \neq 0$.

(III) $k > 1$.

(IIIa) $\mathrm{L}^k \Delta^{d-1} f \neq 0$, if $d \geq 1$, and $\mathrm{L}\Delta^d f = 0$.
(IIIb) $\mathrm{L}\Delta^d f \neq 0$ and $\mathrm{L}^k \Delta^d f = 0$.
(IIIc) $\mathrm{L}^k \Delta^d f \neq 0$.
(IIId) $d \geq 1$ and $\mathrm{L}^k \Delta^{d-1} f = 0$.

To translate the classification in Section 3.4, it suffices to recall the notation in (1.7) and (1.9). The operators X and Y restricted to M_k yield nonzero multiples of L_k and R_k. In Cases IIIb and IIIc, we have removed extraneous conditions. For example, in Case IIIc $\mathrm{L}^k \Delta^d f \neq 0$ implies $\mathrm{L}^k \Delta^{d-1} f \neq 0$, which is therefore omitted.

Figure 1: Translation between labels assigned by Bringmann–Kudla and representation theoretic labels emerging in Section 2, including page references to their modular and representation theoretic realizations.

BK label	Ia	Ib	Ic	Id	IIa	IIb	IIIa	IIIb	IIIc	IIId
mod. form on p.	37	38	38	39	39	40	40	40	41	41
repr. label	GIa	GIc	GId	GIb	CIa	CIb	GIIa	GIIb	GIIc	GIId
repr. on p.	13	14	14	13	15	16	14	14	14	14

4 Construction of spectral derivatives and modular realizations

In this section we give an existence theorem for spectral derivatives. In particular, we employ our theorem to provide examples that realize all possible modules that occur in our classification.

4.1 Commutation relations for differential operators To perform the calculations in Section 4.5, we need several algebraic relations for the Maaß operators. We preserve the subscripts of all operators, but when viewing Δ, L, and R as graded operators they can be suppressed for clarity. In particular, the next relations can be verified by merely using the commutator $[\mathrm{L},\mathrm{R}] = -k$, which follows from (3.1) and in which k on the right hand side is viewed as a graded scalar as well. We have the commutator relations for the Laplace operator:

$$\begin{aligned} \Delta_{k-2r}\,\mathrm{L}_k^r &= \mathrm{L}_k^r\,\big(\Delta_k - r(k-r-1)\big),\\ \Delta_{k+2r}\,\mathrm{R}_k^r &= \mathrm{R}_k^r\,\big(\Delta_k + r(k+r-1)\big); \end{aligned} \tag{4.1}$$

And for the Maaß lowering and raising operators:

$$\begin{aligned} \mathrm{R}_{k-2r}\mathrm{L}_k^r &= -\mathrm{L}_k^{r-1}\,\big(\Delta_k - (r-1)(k-r)\big),\\ \mathrm{L}_{k+2r}\,\mathrm{R}_k^r &= -\mathrm{R}_k^{r-1}\big(\Delta_k + r(k+r-1)\big). \end{aligned} \tag{4.2}$$

The Bol Identity asserts that for an integer $k \le 0$ and a weight-k harmonic function f we have

$$\mathrm{R}_k^{1-k} f = (2\pi\partial_\tau)^{1-k} f. \tag{4.3}$$

A straightforward calculation shows that for smooth functions f and g and integers k_1, k_2 we have

$$\Delta_{k_1+k_2}(f\cdot g) = (\Delta_{k_1} f)\cdot g + f\cdot(\Delta_{k_2} g) - (\mathrm{R}_{k_1} f)\cdot(\mathrm{L}_{k_2} g) - (\mathrm{L}_{k_1} f)\cdot(\mathrm{R}_{k_2} g). \tag{4.4}$$

Proposition 4.1. *The intertwining relations for the flipping operator and the Maaß operators, $k \le 0$, are*

$$\begin{aligned} \mathrm{L}_k\,\mathrm{F}_k\Delta_k &= -(k-2)(k-1)\mathrm{F}_{k-2}\,\mathrm{L}_k,\\ -k(k+1)\,\mathrm{R}_k\,\mathrm{F}_k &= \mathrm{F}_{k+2}\,\mathrm{R}_k(\Delta_k + k). \end{aligned} \tag{4.5}$$

Proof. We begin with the first identity and evaluate by a short calculation

$$\mathrm{L}_k\,\frac{y^{-k}}{(-k)!}\overline{\mathrm{R}_k^{-k}\Delta_k f} = \frac{y^{2-k}}{(-k)!}\,\overline{\mathrm{R}_k^{-k+1}\Delta_k f}.$$

Now we write $\Delta_k = -\mathrm{R}_{k-2}\mathrm{L}_k$ and see

$$\frac{y^{-k+2}}{(-k)!}\,\overline{\mathrm{R}_k^{-k+1}\Delta_k f} = \frac{-y^{-k+2}}{(-k)!}\,\overline{\mathrm{R}_k^{-k+1}\mathrm{R}_{k-2}\mathrm{L}_k f} = -(-k+2)(-k+1)\,\mathrm{F}_{k-2}\mathrm{L}_k f.$$

For the second identity we calculate

$$\mathrm{R}_k\mathrm{F}_k f = \frac{y^{-k-2}}{(-k)!}\,\overline{\mathrm{L}_{-k}\mathrm{R}_k^{-k} f}.$$

Then we use (4.1) and obtain

$$\frac{y^{-k-2}}{(-k)!}\,\overline{\mathrm{R}_k^{-k-1}\big(\Delta_k - k(k-k-1)\big)f} = \frac{-1}{(-k)(-k-1)}\,\mathrm{F}_{k+2}\mathrm{R}_k\big(\Delta_k + k\big)f. \qquad \blacksquare$$

Proposition 4.2. *We have*

$$\begin{aligned}\Delta_k \mathrm{F}_k &= \mathrm{F}_k \Delta_k,\\ \mathrm{F}_k \mathrm{F}_k &= \tfrac{(-1)^{-k}}{(-k)!^2} \big(\Delta_k + 1k\big)\big(\Delta_k + 2(k+1)\big) \cdots \big(\Delta_k + (-k)(-1)\big).\end{aligned} \tag{4.6}$$

Proof. For the first identity we write $f = \Delta_k g$ locally and use both identities in (4.5) as well as (4.1) to obtain

$$\begin{aligned}\Delta_k \mathrm{F}_k f &= -\mathrm{R}_{k-2} \mathrm{L}_k \mathrm{F}_k \Delta_k g = \mathrm{R}_{k-2} (k-2)(k-1)\, \mathrm{F}_{k-2} \mathrm{L}_k g\\ &= \tfrac{-(k-2)(k-1)}{(k-2)(k-1)} \mathrm{F}_k \mathrm{R}_{k-2} (\Delta_{k-2} + k - 2)\, \mathrm{L}_k g\\ &= -\mathrm{F}_k (\Delta_k - k + 2 + k - 2)\, \mathrm{R}_{k-2} \mathrm{L}_k g = \mathrm{F}_k \Delta_k^2 g = \mathrm{F}_k \Delta_k f.\end{aligned}$$

For the second identity we use that for any real-analytic function g we have the identity

$$\mathrm{R}_k \big(y^{-k} \overline{\mathrm{R}_{-k-2} g}\big) = y^{-k-2} \overline{(\Delta_{-k-2} + k + 2\, g}.$$

A repeated application of this identity yields the result. ∎

A direct calculation shows that for smooth functions f we have

$$\mathrm{L}_k \mathrm{F}_k f = \frac{y^{2-k}}{(-k)!} \overline{\mathrm{R}_k^{1-k} f}. \tag{4.7}$$

As a consequence of Proposition 4.2 if $\Delta_k f = 0$, we have $\mathrm{F}_k \mathrm{F}_k f = f$ and, compare for example with Proposition 5.14 of [6],

$$\mathrm{R}_k^{1-k} \mathrm{F}_k f = (-k)! y^{k-2} \overline{\mathrm{L}_k f}. \tag{4.8}$$

4.2 Eisenstein and Poincaré series In this section we revisit some Eisenstein and Poincaré series that we will later use as an input for spectral families when constructing examples of polyharmonic Maaß forms. Let $\Gamma_\infty \subset \mathrm{SL}_2(\mathbb{Z})$ be the subgroup of upper triangular matrices.

We first define the Eisenstein series of weight $k \in \mathbb{Z}$. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - \frac{k}{2}$. Then we set

$$E_k(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} y^s \big|_k \gamma. \tag{4.9}$$

Via analytic continuation $E_k(\tau,s)$ extends to all $s \in \mathbb{C}$ except for possible simple poles. We set $E_k(\tau) = E_k(\tau,0)$. The Maaß lowering and raising operators act as

$$\begin{aligned} \mathrm{L}_k\, E_k(\tau,s) &= s\, E_{k-2}(\tau,s+1),\\ \mathrm{R}_k\, E_k(\tau,s) &= (s+k)\, E_{k+2}(\tau,s-1). \end{aligned} \tag{4.10}$$

In particular, we have $\Delta_k\, E_k(\tau,s) = s(1-k-s)E_k(\tau,s)$. Moreover, we have

$$y^k\, \overline{E_k(\,\cdot\,,s)} = E_{-k}(\,\cdot\,,\overline{s}+k). \tag{4.11}$$

Now we recall some facts on Poincaré series with exponential growth at the cusps following [12]. We let $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ denote the usual Whittaker functions (see p. 190 of [1]). For integers k and $m \neq 0$, $\tau = x+iy \in \mathbb{H}$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we define

$$F_{k,m}(z,s) = \frac{1}{2\Gamma(2s)} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \Big(\big(-\mathrm{sgn}(m)\big)^{1-k} \big(4\pi|m|y\big)^{-\frac{k}{2}} M_{\mathrm{sgn}(m)\frac{k}{2},\, s-\frac{1}{2}}\big(4\pi|m|y\big)\, e(mx) \Big)\Big|_k \gamma. \tag{4.12}$$

This Poincaré series converges for $\mathrm{Re}(s) > 1$ and is an eigenfunction of Δ_k with eigenvalue $s(1-s) + (k^2 - 2k)/4$.

We recall its behavior under the Maaß raising and lowering operators.

Proposition 4.3. *We have*

$$\begin{aligned} \mathrm{R}_k\, F_{k,m}(\tau,s) &= 4\pi|m| \big(s + \tfrac{k}{2}\big)\, F_{k+2,m}(\tau,s),\\ \mathrm{L}_k\, F_{k,m}(\tau,s) &= \frac{1}{4\pi|m|} \big(s - \tfrac{k}{2}\big)\, F_{k-2,m}(\tau,s). \end{aligned} \tag{4.13}$$

Proof. Since L_k and R_k commute with the slash operator, it suffices to show the identity on the corresponding Whittaker functions. We use equations (13.4.10), (13.4.11) and (13.1.32) in [1] which imply the desired identity. Note that parts of these identities were already proven in [14] and [2]. ∎

Proposition 4.4. *Complex conjugation yields*

$$y^k\, \overline{F_{k,m}(\tau,s)} = (-1)^{1-k} (4\pi|m|)^{-k}\, F_{-k,-m}(\tau,\overline{s}). \tag{4.14}$$

Proof. We have

$$\begin{aligned}
& y^k \overline{\Big(\frac{1}{2\Gamma(2s)} \big(-\operatorname{sgn}(m) \big)^{1-k} \big(4\pi |m| y \big)^{-\frac{k}{2}} M_{\operatorname{sgn}(m) \frac{k}{2}, s - \frac{1}{2}} \big(4\pi |m| y \big)\, e(mx) \Big) \big|_k \gamma} \\
= {} & y^k \frac{1}{2\Gamma(2\overline{s})} \big(-\operatorname{sgn}(m) \big)^{1-k} \big(4\pi |m| \operatorname{Im}(\gamma\tau) \big)^{-\frac{k}{2}} \\
& \quad \times M_{\operatorname{sgn}(m) \frac{k}{2}, \overline{s} - \frac{1}{2}} \big(4\pi |m| \operatorname{Im}(\gamma\tau) \big)\, e(-m \operatorname{Re}(\gamma\tau)) (c\overline{\tau} + d)^{-k}.
\end{aligned}$$

Note that $\operatorname{Im}(\gamma\tau) = y(c\tau + d)^{-1}(c\overline{\tau} + d)^{-1}$. A short calculation then yields the desired result. ∎

We also recall the existence of spectral families of Poincaré series (compare [11] and [30]).

Proposition 4.5. *Let f be a harmonic weak Maaß form of weight $k \le 0$. There exists an open neighborhood U in $\mathbb{C}$ of $1 - k/2$ and a holomorphic family of functions $(f_s)_{s \in U}$ on $\mathbb{H}$, where f_s is a weak Maaß form of weight k of eigenvalue $s(1-s) + (k^2 - 2k)/4$, and $f_{1-k/2} = f$.*

4.3 Derivatives of spectral families: constant weight Differentials of spectral families are one tool to provide higher depth Maaß forms. We briefly revisit the case of fixed weight for comparison with the later approach that allows us to change the weight. Prototypical f_s that fit into the following setup are provided by Eisenstein series and Poincaré series.

Lemma 4.6. *Let k be an integer, and $U \subseteq \mathbb{C}$ be an open neighborhood of 0. Given functions $f_s : \mathbb{H} \longrightarrow V$ for a complex vector space V that are smooth in s and τ, we assume that for all $\tau \in \mathbb{H}$ and $s \in U$ we have*

$$\Delta_k f_s = s(1 - k - s) f_s.$$

Then with

$$f^{(d)} := \big(\partial_s^d f_s \big)_{s=0}$$

we have

$$\Delta_k f^{(d)} = d(1-k) f^{(d-1)} - d(d-1) f^{(d-2)}. \tag{4.15}$$

Proof. Since $f_s(\tau)$ is smooth in s and τ, then ∂_s and Δ_k, which is a differential operator with respect to τ, intertwine. For simplicity, we set $\partial_s^d f_s = 0$ if $d < 0$. By the product rule, we have

$$\begin{aligned}\Delta_k \partial_s^d f_s &= \partial_s^d \Delta_k f_s = \partial_s^d s(1-k-s) f_s \\ &= s(1-k-s)\partial_s^d f_s + d(1-k-2s)\partial_s^{d-1} f_s - d(d-1)\partial_s^{d-2} f_s.\end{aligned}$$

Inserting $s = 0$ yields the statement. ∎

We can employ this lemma to produce polyharmonic Maaß forms from spectral families. More specifically, we obtain a preimage of $f^{(0)}$ under Δ_k^d. Note that the case $k = 1$ requires special treatment.

Corollary 4.7. *Let k and $f^{(d)}$ be as in Lemma 4.6. Then*

$$\Delta_k^d \frac{1}{d!(1-k)^d} f^{(d)} = f^{(0)}, \quad \text{if } k \neq 1; \qquad \Delta_k^d \frac{(-1)^d}{(2d)!} f^{(2d)} = f^{(0)}, \quad \text{if } k = 1.$$

4.4 Altering weights of harmonic Maaß forms The construction of polyharmonic Maaß forms in Section 4.3 preserves the weight k of f_s. We next derive a formalism that allows us to produce polyharmonic Maaß forms of weight $k \pm m$ for $m \in \mathbb{Z}_{\geq 0}$ by extending the approach of Section 4.3.

Lemma 4.8. *Let k and $m \geq 0$ be integers, and consider a smooth function $f : \mathbb{H} \longrightarrow V$ for a complex vector space V. We assume that $\Delta_k f = \alpha f$ for some $\alpha \in \mathbb{C}$.*

Writing for integers $0 \leq r \leq m$

$$f_{\mathrm{L},r} := \mathfrak{e}_{r,m-r} \mathrm{L}_k^{m-r} f \quad \text{and} \quad f_{\mathrm{R},r} := \mathfrak{e}_{r,m-r} \mathrm{R}_k^r f$$

and $f_{\mathrm{L},r} = f_{\mathrm{R},r} = 0$ for $r < 0$ and $r > m$, we have

$$\begin{aligned}\Delta_{k-m} f_{\mathrm{L},r} &= \big(\alpha + (m-r)(m-2r-k)\big) f_{\mathrm{L},r} \\ &\quad - f_{\mathrm{L},r-1} + (r+1)(m-r)\big(\alpha + (m-r-1)(m-r-k)\big) f_{\mathrm{L},r+1}, \\ \Delta_{k+m} f_{\mathrm{R},r} &= \big(\alpha - r(m-2r-k) - m\big) f_{\mathrm{R},r} \\ &\quad + \big(\alpha + r(r-1+k)\big) f_{\mathrm{R},r-1} - (r+1)(m-r) f_{\mathrm{R},r+1}.\end{aligned}$$

Proof. We will apply (4.4) and simplify the contributions arising from the first two terms. To this end, recall from (3.5) the Laplace eigenvalues $-(r+1)(m-r)$ of $\mathfrak{e}_{r,m-r}$. From (4.1), we find that

$$\begin{aligned}\Delta_{k+2m-2r}\,\mathrm{L}_k^{m-r} f &= \big(\alpha + (m-r)(m-r-k+1)\big)\,\mathrm{L}_k^{m-r} f,\\ \Delta_{k+2r}\,\mathrm{R}_k^{r} f &= \big(\alpha + r(r-1+k)\big)\,\mathrm{R}_k^{r} f.\end{aligned}$$

The images of $\mathfrak{e}_{r,m-r}$ under the Maaß operators are given in (3.4). Finally, we obtain from (4.2) that

$$\begin{aligned}-\mathrm{R}_{k-2m+2r}\,\mathrm{L}_k^{m-r} f &= \big(\alpha - (m-r-1)(m-r-k)\big)\,\mathrm{L}_k^{m-r-1} f,\\ -\mathrm{L}_{k-2r}\,\mathrm{R}_k^{r} f &= \big(\alpha - r(r-1+k)\big)\,\mathrm{R}_k^{r-1} f.\end{aligned}$$

∎

Proposition 4.9. *Given a harmonic weak Maaß form f of weight k and a non-negative integer $m \ge k$, set*

$$f_{\mathrm{L}} = \sum_{r=0}^{\min\{m,m-k\}} \frac{1}{(m-r)!(m-r-k)!}\,\mathfrak{e}_{r,m-r}\mathrm{L}^{m-r} f.$$

If $m < k$, then set

$$f_{\mathrm{L}} = \sum_{r=0}^{m} \frac{1}{(m-r)!(1-k)_{m-r}}\,\mathfrak{e}_{r,m-r}\mathrm{L}^{m-r} f,$$

where $(a)_r = a(a+1)\cdots(a+r-1)$ is the Pochhammer symbol.

We have $\Delta_{k-m} f_{\mathrm{L}} = 0$. Further, if f is not holomorphic then $f_{\mathrm{L}} \ne 0$ and any linear combination of the functions $\mathfrak{e}_{r,m-r}\mathrm{L}^{m-r} f$ that vanishes under Δ_{k-m} is a scalar multiple of f_{L}.

Proof. The case $m = 0$ is vacuous, and we thus can and will assume that m is positive. To shorten notation, we adopt the notation $f_{\mathrm{L},r}$ from Lemma 4.8, write c_r for coefficient of $f_{\mathrm{L},r}$ in the definition of f_{L}, and write c'_r for coefficient of $f_{\mathrm{L},r}$ in $\Delta_{k-m} f_{\mathrm{L}}$. We will use repeatedly that for $r < \min\{m, m-k\}$ if $m \ge k$ and for $0 \le r < m$ if $m < k$ we have

$$c_{r+1} = (m-r)(m-r-k)\,c_r.$$

We apply Lemma 4.8 with $\alpha = 0$ and the recursion equation for c_r to find that for $0 \le r < \min\{m, m-k\}$ if $m \ge k$ and for $0 \le r < m$ if $m < k$ we have

$$\begin{aligned} c_r' &= (m-r)(m-2r-k)c_r - c_{r+1} \\ &\qquad + (r-1+1)(m-r+1)(m-r+1-1)(m-r+1-k)c_{r-1} \\ &= \Big(\frac{m-2r-k}{m-r-k} - 1 + \frac{r}{m-r-k}\Big)\, c_{r+1} = 0. \end{aligned}$$

We need a case distinction to check the remaining coefficients c_r'. If $m = k$, we have

$$c_0' = (m-0)(m-0-k)c_0 = 0 \quad \text{and} \quad c_1' = (0+1)(m-0)(m-0-1)(m-0-k)c_0 = 0.$$

If $m \ne k$, we have

$$c_0' = (m-0)(m-0-k)c_0 - c_1 = (1-1)c_1 = 0$$

by the recursion for c_r. Still assuming that $m \ne k$, we have for $r = \min\{m, m-k\}$ if $m \ge k$ and $r = m$ if $m < k$ that

$$\begin{aligned} c_r' &= (m-r)(m-2r-k)c_r + (r-1+1)(m-r+1)(m-r+1-1)(m-r+1-k)c_{r-1} \\ &= \big((m-r)(m-2r-k) + r(m-r)\big)\, c_r = (m-r-k)c_r = 0. \end{aligned}$$

Finally, we consider $m > k > 0$, in which case we have for $r = m-k+1$

$$c_r' = (r-1+1)(m-r+1)(m-r+1-1)(m-r+1-k)c_{r-1} = 0.$$

This shows that $\Delta_{k-m} f_{\mathrm{L}}$ vanishes.

To prove the second part of the proposition, we consider the vector space

$$\mathscr{F} = \mathrm{span}\,\mathbb{C}\big\{\mathfrak{e}_{r,m-r} \mathrm{L}^{m-r} f \,:\, 0 \le r \le m\big\},$$

which by Lemma 4.8 with $\alpha = 0$ carries an action of Δ_{k-m}. Also from Lemma 4.8 and the contribution of $\Delta_{k-m} f_{\mathrm{L},r}$ to $f_{\mathrm{L},r-1}$ stated there, we see that Δ_{k-m} yields a surjective map from $\mathscr{F}$ to $\mathscr{F} / \mathbb{C}\mathfrak{e}_{m,0} f$. In particular, its kernel has dimension at most 1 and is thus spanned by f_{L}. ∎

Example 4.10. The vector-valued modular form $\mathfrak{e}_{m,0}$ of weight $-m$ from Example 3.4 that appears in case I(a) of Bringmann–Kudla's classification matches the case $k = 0$ for the constant modular form $f = 1$ of Proposition 4.11. Note that $\mathrm{L}^{m-r} 1 = 0$ for $0 \le r < m$.

Proposition 4.11. *Given a harmonic weak Maaß form f of weight k and an integer $m > -k$, set*

$$f_{\mathrm{R}} = \sum_{r=\max\{0,1-k\}}^{m} \frac{1}{(m-r)!(r+k-1)!}\, \mathfrak{e}_{r,m-r} \mathrm{R}_k^r f.$$

If $m \le -k$, set

$$f_{\mathrm{R}} = \sum_{r=0}^{m} \frac{1}{(m-r)!(k)_r}\, \mathfrak{e}_{r,m-r} \mathrm{R}_k^r f,$$

where $(a)_r$ is the Pochhammer symbol as in Proposition 4.9.

We have $\Delta_{k+m} f_{\mathrm{R}} = 0$. Further, if $y^k \overline{f}$ is not holomorphic then $f_{\mathrm{R}} \ne 0$ and any linear combination of the $\mathfrak{e}_{r,m-r} \mathrm{R}_k^r f$ that vanishes under Δ_{k+m} is a scalar multiple of f_{R}.

Proof. The proof is analogous to the one of Proposition 4.9. We write c_r for coefficient of $f_{\mathrm{R},r}$ in the definition of f_{R}, and write c'_r for coefficient of $f_{\mathrm{R},r}$ in $\Delta_{k+m} f_{\mathrm{R}}$, and obtain the relation

$$(r+k)c_{r+1} = (m-r)c_r$$

for the nonzero coefficient of $\mathfrak{e}_{r,m-r} \mathrm{R}_k^r f$ in f_{R}.

Lemma 4.6 with $\alpha = 0$ yields that for $0 < r < \min\{m, 1-k\}$ if $m > -k$ and for $0 < r < m$ if $m \le -k$ we have

$$\begin{aligned} c'_r &= \big(-r(m-2r-k) - m\big)c_r + (r+1)(r+k)c_{r+1} - r(m-r+1)c_{r-1} \\ &= \big(-r(m-2r-k) - m\big)c_r + (r+1)(m-r)c_r - r(r-1+k)c_r = 0. \end{aligned}$$

The special cases for $r = 0$, $r = 1-k$, $r = -k$, and $r = m$ follow the same pattern.

To see that Δ_{k+m} is a linear transformation of rank at least m on

$$\mathscr{F} = \mathrm{span}\,\mathbb{C}\big\{\mathfrak{e}_{r,m-r} \mathrm{R}^r f : 0 \le r \le m\big\},$$

we note that it is surjective onto $\mathscr{F} / \mathbb{C}\mathfrak{e}_{0,m} f$ by inspection of the contribution of $f_{\mathrm{R},r}$ to $f_{\mathrm{R},r+1}$ in Lemma 4.8. ∎

Example 4.12. The vector-valued Eisenstein series of weight $2+m$ in case III(b) of Bringmann–Kudla's classification matches the case $k = 2$ of Proposition 4.11. In (6.10) of [7] they consider

$$\sum_{r=0}^{m} \frac{m!}{(r+1)!(m-r)!}\, \mathfrak{e}_{r,m-r} \mathrm{R}^r E_2, \tag{4.16}$$

where E_2 is the modular Eisenstein series of weight 2 and level 1.

4.5 Derivatives of spectral families: altering weight In preparation to the construction of polyharmonic Maaß forms, we consider the action of the Laplace operator on spectral families.

Lemma 4.13. *Let k and $m \geq 0$ be integers, and $U \subseteq \mathbb{C}$ be an open neighborhood of 0. Given functions $f_s : \mathbb{H} \longrightarrow V$ for a complex vector space V that are smooth in s and τ, we assume that for all $\tau \in \mathbb{H}$ and $s \in U$ we have*

$$\Delta_k f_s = s(1-k-s) f_s.$$

We set

$$f_{\mathrm{L},r}^{(d)} := \Big(\partial_s^d\, \mathfrak{e}_{r,m-r}\, \mathrm{L}_k^{m-r} f_s\Big)_{s=0} \quad \text{and} \quad f_{\mathrm{R},r}^{(d)} := \Big(\partial_s^d\, \mathfrak{e}_{r,m-r}\, \mathrm{R}_k^{r} f_s\Big)_{s=0},$$

and if $d < 0$, $r < 0$, or $r > m$ set $f_{\mathrm{L},r}^{(d)} = f_{\mathrm{R},r}^{(d)} = 0$. Then we have

$$\begin{aligned}
\Delta_{k-m} f_{\mathrm{L},r}^{(d)} &= (m-r)(m-2r-k)\, f_{\mathrm{L},r}^{(d)} \\
&\quad - f_{\mathrm{L},r-1}^{(d)} + (r+1)(m-r)(m-r-1)(m-r-k)\, f_{\mathrm{L},r+1}^{(d)} \\
&\quad + d(1-k)\, f_{\mathrm{L},r}^{(d-1)} + d(r+1)(m-r)(1-k)\, f_{\mathrm{L},r+1}^{(d-1)} \\
&\quad - d(d-1)\, f_{\mathrm{L},r}^{(d-2)} - d(d-1)(r+1)(m-r)\, f_{\mathrm{L},r+1}^{(d-2)}, \\
\Delta_{k+m} f_{\mathrm{R},r}^{(d)} &= -\big(r(m-2r-k)+m\big)\, f_{\mathrm{R},r}^{(d)} \\
&\quad + r(r-1+k)\, f_{\mathrm{R},r-1}^{(d)} - (r+1)(m-r)\, f_{\mathrm{R},r+1}^{(d)} \\
&\quad + d(1-k)\, f_{\mathrm{R},r}^{(d-1)} + d(1-k)\, f_{\mathrm{R},r-1}^{(d-1)} \\
&\quad - d(d-1)\, f_{\mathrm{R},r}^{(d-2)} - d(d-1)\, f_{\mathrm{R},r-1}^{(d-2)}.
\end{aligned}$$

Proof. Since $f_s(\tau)$ is smooth in s and τ, we can intertwine differentials with respect to s and τ. In particular, we can apply Lemma 4.8 to f_s with $\alpha = s(1-k-s)$ to compute

$$\begin{aligned}
\Delta_{k-m}\big(\partial_s^d\, \mathfrak{e}_{r,m-r}\, \mathrm{L}_k^{m-r} f_s\big) &= \partial_s^d\big(\Delta_{k-m}\, \mathfrak{e}_{r,m-r}\, \mathrm{L}_k^{m-r} f_s\big) \quad \text{and} \\
\Delta_{k+m}\big(\partial_s^d\, \mathfrak{e}_{r,m-r}\, \mathrm{R}_k^{r} f_s\big) &= \partial_s^d\big(\Delta_{k+m}\, \mathfrak{e}_{r,m-r}\, \mathrm{R}_k^{r} f_s\big)
\end{aligned}$$

The result follows after simplifying the resulting right hand side in Lemma 4.8 and setting $s = 0$. ■

We are now ready to produce polyharmonic Maaß forms from spectral families.

Theorem 4.14. *Let $U \subseteq \mathbb{C}$ be an open neighborhood of 0, and k and $m \ge 0$ be integers. Given functions $f_s : \mathbb{H} \longrightarrow V$ for a complex vector space V that are smooth in s and τ, we assume that for all $\tau \in \mathbb{H}$ and $s \in U$, we have*

$$\Delta_k f_s = s(1-k-s) f_s.$$

For integers $0 \le r \le m$ and $d \ge 0$, we write

$$f_{\mathrm{L},r}^{(d)} := \Big(\partial_s^d \, \mathfrak{e}_{r,m-r} \, \mathrm{L}_k^{m-r} f_s \Big)_{s=0} \quad \text{and} \quad f_{\mathrm{R},r}^{(d)} := \Big(\partial_s^d \, \mathfrak{e}_{r,m-r} \, \mathrm{R}_k^{r} f_s \Big)_{s=0},$$

and let f_{L} and f_{R} be as in Propositions 4.9 and 4.11.

Assume that $k \le 0$ or $k-m > 1$. Given an integer $d \ge 0$ there are functions

$$f_{\mathrm{L}}^{(d)} \in \mathrm{span}\, \mathbb{C} \big\{ f_{\mathrm{L},r}^{(d-t)} : 0 \le r \le m, 0 \le t \le d \big\}$$

with

$$\Delta_{k-m}^{d} f_{\mathrm{L}}^{(d)} = f_{\mathrm{L}}.$$

Assume that $k > 1$ or $k+m < 1$. Given an integer $d \ge 0$ there are functions

$$f_{\mathrm{R}}^{(d)} \in \mathrm{span}\, \mathbb{C} \big\{ f_{\mathrm{R},r}^{(d-t)} : 0 \le r \le m, 0 \le t \le d \big\}$$

with

$$\Delta_{k+m}^{d} f_{\mathrm{R}}^{(d)} = f_{\mathrm{R}}.$$

Proof. The case $m = 0$ follows from Corollary 4.7. We therefore assume that m is positive, and proceed with the following strategy. The cases of $f_{\mathrm{L}}^{(d)}$ and $f_{\mathrm{R}}^{(d)}$ differ only in the last step. We start by identifying the $\mathbb{C}[\Delta_k]$-modules generated by $f_{\mathrm{L},r}^{(d)}$, $0 \le r \le m$, with quotients of $\mathbb{C}[T] \otimes V$ for a fixed complex vector space V of dimension $m+1$, which is independent of d. While this is merely a matter of renormalization, it allows us to relate the functions $f_{\mathrm{L}}^{(d)}$ for varying d to each other. Specifically, it enables us to reformulate the statement of the theorem in terms of generalized eigenvectors. Using the grading with respect to powers of T, we can then reduce our considerations to a problem concerning the interplay of three endomorphisms of V. We solve it by inspection of ranks and images via a coordinate projection. The ideas for $f_{\mathrm{R}}^{(d)}$ remain the same, but in the last step we use an alternating trace instead of a coordinate projection.

We consider the case of f_{L}. We let $V = \mathbb{C}^{m+1}$ with basis $v_0, \ldots, v_m$ and set

$$W_d := \big(\mathbb{C}[T]/T^{d+1}\big) \otimes V.$$

These are modules for $R = \mathbb{C}[T] \otimes \mathrm{End}(V)$ with R-module homomorphisms

$$W_d \hookrightarrow W_{d+1},\, w \longmapsto Tw \quad \text{and} \quad W_{d+1} \twoheadrightarrow W_d,\, w \longmapsto w \pmod{T^{d+1}}.$$

We identify V with W_0.

The vector space isomorphisms

$$\phi_d : W_d \longrightarrow \mathrm{span}\,\mathbb{C}\big\{f_{\mathrm{L},r}^{(d-t)} : 0 \le r \le m, 0 \le t \le d\big\},\, T^t v_r \longmapsto \frac{1}{(d-t)!}\, f_{\mathrm{L},r}^{(d-t)}$$

allow us to reformulate the formula for Δ_{k-m} in Lemma 4.13 as an endomorphism of W_d. Specifically, we have the pullback

$$\phi_d^* \Delta_{k-m} = A + TB + T^2 C$$

with linear transformations $A, B, C \in \mathrm{End}(V)$ defined by

$$\begin{aligned}
A v_r &= (m-r)(m-2r-k)\, v_r \;-\; v_{r-1} \;+\; (r+1)(m-r)(m-r-1)(m-r-k)\, v_{r+1},\\
B v_r &= (1-k)\, v_r \;+\; (r+1)(m-r)(1-k)\, v_{r+1},\\
C v_r &= -\, v_r \;-\; (r+1)(m-r)\, v_{r+1},
\end{aligned}$$

where we set $v_{-1} = v_{m+1} = 0$ to simplify notation. Since d does not appear in these equations, we conclude that these pullbacks lift to an element Δ of R. We consider Δ as an element of $\mathrm{End}(W_d)$ for any d, depending on the context, and observe it commutes with $W_d \hookrightarrow W_{d+1}$ and $W_{d+1} \twoheadrightarrow W_d$.

Since we assume that $k \le 0$ or $k > m$, Proposition 4.9, when translated to the current notation via ϕ_0, informs us that the kernel of A is spanned by a vector $w_0 = \sum_r c_r v_r$, $c_r \in \mathbb{C}$, that corresponds to f_{L} with $c_m \ne 0$ and $(1-k)c_{m-1} = c_m$. Note that when referring to c_{m-1}, we use our assumption that $m > 0$. We have $k \ne 1$, since $k \le 0$ or $k > m$.

From the formula for Av_r with $r = m-1$ and $r = m$, we infer that the image of A is spanned by $v_0, \ldots, v_{m-1}$. Since $w_0 \in \ker A$ receives a nontrivial contribution from v_m, we conclude that $\ker A = \ker A^2$. In other words, the generalized eigenspace of A associated with the eigenvalue zero has dimension one.

The action of Δ on W_d preserves the filtration $W_d \supset TW_{d-1} \supset \cdots T^d W_0$. From the expression $\Delta = A + TB + T^2 C$, we see that its action on the associated graded space

$$W_d / TW_{d-1} \oplus TW_{d-1}/T^2 W_{d-2} \oplus \cdots \oplus T^d W_0$$

coincides with the action of A. We conclude that zero is an eigenvalue of $\Delta \in \mathrm{End}(W_d)$ with multiplicity $d+1$.

We next show the existence of generalized eigenvectors $w_d \in W_d$ for $\Delta \in \mathrm{End}(W_d)$ of eigenvalue 0 and exact depth d in W_d. Note that the case $d = 0$ follows with $w_0 \in W_0$. We consider the case $d = 1$. It suffices to construct a preimage of $T w_0 \in W_1$ under Δ. To this end, we verify by a calculation that $((m+1-k)-B)w_0$ lies in the image of A. We let v_1 be a preimage and set $w_1 = w_0 + T v_1 \in W_1$. Then we have

$$\Delta(w_0 + T v_1) \equiv (A + TB)(w_0 + T v_1) \equiv T(B w_0 + A v_1) \equiv T(m+1-k) w_0 \pmod{T^2}.$$

The assumption $k \le 0$ or $k - m > 1$ guarantees that the right hand side does not vanish. That is, $w_1 = w_0 + T v_1$ is a generalized eigenvector of exact depth 1. For clarity, we recall that the multiplicity of an eigenvalue is the exponent of the corresponding term in the factorization of the characteristic polynomial, while the exact depth is the exponent in the minimal polynomial minus one.

Now by induction on $d \ge 2$, we assume that the desired vector w_{d-1} exists. We recall that $\Delta \in \mathrm{End}(W_d)$ has eigenvalue 0 with multiplicity $d+1$. We will first show that every $v \in W_d$ with $\Delta^d v = 0$ lies in $T W_{d-1} \subset W_d$. By contraposition, we suppose that there were $v \in W_d \setminus T W_{d-1}$ with $\Delta^d v = 0$. Then $v \pmod{T}$ in $W_d / T W_{d-1}$, i.e., the 0-th coefficient of v with respect to T, is nonzero, but lies in the kernel of Δ^d. Since Δ acts on $W_d / T W_{d-1} \cong W_0$ as A and since $\ker A = \ker A^d$ is spanned by w_0, the 0-th coefficient of v is a nonzero multiple of w_0. That is, we can assume that

$$v = w_0 + \sum_{i=1}^{d} T^i v_i.$$

We next consider $v \pmod{T^2}$ in $W_d / T^2 W_{d-2} \cong W_1$. Recall that the generalized eigenspace of $\Delta \in \mathrm{End}(W_1)$ associated with the eigenvalue 0 is 2-dimensional and spanned by $T w_0$ and w_1. We thus have $v \equiv w_1 + T c w_0 \pmod{T^2}$ for some $c \in \mathbb{C}$, and $\Delta v \equiv T w_0 \pmod{T^2}$.

Since $\Delta^d v = 0$, we see that $\Delta v \in T W_{d-1}$ lies in the kernel of Δ^{d-1}. By the induction hypothesis, there is a generalized eigenvector $w_{d-1} \in W_{d-1}$ with $\Delta^{d-1} w_{d-1} = w_0 \ne 0$. In particular, the kernel of Δ^{d-1} acting on W_{d-1} is contained in $T W_{d-2}$. Since further $T W_{d-1} \cong W_{d-1}$ as $\mathbb{C}[\Delta]$-modules, we conclude that $\Delta v \in T^2 W_{d-2}$. This implies that $\Delta v \equiv 0 \pmod{T^2}$, a contradiction.

We have shown by contradiction that the kernel Δ^d acting on W_d is contained in $T W_{d-1}$ and thus has dimension d. We conclude that there is a generalized eigenvector $w_d \in W_d$ of exact depth $d+1$. Its image under ϕ_d yields $f_{\mathrm{L}}^{(d)}$. This finishes the proof in the case of f_{L} and $k \ne 1$.

In the case of f_{R}, we follow the same pattern with

$$\phi_d : W_d \longrightarrow \operatorname{span}\mathbb{C}\big\{f_{\mathrm{R},r}^{(d-t)} : 0 \le r \le m, 0 \le t \le d\big\},\ T^t v_r \longmapsto \frac{1}{(d-t)!} f_{\mathrm{R},r}^{(d-t)}$$

and $\Delta = A + TB + T^2C \equiv \phi_d^* \Delta_{k+m}$, where

$$\begin{aligned}
A v_r &= -\big(r(m-2r-k)+m\big) v_r \,+\, r(r-1+k) v_{r-1} \,-\, (r+1)(m-r) v_{r+1},\\
B v_r &= (1-k) v_r \,+\, (1-k) v_{r-1},\\
C v_r &= -\, v_r \,-\, v_{r-1}.
\end{aligned}$$

The image of A consists of all vectors $v = \sum c_r v_r$ with vanishing alternating trace

$$\widetilde{\mathrm{tr}}(v) := \sum (-1)^r c_r.$$

Proposition 4.11 provides an element w_0 in the kernel of A. We want to verify that w_0 does not lie in the image of A, that is, $\widetilde{\mathrm{tr}}(w_0) \neq 0$. In the case $k \le -m$, we have $k < 0$ and $\widetilde{\mathrm{tr}}(w_0) \neq 0$ follows from the inspection of the expression for f_{R} in Proposition 4.11, since the coefficients of w_0 with respect to $v_0, \ldots, v_{m+1}$ have alternating sign. We consider the case $k > 1$, which implies $m > -k$. Then we need to check that the following expression does not vanish:

$$\widetilde{\mathrm{tr}}(w_0) = \sum_{r=0}^{m} \frac{(-1)^r}{(m-r)!(r+k-1)!}.$$

We multiply by $(-1)^{k-1}(m+k-1)!$ and shift r by $k-1$ to obtain a tail of the binomial expansion of $0 = (1-1)^{m+k-1}$:

$$\sum_{r=k-1}^{m+k-1} \binom{m+k-1}{r} (-1)^r = -\sum_{r=0}^{k-1} \binom{m+k-1}{r} (-1)^r.$$

If $2m \le m+k-1$, the summands on the left hand side have monotone absolute value, otherwise the ones on the right hand side do. We conclude that the sum is nonzero as desired by grouping successive terms.

To finish the proof, we show the existence of generalized eigenvectors $w_d \in W_d$ as before. We can choose $w_1 = w_0 - T v_1$, where v_1 is a preimage under A of $((1-k)c_0 / \widetilde{\mathrm{tr}}(w_0) - B) w_0$, where $w_0 = \sum c_r v_r$. The inductive proof of existence of w_d for $d \ge 2$ follows the same line as before. ∎

Remark 4.15. The maps $W_{d-1} \hookrightarrow W_d$ and $W_d \twoheadrightarrow W_{d-1}$ can be used to iteratively determine the vectors w_d. More specifically, w_0 is given by Propositions 4.9 and 4.11. Further, for $d \geq 1$ we have $w_d \equiv w_{d-1} \pmod{T^d}$ and Δw_d lies in the span of $T^t w_{d-t}$ for $1 \leq t \leq d$. That is, we can set $w_d = w_{d-1} + T^d v$ for some $v \in V$ that is unique up to scalar multiples of w_0, and determine Av uniquely from

$$\Delta w_{d-1} + T^d Av \in \operatorname{img}\Delta \cap \operatorname{span}\mathbb{C}\{T^t w_{d-t} : 1 \leq t \leq d\}.$$

Note that in this step we use $\operatorname{img} A \cap \ker A = \{0\}$.

5 Modular realizations

We next provide modular realizations for each representation that we found in Section 3.4 by providing the correct input to Corollary 4.7 and Theorem 4.14. Both require a spectral family f_s. We primarily use spectral families that specialize at $s = 0$ to the modular realizations provided by Bringmann–Kudla in the case of $d = 0$. Case IIId, which does not occur for $d = 0$, can be constructed from Case Ib.

We restrict ourselves to congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, but with the exception of Case IIb the construction extends in a straightforward way to all co-compact Fuchsian groups of $\mathrm{SL}_2(\mathbb{R})$ with at least one cusp.

We calculated the examples in this section via an implementation of Remark 4.15 in the computer algebra system Nemo [22]. The code is available on the third named author's homepage. We use the notation

$$\begin{aligned} E^{(d)}_{k,s_0}(\tau) &= \big(\partial_s^d E_k(\tau,s)\big)_{s=s_0}, & E^{(d)}_{k}(\tau) &= E^{(d)}_{k,0}(\tau), \\ P^{(d)}_{k,n,s_0}(\tau) &= \big(\partial_s^d F_{k,n}(\tau, 1-\tfrac{k}{2}-s)\big)_{s=s_0}, & P^{(d)}_{k,n}(\tau) &= P^{(d)}_{k,n,0}(\tau). \end{aligned} \tag{5.1}$$

The primary purpose of our implementation is to determine linear combinations of these functions that fall under Cases Ia–IIId.

The cases in this section are labeled in a compatible way with Bringmann–Kudla; See Table 3.5 for a translation between these labels and the representation theoretic labels that we employed in Section 3.4.

5.1 Case Ia We provide a polyharmonic weak Maaß form f of exact depth d and weight $k < 1$ with $\mathrm{L}_k \Delta_k^d f = 0$ and $\mathrm{R}_k^{1-k} \Delta_k^d f = 0$.

The case of $k = 0$ and $d = 0$ is classical: We have the modular form $f(\tau) = 1$, which vanishes under L_0 and R_0. Bringmann and Kudla extended this to all $k \leq 0$

via a vector-valued construction, which also falls under the scope of Proposition 4.9. Specifically, we have a modular form $\mathfrak{e}_{-k,0}$ of weight $k \le 0$ that vanishes under L_k and R_k^{1-k} by (3.4).

Theorem 4.14 gives the existence of preimages of $\mathfrak{e}_{-k,0}$ under Δ_k^d when setting $f_s = E_0(\,\cdot\,, s)$, $m = -k$, and $k = 0$.

Example 5.1. We obtain a modular realization $f^{(2)}$ of this case in weight -3 and depth 2 using pure Nemo code with $f^{(0)} = \Delta_{-3}^2 f^{(2)}$ given by

$$\begin{aligned}
&\tfrac{1}{72}\,\mathfrak{e}_{0,3}\,\mathrm{L}^3 E_0^{(2)} + \tfrac{1}{8}\,\mathfrak{e}_{1,2}\,\mathrm{L}^2 E_0^{(2)} + \tfrac{1}{2}\,\mathfrak{e}_{2,1}\,\mathrm{L}^1 E_0^{(2)} + \tfrac{1}{2}\,\mathfrak{e}_{3,0}\,E_0^{(2)} \\
&\quad + \tfrac{11}{216}\,\mathfrak{e}_{0,3}\,\mathrm{L}^3 E_0^{(1)} + \tfrac{3}{8}\,\mathfrak{e}_{1,2}\,\mathrm{L}^2 E_0^{(1)} + \mathfrak{e}_{2,1}\,\mathrm{L}^1 E_0^{(1)} \\
&= \tfrac{1}{2}\,\mathfrak{e}_{3,0}\,E_{0,0}^{(2)} + \tfrac{1}{18}\mathfrak{e}_{0,3}\,E_{-6,3}^{(1)} + \tfrac{1}{4}\mathfrak{e}_{1,2}\,E_{-4,2}^{(1)} + \mathfrak{e}_{2,1}\,E_{-2,1}^{(1)} \\
&\quad + \tfrac{5}{27}\,\mathfrak{e}_{0,3}\,E_{-6,3}^{(0)} + \tfrac{5}{8}\mathfrak{e}_{1,2}\,E_{-4,2}^{(0)} + \mathfrak{e}_{2,1}\,E_{-2,1}^{(0)}.
\end{aligned}$$

5.2 Case Ib We provide a polyharmonic weak Maaß form f of exact depth d and weight $k < 1$ with $\mathrm{L}_k \Delta_k^d f = 0$, and $\mathrm{R}_k^{1-k} \Delta_k^d f \ne 0$.

Bringmann–Kudla realized Case Ib for $d = 0$ in terms of weakly holomorphic modular forms (excluding constants in weight $k = 0$). Indeed, a weakly holomorphic modular form f with nonzero principal part behaves as required under L_k and R_k^{1-k}. We can obtain further modular realizations $\mathfrak{e}_{m,0} f$ in depth 0 and weight $k-m$ by Proposition 4.9 for such f and a non-negative integer m. Indeed, we have $\mathrm{L}\,\mathfrak{e}_{m,0} f = \mathfrak{e}_{m,0}\,\mathrm{L} f = 0$ and $\mathrm{R}^{1-k+m}\,\mathfrak{e}_{m,0} f$ can be written as a linear combination $\sum_r c_r \mathfrak{e}_{r,m-r} \mathrm{R}^{1-k-r} f$, $c_r \in \mathbb{C}$, with nonzero c_0. Since we have $\mathfrak{e}_{0,m}\,\mathrm{R}^{1-k} f \ne 0$ by the Bol Identity, we conclude that $\mathrm{R}^{1-k+m}\,\mathfrak{e}_{m,0} f \ne 0$.

Given such a weakly holomorphic modular form f there is a holomorphic family f_s, $s \in \mathbb{C}$, with $f = f_0$ and $\Delta f_s = s(1-s-k)\,f_s$ after substituting $1 - \frac{k}{2} - s$ for s in Proposition 4.5. This family can be explicitly constructed via Poincaré series. The desired modular realization of Case Ib therefore exists for all $k \le 0$ by Corollary 4.7 applied to this family f_s and given k. The vector-valued generalizations $\mathfrak{e}_{m,0} f_s$ give rise to preimages under Δ^d of $\mathfrak{e}_{m,0} f$ by Theorem 4.14 applied to f_s, k, and m.

5.3 Case Ic We provide a polyharmonic weak Maaß form f of exact depth d and weight $k < 1$ with $\mathrm{L}_k \Delta_k^d f \ne 0$, and $\mathrm{R}_k^{1-k} \Delta_k^d f = 0$.

Recall the flipping operator from (3.2). Given a polyharmonic Maaß form of exact depth d and weight k that realizes Case Ib, $\mathrm{F}_k f$ realizes Case Ic in the same depth and weight. The depth of f and $\mathrm{F}_k f$ coincides by the commutator relation in (4.6).

We apply Equations (4.8) and (4.7) to find that

$$\mathrm{L}_k\,\Delta_k^d\,\mathrm{F}_k\,f = \mathrm{L}_k\,\mathrm{F}_k\,\Delta_k^d\,f = \frac{y^{2-k}}{(-k)!}\,\overline{\mathrm{R}^{1-k}\,\Delta_k^d f} \neq 0$$

and likewise $\mathrm{R}_k^{1-k}\,\Delta_k^d\,\mathrm{F}_k\,f = 0$.

5.4 Case Id We provide a polyharmonic weak Maaß form f of exact depth d and weight $k < 1$ with $\mathrm{L}_k\,\Delta_k^d\,f \neq 0$, and $\mathrm{R}_k^{1-k}\,\Delta_k^d\,f \neq 0$.

As in the work of Bringmann–Kudla, we can obtain this form from polyharmonic weak Maaß forms f_{Ib} and f_{Ic} that realize Cases Ib and Ic. Then $f = f_{\mathrm{Ib}} + f_{\mathrm{Ic}}$ is harmonic of depth d and satisfies

$$\begin{aligned} \mathrm{L}_k\Delta_k^d\,f &= \mathrm{L}_k\Delta_k^d\,f_{\mathrm{Ib}} + \mathrm{L}_k\Delta_k^d\,f_{\mathrm{Ic}} = \mathrm{L}_k\Delta_k^d\,f_{\mathrm{Ic}} \neq 0,\\ \mathrm{R}_k\Delta_k^d\,f &= \mathrm{R}_k\Delta_k^d\,f_{\mathrm{Ib}} + \mathrm{R}_k\Delta_k^d\,f_{\mathrm{Ic}} = \mathrm{R}_k\Delta_k^d\,f_{\mathrm{Ib}} \neq 0. \end{aligned}$$

Another construction is given by Eisenstein series. Recall that $E_k(\cdot, 1-k)$ is harmonic. We have $\mathrm{L}\,E_k(\cdot, 1-k) = E_{k-2}(\cdot, -k) \neq 0$ and $\mathrm{R}_k^{1-k}\,E_k(\cdot, 1-k) = (1-k)!E_{2-k} \neq 0$ by (4.10).

For any non-negative integer m, Proposition 4.9 applied to $E_k(\cdot, 1-k)$ yields further realizations in weight $k-m$. If $m \le -k$, then Proposition 4.11 yields yet further realizations in weight $k+m$. Note that this includes the choice $m = -k$, which provides modular realizations in weight 0.

If $k < 0$ we obtain modular realization in weight k and depth d from Corollary 4.7 applied to $E_k(\cdot, 1-k-s)$. The vector-valued realizations in depth 0 and weight $k \pm m$ arising from $E_k(\cdot, 1-k)$ via Propositions 4.9 and 4.11, can be extended to higher depth by virtue of Theorem 4.14.

Remark 5.2. The exceptional role of $k = 0$ in Case Id is connected to the realization of Case Ia via $E_0(\cdot, 0)$.

5.5 Case IIa We provide a polyharmonic weak Maaß form f of exact depth d and weight $k = 1$ with $\mathrm{L}_k\,\Delta_k^d\,f = 0$.

In analogy with Case Ib, weakly holomorphic modular forms provide modular realizations of Case IIa in depth $d = 0$. Corollary 4.7 in conjunction with Proposition 4.5 allows us to extend them to positive depth. More specifically, given a weakly holomorphic modular form f of weight $k = 1$, as in Case Ib Proposition 4.5 yields a spectral family f_s with $f_0 = f$ and $\Delta_1\,f_s = s(1-1-s)\,f_s = -s^2\,f_s$. Note that as opposed to Cases Ia–Id, only even iterated derivatives $\big(\partial_s^{2d-2t}\,f_s\big)_{s=0}$, $0 \le t \le d$, of f_s occur when applying Corollary 4.7 to f_s.

5.6 Case IIb We provide a polyharmonic weak Maaß form f of exact depth d and weight $k = 1$ with $\mathrm{L}_k \Delta_k^d f \neq 0$.

Bringmann–Kudla realized depth 0 of Case IIb via incoherent Eisenstein series associated to prime fundamental discriminants $-D < 0$ and the function

$$\phi_D^-\big(\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big)\big) = -i\sqrt{D}\big(\tfrac{-D}{c}\big), \text{ if } \gcd(D,c) = 1; \quad \phi_D^-\big(\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big)\big) = \big(\tfrac{-D}{a}\big), \text{ otherwise,}$$

where $(-D/c)$ is the quadratic symbol. In the next definition, the normalization of s differs from the one in [7] in order to obtain a spectral family that satisfies the usual eigenvalue equation $\Delta_1 E_D^-(\tau,s) = s(1-1-s)E_D^-(\tau,s) = -s^2 E_D^-(\tau,s)$:

$$E_D^-(\tau,s) := \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \phi_D^-(\gamma)\, y^s\big|_1 \gamma.$$

Since $E_D^-(\tau,s)$ vanishes at $s = 0$, the next definition features the exponent $d+1$ as opposed to d. We set

$$E_D^{-(d)}(\tau) := \big(\partial_s^{d+1} E_D^-(\tau)\big)_{s=0}.$$

A modular realization for $d = 0$ is given by $E_D^{-(0)}(\tau)$. We can apply Corollary 4.7 to the spectral family $E_D^-(\cdot,s)/s$ in weight $k = 1$ to obtain modular realizations in positive depth d from linear combinations of $E_D^{-(2d-2t)}$ for $0 \le t \le d$.

5.7 Case IIIa We provide a polyharmonic weak Maaß form f of exact depth d and weight $k > 1$ with $\mathrm{L}_k^k \Delta_k^{d-1} f \neq 0$, if $d > 0$, and $\mathrm{L}_k \Delta_k^d f = 0$.

A modular realization in depth 0 is provided by holomorphic modular forms. For general d, let f be a modular realization of Case Id in depth $d > 0$ and weight $2-k$. We claim that then $\mathrm{R}_{2-k}^{k-1} f$ is a modular realization of Case IIIa in depth d and weight k. Indeed, by (4.1) and (4.2) we have

$$\begin{aligned} \mathrm{L}_k \Delta_k^d \mathrm{R}_{2-k}^{k-1} f &= \mathrm{L}_k \mathrm{R}_{2-k}^{k-1} \Delta_{2-k}^d f = -\mathrm{R}_{2-k}^{k-2} \Delta_{2-k}^{d+1} f = 0, \\ \mathrm{L}_k^k \Delta_k^{d-1} \mathrm{R}_{2-k}^{k-1} f &= \mathrm{L}_k^k \mathrm{R}_{2-k}^{k-1} \Delta_{2-k}^{d-1} f = (k-1)!(k-2)!\,\mathrm{L}_{2-k} \Delta_{2-k}^d f \neq 0. \end{aligned}$$

5.8 Case IIIb We provide a polyharmonic weak Maaß form f of exact depth d and weight $k > 1$ with $\mathrm{L}_k \Delta^d f \neq 0$ and $\mathrm{L}_k^k \Delta_k^d f = 0$.

A modular realization in depth 0 and weight 2 is given by the weight-2 Eisenstein series E_2. We have $\mathrm{L} E_2 = \frac{3}{\pi} \neq 0$ and $\mathrm{L}^2 E_2 = 0$ as required. For any positive integer m

Proposition 4.11 applied to E_2 yields a modular realization in weight $2+m$, which has already appeared in the work of Bringmann–Kudla:

$$\sum_{r=0}^{m} \frac{1}{(m-r)!(r+1)!} \mathfrak{e}_{r,m-r} \mathrm{R}_2^r E_2.$$

We employ Theorem 4.14 to the spectral family $E_2(\cdot\,,s)$ of weight $k=2$ and any non-negative integer m to obtain modular realization of given depth d and weight $2+m$.

5.9 Case IIIc We provide a polyharmonic weak Maaß form f of exact depth d and weight $k>1$ with $\mathrm{L}_k^k \Delta_k^d f \neq 0$.

Bringmann–Kudla utilized Poincaré series to provide a modular realization. They fit, more generally, in the following framework. Let f be a modular realization of Case Ic in depth $d+1$ and weight $2-k$. We claim that then $\mathrm{R}_{2-k}^{k-1} f$ is a modular realization of Case IIIc in depth d and weight k. Similar to the treatment of Case IIIa, we employ (4.1) and (4.2) to find that if $d>0$ then

$$\begin{aligned} \mathrm{L}_k^k \Delta_k^d \mathrm{R}_{2-k}^{k-1} f &= \mathrm{L}_k^k \mathrm{R}_{2-k}^{k-1} \Delta_{2-k}^d f = (k-1)!(k-2)!\, \mathrm{L}_k \Delta_{2-k}^{d+1} f \neq 0, \\ \Delta_k^{d+1} \mathrm{R}_{2-k}^{k-1} f &= \mathrm{R}_{2-k}^{k-1} \Delta_{2-k}^{d+1} f = 0. \end{aligned}$$

5.10 Case IIId We provide a polyharmonic weak Maaß form f of exact depth $d>0$ and weight $k>1$ with $\mathrm{L}^k \Delta^{d-1} f = 0$. Recall that this case does not appear in depth 0 and thus is absent from the classification of Bringmann–Kudla.

Let f be a modular realization of Case Ib in depth $d+1$ and weight $2-k$. We claim that then $\mathrm{R}_{2-k}^{k-1} f$ is a modular realization of Case IIId in depth d and weight k. Repeating the computation of Case IIIc based on (4.1) and (4.2), we confirm that

$$\begin{aligned} \mathrm{L}_k^k \Delta_k^{d-1} \mathrm{R}_{2-k}^{k-1} f &= \mathrm{L}_k^k \mathrm{R}_{2-k}^{k-1} \Delta_{2-k}^{d-1} f = (k-1)!(k-2)!\, \mathrm{L}_{2-k} \Delta_{2-k}^{d} f = 0, \\ \Delta_k^{d} \mathrm{R}_{2-k}^{k-1} f &= \mathrm{R}_{2-k}^{k-1} \Delta_{2-k}^{d} f \neq 0. \end{aligned}$$

Observe that the first equality also implies that $\mathrm{R}_{2-k}^{k-1} f$ vanishes under Δ_k^{d+1}.

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards Applied Mathematics Series, No. 55. For sale by the Superintendent of Documents. U. S. Government Printing Office, Washington, D.C., 1964.

[2] C. Alfes. Formulas for the coefficients of half-integral weight harmonic Maaßforms. *Math. Z.* 277.3-4 (2014).

[3] J. N. Bernstein and S. I. Gelfand. Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. *Compos. Math.* 41 (1980).

[4] V. Bondarenko. *Bundles of semi-chains and their representations.* Preprint no. 88.60, Inst. Math. Acad. Sci. of Ukraine, Kiev. 1988.

[5] V. M. Bondarenko. Representations of bundles of semichained sets, and their applications. *St. Petersbg. Math. J.* 3.5 (1991).

[6] K. Bringmann, A. Folsom, K. Ono, and L. Rolen. *Harmonic Maass forms and mock modular forms: theory and applications.* Vol. 64. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2017.

[7] K. Bringmann and S. Kudla. A classification of harmonic Maass forms. *Math. Ann.* 370.3-4 (2018).

[8] K. Bringmann and K. Ono. The $f(q)$ mock theta function conjecture and partition ranks. *Invent. Math.* 165.2 (2006).

[9] F. Brown. A class of non-holomorphic modular forms III: real analytic cusp forms for $\mathrm{SL}_2(\mathbb{Z})$. *Res. Math. Sci.* 5.3 (2018).

[10] F. Brown. *Multiple modular values and the relative completion of the fundamental group of* $\mathscr{M}_{1,1}$. arXiv:1407.51673. 2017.

[11] J. H. Bruinier, J. Funke, and Ö. Imamoḡlu. Regularized theta liftings and periods of modular functions. *J. Reine Angew. Math.* 703 (2015).

[12] J. H. Bruinier. *Borcherds products on O(2, l) and Chern classes of Heegner divisors.* Vol. 1780. Lecture Notes in Mathematics. Berlin: Springer, 2002.

[13] J. H. Bruinier and J. Funke. On two geometric theta lifts. *Duke Math. J.* 125.1 (2004).

[14] J. H. Bruinier and K. Ono. Algebraic formulas for the coefficients of half-integral weight harmonic weak Maass forms. *Adv. Math.* 246 (2013).

[15] I. Burban and Y. Drozd. Derived categories of nodal algebras. *J. Algebra* 272.1 (2004).

[16] I. Burban and Y. Drozd. *Non-commutative nodal curves and derived tame algebras.* arXiv:1805.05174. 2018.

[17] W. W. Crawley-Boevey. Functorial filtrations. II: Clans and the Gelfand problem. *J. Lond. Math. Soc., II. Ser.* 40.1 (1989).

[18] C. W. Curtis and I. Reiner. *Methods of representation theory with applications to finite groups and orders. Volume 1.* New York etc.: John Wiley & Sons, 1990.

[19] Y. A. Drozd. Modules over hereditary orders. *Mat. Zametki* 29.6 (1981).

[20] W. Duke, Ö. Imamoḡlu, and Á. Tóth. Cycle integrals of the j-function and mock modular forms. *Ann. of Math. (2)* 173.2 (2011).

[21] T. Eguchi, H. Ooguri, and Y. Tachikawa. Notes on the $K3$ surface and the Mathieu group M_{24}. *Exp. Math.* 20.1 (2011).

[22] C. Fieker, W. Hart, T. Hofmann, and F. Johansson. Nemo/Hecke: Computer Algebra and Number Theory Packages for the Julia Programming Language. *Proceedings of the 2017 ACM on International Symposium on Symbolic and Algebraic Computation.* ACM, New York, USA, 2017.

[23] T. Gannon. Much ado about Mathieu. *Adv. Math.* 301 (2016).

[24] I. M. Gel'fand. *The cohomology of infinite dimensional Lie algebras; some questions of integral geometry.* Actes Congr. internat. Math. 1970, 1, 95-111 (1971). 1971.

[25] W. Gnedin. Tame matrix problems in Lie theory and commutative algebra. PhD thesis. University of Cologne, 2016.

[26] M. B. Green and P. Vanhove. The Low-energy expansion of the one loop type II superstring amplitude. *Phys. Rev. D* 61 (2000).

[27] S. S. Kudla, M. Rapoport, and T. Yang. On the derivative of an Eisenstein series of weight one. *Internat. Math. Res. Notices* 7 (1999).

[28] J. C. Lagarias and R. C. Rhoades. Polyharmonic Maass forms for $\mathrm{PSL}(2,\mathbb{Z})$. *Ramanujan J.* 41.1-3 (2016).

[29] T. Matsusaka. Polyharmonic weak Maass forms of higher depth for $\mathrm{SL}_2(\mathbb{Z})$. *Ramanujan J.* 51.1 (2020).

[30] R. Strathausen. Master Thesis, Paderborn University. 2021.

[31] J.-L. Verdier. Sur les intégrales attachées aux formes automorphes (d'après Goro Shimura). *Séminaire Bourbaki, Vol. 6.* Soc. Math. France, Paris, 1995.

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