ON THE ARITHMETIC OF THE DISCRIMINANT FOR CUBIC SURFACES

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Abstract. The 27 lines on a smooth cubic surface over $\mathbb{Q}$ are acted upon by a finite quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We construct explicit examples such that the operation is via the index two subgroup of the maximal possible group. This is the simple group of order 25 920. Our examples are given in pentahedral normal form with rational coefficients. On the corresponding parameter space, we search for rational points, discuss their asymptotic, and construct an accumulating subvariety.

1. Introduction

1.1. Let $\mathscr{S} \subset \mathbf{P}^3$ be a smooth cubic surface over an algebraically closed field. It is well known that there are exactly 27 lines on $\mathscr{S}$. The intersection matrix of these lines is essentially the same for every smooth cubic surface. The group of all permutations of the 27 lines respecting the intersection matrix is isomorphic to the Weyl group $W(E_6)$.

For a smooth cubic surface $S \subset \mathbf{P}^3$ over $\mathbb{Q}$, the 27 lines are, in general, not defined over $\mathbb{Q}$, but over an algebraic field extension L. If L is chosen to be the minimal such field then the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is a subgroup of $W(E_6)$.

1.2. In a previous article [4], we described a strategy how to find explicit examples of cubic surfaces over $\mathbb{Q}$ such that the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is exactly the index two subgroup $D^1W(E_6) \subset W(E_6)$. This is the simple group of order 25 920.

Our approach was as follows. We considered cubic surfaces in pentahedral normal form with rational coefficients. For these, we studied the discriminant Δ. We showed that $\mathrm{Gal}(L/\mathbb{Q})$ is contained in the index two subgroup if and only if $(-3)\Delta$ is a perfect square in $\mathbb{Q}$.

1.3. This leads to a point search on the double covering of $\mathbf{P}^4$ ramified at the degree 32 discriminantal variety. A generalized Cremona transform reduces the degree to eight.

In the present article, we will discuss the asymptotic of the $\mathbb{Q}$-rational points of bounded height on the resulting double covering and construct an accumulating subvariety. A final section is devoted to the problem to what extent this subvariety is unique.

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2. The discriminant and the index two subgroup

The goal of this section is to fix notation and to recall some facts on cubic surfaces and their discriminant.

2.1. One way to write down a cubic surface in explicit form is the so-called *pentahedral normal form.* Denote by $S^{(a_0,\dots,a_4)}$ the cubic surface given in $\mathbf{P}^4$ by the system of equations

$$\begin{aligned} a_0X_0^3 + a_1X_1^3 + a_2X_2^3 + a_3X_3^3 + a_4X_4^3 &= 0\,, \\ X_0 + \quad X_1 + \quad X_2 + \quad X_3 + \quad X_4 &= 0\,. \end{aligned}$$

Remark 2.2. A general cubic surface over an algebraically closed field may be brought into pentahedral normal form over that field. Further, the coefficients are unique up to permutation and scaling. This is a classical result, which was first observed by J. J. Sylvester [9]. A proof is given in [2]. Cubic surfaces in pentahedral normal form with rational coefficients are, however, special to a certain extent.

Definition 2.3. The expression

$$\Delta(S^{(a_0,\dots,a_4)}) :=$$
$$a_0^8 \cdot \ldots \cdot a_4^8 \cdot$$
$$\prod_{i_1,i_2,i_3,i_4\in\{0,1\}} \Big(\frac{1}{\sqrt{a_0}} + (-1)^{i_1}\frac{1}{\sqrt{a_1}} + (-1)^{i_2}\frac{1}{\sqrt{a_2}} + (-1)^{i_3}\frac{1}{\sqrt{a_3}} + (-1)^{i_4}\frac{1}{\sqrt{a_4}}\Big)$$

is called the *discriminant* of the cubic surface $S^{(a_0,\dots,a_4)}$. Instead of $\Delta(S^{(a_0,\dots,a_4)})$, we will usually write $\Delta(a_0,\dots,a_4)$.

Facts 2.4. a) $\Delta \in \mathbb{Q}[a_0,\dots,a_4]$ *is a symmetric polynomial, homogeneous of degree* 32*, and absolutely irreducible.*

b) *The cubic surface* $S^{(a_0,\dots,a_4)}$ *is non-singular if and only if* $\Delta(a_0,\dots,a_4) \neq 0$.

Proof. [4, Lemma 2.5 and Corollary 2.10]. □

Theorem 2.5. *Let* $a_0,\dots,a_4 \in \mathbb{Q}$ *such that* $\Delta(a_0,\dots,a_4) \neq 0$*. Then, the Galois group operating on the 27 lines on* $S^{(a_0,\dots,a_4)}$ *is contained in the index two subgroup* $D^1W(E_6) \subset W(E_6)$ *if and only if* $(-3)\Delta(a_0,\dots,a_4) \in \mathbb{Q}$ *is a perfect square.* □

Remark 2.6. This result was essentially known to H. Burkhardt [1, p. 341] in 1893. Burkhardt gives credit to C. Jordan [6], who was the first to study the automorphism group of the configuration of the 27 lines on a cubic surface. In [4, Theorem 2.12], we give a modern proof.

Proposition 2.7 (The two constraints)**.** *Suppose* $a_0,\dots,a_4 \in \mathbb{Z}$ *are such that* $\gcd(a_0,\dots,a_4) = 1$ *and* $(-3)\Delta(a_0,\dots,a_4) \neq 0$ *is a perfect square in* $\mathbb{Q}$.

a) *Then,* $a_0,\dots,a_4$ *all have the same sign.*

b) *Further, for every prime number* $p \equiv 2 \pmod 3$*, all the p-adic valuations* $\nu_p(a_0),\dots,\nu_p(a_4)$ *are even.*

Proof. This is shown in [4, Proposition 3.3]. □

Fact 2.8. *There is a form* Δ' *homogeneous of degree* 8 *such that*

$$\Delta(a_0,\dots,a_4) = (a_0 \cdot \ldots \cdot a_4)^8 \cdot \Delta'(1/a_0,\dots,1/a_4)\,.$$

Proof. The octic Δ' is given by the formula

$$\Delta'(x_0, \ldots, x_4) := \prod_{i_1,i_2,i_3,i_4 \in \{0,1\}} \left(\sqrt{x_0} + (-1)^{i_1}\sqrt{x_1} + (-1)^{i_2}\sqrt{x_2} + (-1)^{i_3}\sqrt{x_3} + (-1)^{i_4}\sqrt{x_4}\right). \qquad \square$$

Definition 2.9. We will call the birational map ι from $\mathbf{P}^4$ to itself, given by

$$(a_0 : \ldots : a_4) \mapsto (1/a_0 : \ldots : 1/a_4)\,,$$

a *generalized Cremona transform.* Note that the standard Cremona transform of $\mathbf{P}^2$ is $(a_0 : a_1 : a_2) \mapsto (1/a_0 : 1/a_1 : 1/a_2)$.

Lemma 2.10. a) $\Delta' \in \mathbb{Q}[x_0, \ldots, x_4]$ *is a symmetric polynomial, homogeneous of degree eight and absolutely irreducible.*

b) *One has* $\Delta'(0, x_1, \ldots, x_4) = D^2$ *for a symmetric, homogeneous quartic form* $D \in \mathbb{Q}[x_1, \ldots, x_4]$.

Proof. See [4, Lemma 4.5]. $\square$

Remarks 2.11. i) The ramification locus $R :=$ "$\Delta' = 0$" is a rational threefold. The parametrization $\iota\colon \mathbf{P}^3 \to R$ given by

$$\iota\colon (t_0 : \ldots : t_3) \mapsto \left(t_0^2 : t_1^2 : t_2^2 : t_3^2 : (t_0 + \ldots + t_3)^2\right)$$

is a finite birational morphism.

ii) The equation $D = 0$ defines the Roman surface of J. Steiner.

3. Rational points on the discriminantal covering

3.1. A point search.

3.1.1. We are interested in smooth cubic surfaces $S^{(a_0,\ldots,a_4)}$ such that the Galois group operating on the 27 lines is exactly equal to $D^1W(E_6)$.

By Theorem 2.5, this implies that $(a_0 : \ldots : a_4) \in \mathbf{P}^4(\mathbb{Q})$ gives rise to a $\mathbb{Q}$-rational point on the discriminantal covering. Further, according to Fact 2.4.b), $(a_0 : \ldots : a_4)$ is supposed not to lie on the ramification locus.

Finally, if two of the coefficients were the same, say $a_0 = a_1$, then $S^{(a_0,\ldots,a_4)}$ allowed the tritangent plane $x_0+x_1 = 0$, which was defined over $\mathbb{Q}$. Consequently, the order of the group acting on the lines could be at most 1152.

Thus, on the double covering $\pi\colon O \to \mathbf{P}^4_{\mathbb{Q}}$, given by

$$w^2 = (-3)\Delta'(x_0, \ldots, x_4)\,,$$

we searched for rational points such that

i) $w \neq 0$,

ii) the five coordinates $x_0, \ldots, x_4$ are pairwise different from each other.

3.1.2. Surprisingly many solutions have been found. It turned out that there are 4 900 907 essentially different solutions up to a height limit of 3000. Under symmetry, they give rise to 120 solutions each. The smallest ones are $(1 : 3 : 7 : 9 : 12)$, $(1 : 3 : 4 : 7 : 13)$, $(1 : 3 : 7 : 12 : 13)$, and $(3 : 7 : 9 : 12 : 13)$. For a few height limits, we indicate the number of solutions up to that limit in the table below.

Table 1. Numbers of solutions up to various height limits

limit	#	limit	#	limit	#	limit	#
10	0	150	4 659	500	93 680	1250	741 701
25	20	200	10 039	600	140 393	1500	1 111 303
50	209	250	17 429	750	236 403	2000	2 088 752
80	892	300	25 778	800	276 409	2500	3 339 244
100	1 481	400	54 331	1000	460 330	3000	4 900 907

Remark 3.1.3. We used the constraints shown above to optimize the searching algorithm. On one hand, it is sufficient to search for solutions such that $0 < x_0 < x_1 < x_2 < x_3 < x_4$. On the other hand, only 751 of the positive integers up to 3000 fulfill the condition that all prime divisors $p \equiv 2 \pmod 3$ have an even exponent.

3.2. The conjecture of Manin.

3.2.1. Let X be a non-singular (weak) Fano variety over $\mathbb{Q}$. Assume that $X(\mathbb{Q}) \neq \emptyset$. Then, the conjecture of Manin [5] makes the following prediction for the number of $\mathbb{Q}$-rational points on X of bounded anticanonical height.

There exists some $\tau > 0$ such that, for every Zariski open set $X^\circ \subseteq X$ that is sufficiently small but non-empty,

$$\#\{x \in X^\circ(\mathbb{Q}) \mid h_{-K}(x) < B\} \sim \tau B \log^r B$$

for $r := \operatorname{rk}\operatorname{Pic}(X) - 1$ and $B \gg 0$. There is a conjectural description [7] for the constant τ, which we will not use here.

Unfortunately, O is singular. In this situation, one has to consider a resolution $\widetilde{O}$ of singularities and compare heights.

Theorem 3.2.2. i) *The singular locus of O is reducible into ten components. The component $S_{(x_0,x_1)}$ is given by*

$$x_0 - x_1 = 0\,, \quad x_2^2 + x_3^2 + x_4^2 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4 = 0\,.$$

The others are obtained by permuting coordinates.

ii) *Let* $\mathrm{pr}\colon \widetilde{O} \to O$ *be the proper and birational morphism obtained by blowing up the ten singular components.*

a) *Then, $\widetilde{O}$ is non-singular, i.e.,* pr *is a resolution of singularities.*

b) *Further,* $\operatorname{rk}\operatorname{Pic}(\widetilde{O}) = 11$.

c) *The canonical divisor of $\widetilde{O}$ is $K = \mathrm{pr}^* K_O$ for $K_O = -\pi^* H$ and H a hyperplane section of $\mathbf{P}^4$.*

Proof. This is shown in [4, Proposition 5.2 and Theorem 5.3]. □

Remark 3.2.3 (The prediction—Manin's conjecture for the double covering O)**.** Theorem 3.2.2.ii.c) implies

$$h_{-K}(y) = h_{-\mathrm{pr}^* K_O}(y) = h_{-K_O}(\mathrm{pr}(y)) = h_{\mathrm{naive},\mathbf{P}^4}(\pi(\mathrm{pr}(y)))$$

for every $y \in \widetilde{O}(\mathbb{Q})$. Manin's conjecture therefore predicts that, for every sufficiently small, non-empty, Zariski open subset $O^\circ \subseteq O$,

$$\#\{x \in O^\circ(\mathbb{Q}) \mid h_{\mathrm{naive},\mathbf{P}^4}(\pi(x)) < B\} \sim \tau B \log^{10} B\,.$$

The reader might want to compare Table 2 below, where the actual numbers are given for a reasonably chosen Zariski open subset.

Remark 3.2.4. The proof given in [4] actually shows that $\mathrm{Pic}(\widetilde{O}) \cong \mathbb{Z}^{11}$ is a trivial $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module. This implies that there is no Brauer-Manin obstruction present on $\widetilde{O}$.

3.3. **Infinitely many solutions.**

Proposition 3.3.1. *There are infinitely many $\mathbb{Q}$-rational points on O. In fact, over the quadric surface Q in $\mathbf{P}^4$, given by $l = q = 0$ for*

$$
\begin{aligned}
l &:= x_0 + x_1 + x_2 - 3x_3 - 3x_4\,,\\
q &:= x_0^2 + x_1^2 + x_2^2 + 9x_3^2 - x_0x_1 - x_0x_2 - 3x_0x_3 - x_1x_2 - 3x_1x_3 - 3x_2x_3\,,
\end{aligned}
$$

the double covering $\pi\colon O \to \mathbf{P}^4_{\mathbb{Q}}$ splits. In particular, there are one or two $\mathbb{Q}$-rational points above each $\mathbb{Q}$-rational point of Q.

Proof. Modulo $\mathscr{I}_Q$, one has actually

$$(-3)\Delta'(x_0,\ldots,x_4) = [\tfrac{64}{3}(x_0 - x_1)(x_0 - x_2)(x_1 - x_2)(x_3 - x_4)]^2\,. \tag{3.1}$$

□

Remarks 3.3.2. i) The difference of the two octic forms in equation (3.1) consists of 495 monomials. To verify the assertion, one may first use the linear equation to eliminate x_4 and then check that the remaining octic form in $x_0, \ldots, x_3$ is divisible by the quadratic form q.

Actually, a simple Gröbner base calculation reveals the fact that equation (3.1) is true even modulo $\mathscr{I}_Q^2$.

ii) There is another proof for Lemma 3.3.1, which is somehow easier from the computational point of view but less canonical. In fact, Q is parametrized by the birational map $\iota\colon \mathbf{P}^2 \dashrightarrow Q$,

$$
\begin{aligned}
&(t_0 : t_1 : t_2) \mapsto\\
&\quad\big((t_0^2 + t_1^2 + t_2^2 - t_0t_1 - t_0t_2 - t_1t_2) : (t_0^2 + t_1^2 + t_2^2 - t_0t_1 + 2t_0t_2 - t_1t_2) :\\
&\qquad : (t_0^2 + t_1^2 + t_2^2 - t_0t_1 - t_0t_2 + 2t_1t_2) : t_2^2 : (t_0^2 + t_1^2 - t_0t_1)\big),
\end{aligned}
$$

being defined over $\mathbb{Q}$. The locus where ι is undefined does not contain any $\mathbb{Q}$-rational point since the quadratic form $t_0^2 + t_1^2 - t_0t_1$ does not represent zero over $\mathbb{Q}$. A direct calculation shows

$$(-3)\Delta'\big(\iota(t_0, t_1, t_2)\big) = [576t_0t_1(t_0 - t_1)t_2^3(t_0^2 + t_1^2 - t_0t_1 - t_2^2)]^2\,.$$

Here, the factor t_0 corresponds to $(x_0 - x_1)$, t_1 to $(x_0 - x_2)$, $(t_0 - t_1)$ to $(x_1 - x_2)$, and $(t_0^2 + t_1^2 - t_0t_1 - t_2^2)$ to $(x_3 - x_4)$. The factor t_2^3 is somehow artificial. For $t_2 = 0$, the parametrization is constant to $(1 : 1 : 1 : 0 : 1)$.

The parametrization ι is actually constructed in a very naive manner. Start with the point $(1 : 1 : 1 : 0 : 1)$ and determine for which value of $\tau \neq 0$ the point

$$(1 : (1 + \tau t_0) : (1 + \tau t_1) : (\tau t_2/3) : (1 + \tau(t_0 + t_1 - t_2)/3))$$

is contained in the quadric surface Q. Many other parametrizations would serve the same purpose.

Remarks 3.3.3. i) The surface Q is obviously symmetric under permutations of $\{x_0, x_1, x_2\}$. It is symmetric under switch of x_3 and x_4, too. All in all, there are ten mutually different copies of Q.

ii) Q is a smooth quadric surface. The two pencils of lines on Q are defined over $\mathbb{Q}(\sqrt{-3})$ and conjugate to each other.

iii) This implies that $\mathrm{Pic}(Q) = \mathbb{Z}$. The Picard group has two generators as soon as the ground field contains $\mathbb{Q}(\sqrt{-3})$.

For quadrics such as Q, Manin's conjecture is proven. Here, for the canonical divisor, one has $K_Q = -2H$ for H the hyperplane section on Q. Hence, the square of the naive height is an anticanonical height. The number of points of naive height $\leq B$ is therefore asymptotically $\tau_Q B^2$ for some constant τ_Q.

This means that $\pi^{-1}(Q) \subset O$ is an example of a so-called *accumulating subvariety.* The growth of the number of rational points on $\pi^{-1}(Q)$ is faster than predicted for a sufficiently small Zariski open subset of O.

Remark 3.3.4. As the height limit of 3000 is too low, most of the rational points found are actually not contained in $\pi^{-1}(Q)$ or one of its copies. Cf. Table 2 below for the numbers of points on O with those over the copies of Q excluded.

TABLE 2. Numbers of solutions, accumulating subvarieties excluded

limit	#	limit	#	limit	#	limit	#
10	0	150	4 101	500	86 897	1250	699 160
25	12	200	8 989	600	130 723	1500	1 049 502
50	156	250	15 760	750	221 187	2000	1 977 863
80	736	300	23 496	800	258 899	2500	3 166 974
100	1 248	400	50 070	1000	432 737	3000	4 651 857

To visualize the growths of the numbers of solutions, we include the following graphs. Observe that the vertical scale is logarithmic.

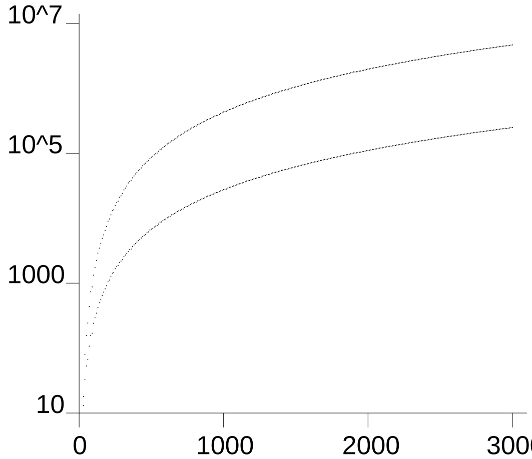

FIGURE 1. Numbers of solutions. The lower graph indicates the accumulating subvarieties, the upper graph their complement.

Remarks 3.3.5. i) The smallest $\mathbb{Q}$-rational points on Q with no two coordinates equal are $(3 : 9 : 12 : 1 : 7)$ and $(1 : 7 : 13 : 3 : 4)$. [4, Algorithm 3.7] shows that, indeed, these two points yield cubic surfaces such that the 27 lines are acted upon by the simple group $D^1W(E_6)$. According to B. L. van der Waerden, this is the generic behaviour on Q.

ii) When testing the cubic surface corresponding to $(3 : 9 : 12 : 1 : 7)$, [4, Algorithm 3.7] works with the primes 19 and 73. Therefore, we have an explicit infinite set of $\mathbb{Q}$-rational points that lead to the group $D^1W(E_6)$. It is given by those points on Q reducing to $(3 : 9 : 12 : 1 : 7)$ modulo both 19 and 73.

3.3.6. Some of the surprising properties of Q are described by the following two facts.

Fact 3.3.7. *Q meets the octic R only within its singular locus. Actually,*

$$Q \cap R \subset S_{(x_0,x_1)} \cup S_{(x_0,x_2)} \cup S_{(x_1,x_2)} \cup S_{(x_3,x_4)}\,.$$

Proof. Suppose $(x_0 : \ldots : x_4) \in Q \cap R$. Then, formula (3.1) implies that $x_0 = x_1$, $x_0 = x_2$, $x_1 = x_1$, or $x_3 = x_4$. The equation $x_0 = x_1$ yields $x_0 = (-x_2 + 3x_3 + 3x_4)/2$. Substituting this into the quadratic relation $q(x_0, \ldots, x_4) = 0$ from the definition of Q shows

$$x_2^2 + x_3^2 + x_4^2 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4 = 0\,.$$

For the relations $x_0 = x_2$, $x_1 = x_2$, and $x_3 = x_4$, the situation is analogous. □

Fact 3.3.8. *Q is tangent to all five coordinate hyperplanes.*

The points of tangency are $(0 : 3 : 3 : 1 : 1)$, $(3 : 0 : 3 : 1 : 1)$, $(3 : 3 : 0 : 1 : 1)$, $(1 : 1 : 1 : 0 : 1)$, *and* $(1 : 1 : 1 : 1 : 0)$. □

3.3.9. The quadric surface Q determines the linear form l uniquely. On the other hand, the quadratic form q is unique only up to a multiple of l. One might have the idea to fix a canonical representative $\underline{q}$ by the requirement that the quadric threefold, given by $\underline{q} = 0$, contain some of the singular components entirely. This is possible to a certain extent.

Fact. a) *There is no quadric threefold in $\mathbf{P}^4$ containing the singular components $S_{(x_0,x_1)}$ and $S_{(x_3,x_4)}$.*

b) *There is, however, a one-dimensional family of quadric threefolds in $\mathbf{P}^4$ containing $S_{(x_0,x_1)}$ and $S_{(x_0,x_2)}$. It is given by $f_t = 0$ for a parameter t and*

$$\begin{aligned} f_t := -tx_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 & - \\ - (1-t)x_0x_1 - (1-t)x_0x_2 + 2x_0x_3 + 2x_0x_4 & + \\ + (1-t)x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4 & = 0\,. \end{aligned}$$

Proof. The statement that a quadric threefold contains $S_{(x_0,x_1)}$ is equivalent to saying it is given by an equation of the form $\underline{q} = 0$ for

$$\underline{q} := a(x_2^2 + x_3^2 + x_4^2 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4) + (a_0x_0 + \ldots + a_4x_4) \cdot (x_0 - x_1)\,.$$

The assumptions of a) yield a linear system of equations that is only trivially solvable. On the other hand, the system of equations for b) leads to a two-dimensional vector space. □

Remark 3.3.10. This family is attached to the rational map $f\colon \mathbf{P}^4 \dashrightarrow \mathbf{P}^1$,

$$\begin{aligned} (x_0 : \ldots : x_4) \mapsto (x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0x_1 - x_0x_2 + 2x_0x_3 + 2x_0x_4 & + \\ + x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4 &) \\ : (x_0^2 - x_0x_1 - x_0x_2 + x_1x_2) & \,. \end{aligned}$$

The map f enjoys the following remarkable properties.

i) Its locus of indeterminacy is equal to $S_{(x_0,x_1)} \cup S_{(x_0,x_2)}$.

ii) The fiber at $t = -1$ is a singular quadric of rank three. The fiber at infinity is reducible into the two hyperplanes $x_0 = x_1$ and $x_0 = x_2$. All other special fibers are smooth.

iii) The special fiber at $t = \frac{1}{3}$ may also be written as

$$4q + (-7x_0 + 5x_1 + 5x_2 + 9x_3 - 3x_4)l = 0\,.$$

In particular, the accumulating subvariety Q is contained within this fiber.

iv) The fiber at $t = \frac{1}{3}$ contains more of the rational points known than any other, even after deleting the accumulating subvarieties. The singular fiber at $t = -1$ follows next.

3.4. **A statistical method.**

3.4.1. We detected the quadric surface Q by a statistical investigation of the rational points found on O.

The theoretical background for this is a supplement to Manin's conjecture. In fact, it is expected that the rational points on $\widetilde{O}$ that lie outside the accumulating subvarieties are equidistributed with respect to a certain measure, the so-called Tamagawa measure. This includes that, for p a prime of good reduction, every $\mathbb{F}_p$-rational point on $\widetilde{O}_p$ is equally likely to occur as the reduction of a $\mathbb{Q}$-rational point.

Thus, in order to detect an accumulating subvarietiy, one might look at the rational points found and search for irregularities in the distribution of their reductions modulo p.

3.4.2. More precisely, our method was as follows. For a fixed good prime p, we counted how many rational points reduced to each of the points modulo p. This simply meant to group the known rational points into residue classes. Thereby, we ignored the points reducing to the ramification locus.

Example 3.4.3. For $p = 53$, most residue classes contained less than 500 points. However, there were exceptional residue classes containing between 1150 and 1250 points. Finally, some residue classes were even more exceptional as they contained more than 2000 points. We illustrate the distribution in the histogram below.

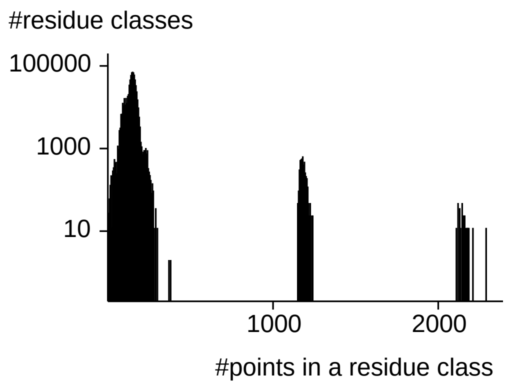

FIGURE 2. Numbers of rational points in the residue classes

When working with a different prime that is approximately of the same size, one ends up with a similar picture.

3.4.4. The $\mathbb{Q}$-rational points on $\widetilde{O}$ are clearly not equidistributed modulo primes such as $p = 53$. Assuming that this phenomenon is explained by an accumulating subvariety, one would expect this subvariety to reduce only to exceptional residue classes. We therefore filtered our data by extracting the rational points that reduce to exceptional residue classes modulo several primes.

This leads to a list of points that are supposed to lie on one or more accumulating subvarieties. To determine a single one of these, we selected an exceptional residue class modulo 53. Among the coordinates of the corresponding points, we searched for linear and quadratic dependencies. The equations for Q, given above, arised in this way.

Remark 3.4.5. Having removed the points lying above Q and its copies, there is no further statistical abnormality to be seen from our data.

Nevertheless, it would be exaggerating to say this might suggest that all accumulating subvarieties have been found. The problem is that an accumulating subvariety could have, say, quadratic growth but a very small constant factor. In such a case, it can not be detected by our statistical method.

4. Accumulating subvarieties

4.1. The goal of this section is to prove that there are no other accumulating subvarieties that are, in a certain sense, similar to Q. Similarity shall include to be a non-degenerate quadric surface, over which the double covering $\pi\colon O \to \mathbf{P}^4_{\mathbb{Q}}$ splits.

In view of the first constraint established above, this implies that the real points on such a quadric surface S are contained in the 16-ant

$$\{ (x_0 : \ldots : x_4) \in \mathbf{P}^4(\mathbb{R}) \mid x_0, \ldots, x_4 \geq 0 \text{ or } x_0, \ldots, x_4 \leq 0 \} .$$

Further, there are strong restrictions for the behaviour at the boundary. By Lemma 2.10.b), we know that Δ' is a perfect square on the coordinate hyperplane H_0, given by $x_0 = 0$. On the other hand, we require $(-3)\Delta'$ to be a perfect square on S.

A way to realize both of these, seemingly contradictory, requirements is to make $S \cap H_0$ a curve of degree two, on which (-3) is the square of a rational function. The only such examples are two lines over $\mathbb{Q}(\sqrt{-3})$ that are conjugate to each other. This implies that S must necessarily be tangent to H_0 and the point of tangency must be a $\mathbb{Q}$-rational point on the ramification locus R.

Theorem 4.2. *Suppose $S \subset \mathbf{P}^4_{\mathbb{Q}}$ is a smooth quadric surface such that the double covering $\pi\colon O \to \mathbf{P}^4_{\mathbb{Q}}$ splits over S. Assume further that S is tangent to the five coordinate hyperplanes $H_0, \ldots, H_4$ and that, for each i, the point of tangency is actually contained in one of the three lines on $H_i \cap R$.*

Then, S is equal to Q or one of its copies under permutation of coordinates.

Remark 4.3. On the Steiner surface $H_0 \cap R$, there are two types of $\mathbb{Q}$-rational points. There are the three lines given by $(0 : r : r : s : s)$ and permutations of the four coordinates to the right. The other $\mathbb{Q}$-rational points are of the form $(0 : t_1^2 : \ldots : t_4^2)$ for $t_1, \ldots, t_4 \in \mathbb{Q}$ such that $t_1 + \ldots + t_4 = 0$.

Lemma 4.4. *Assume S is as in Theorem 4.2. Further, write*

$$P^{(0)} := (0 : x_1^{(0)} : x_2^{(0)} : x_3^{(0)} : x_4^{(0)})$$

for the point of tangency of S with the coordinate hyperplane H_0.
Then, $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)} \neq 0$.

Proof. Assume, to the contrary, that $x_1^{(0)} = 0$. The assumption on the type of the points of tangency made in Theorem 4.2 implies that one more coordinate must vanish. Without restriction, we may assume $P^{(0)} = (0 : 0 : 0 : 1 : 1)$. The tangent plane at $P^{(0)}$ is given by $x_0 = 0$ and another linear relation $C_1x_1 + \ldots + C_4x_4 = 0$. Whatever the coefficients are, there is a tangent vector $(v_0, \ldots, v_4)$ such that $v_1 < 0$ or $v_2 < 0$. The implicit function theorem yields a real point $(x_0 : \ldots : x_3 : 1) \in S(\mathbb{R})$ satisfying $x_1 < 0$ or $x_2 < 0$. This is a contradiction. □

Lemma 4.5. *Assume that the quadric surface S is tangent to the coordinate hyperplanes H_0, H_1, and H_2 in $(0 : x_1^{(0)} : x_2^{(0)} : x_3^{(0)} : x_4^{(0)})$, $(x_0^{(1)} : 0 : x_2^{(1)} : x_3^{(1)} : x_4^{(1)})$, and $(x_0^{(2)} : x_1^{(2)} : 0 : x_3^{(2)} : x_4^{(2)})$, respectively.*
Then,

$$x_1^{(0)}x_2^{(1)}x_0^{(2)} - x_2^{(0)}x_0^{(1)}x_1^{(2)} = 0$$

or

$$\begin{aligned}
x_1^{(0)}x_2^{(1)}x_0^{(2)} + x_2^{(0)}x_0^{(1)}x_1^{(2)} &= 0\,,\\
x_1^{(0)}x_0^{(1)}x_3^{(2)} - x_1^{(0)}x_0^{(2)}x_3^{(1)} - x_0^{(1)}x_1^{(2)}x_3^{(0)} &= 0\,,\\
x_1^{(0)}x_2^{(1)}x_3^{(2)} + x_2^{(0)}x_1^{(2)}x_3^{(1)} - x_2^{(1)}x_1^{(2)}x_3^{(0)} &= 0\,,\\
x_2^{(0)}x_0^{(1)}x_3^{(2)} - x_2^{(0)}x_0^{(2)}x_3^{(1)} + x_2^{(1)}x_0^{(2)}x_3^{(0)} &= 0\,.
\end{aligned}$$

Proof. The linear equation by which S is defined may be written

$$L_0x_0 + L_1x_1 + L_2x_2 + L_3x_3 + L_4x_4 = 0\,. \tag{4.1}$$

We distinguish three cases.

First case. $L_4 \neq 0$.

Then, we may use the linear equation (4.1) to eliminate x_4 from the quadratic equation. Write

$$\begin{aligned}
&Q_0x_0^2 + Q_1x_1^2 + Q_2x_2^2 + Q_3x_3^2 +\\
&\qquad + Q_4x_0x_1 + Q_5x_0x_2 + Q_6x_0x_3 + Q_7x_1x_2 + Q_8x_1x_3 + Q_9x_2x_3 \,=\, 0\,.
\end{aligned}$$

Tangency of H_0 at $(0 : x_1^{(0)} : x_2^{(0)} : x_3^{(0)} : x_4^{(0)})$ means that the two linear forms

$$\begin{aligned}
&(Q_4x_1^{(0)} + Q_5x_2^{(0)} + Q_6x_3^{(0)})x_0 + (2Q_1x_1^{(0)} + Q_7x_2^{(0)} + Q_8x_3^{(0)})x_1 +\\
&\qquad + (2Q_2x_2^{(0)} + Q_7x_1^{(0)} + Q_9x_3^{(0)})x_2 + (2Q_3x_3^{(0)} + Q_8x_1^{(0)} + Q_9x_2^{(0)})x_3
\end{aligned}$$

and

$$L_0x_0 + L_1x_1 + L_2x_2 + L_3x_3 + x_4$$

together generate x_0. This enforces the linear relations

$$\begin{aligned}
2x_1^{(0)}Q_1 + x_2^{(0)}Q_7 + x_3^{(0)}Q_8 &= 0\,,\\
2x_2^{(0)}Q_2 + x_1^{(0)}Q_7 + x_3^{(0)}Q_9 &= 0\,,\\
2x_3^{(0)}Q_3 + x_1^{(0)}Q_8 + x_1^{(0)}Q_9 &= 0\,.
\end{aligned} \tag{4.2}$$

The two other points of tangency yield relations that are completely analogous. Altogether, we find the homogeneous linear system of equations associated with the 9×10-matrix

$$\left(\begin{array}{cccccccccc} 0 & 2x_1^{(0)} & 0 & 0 & 0 & 0 & 0 & x_2^{(0)} & x_3^{(0)} & 0 \\ 0 & 0 & 2x_2^{(0)} & 0 & 0 & 0 & 0 & x_1^{(0)} & 0 & x_3^{(0)} \\ 0 & 0 & 0 & 2x_3^{(0)} & 0 & 0 & 0 & 0 & x_1^{(0)} & x_2^{(0)} \\ 2x_0^{(1)} & 0 & 0 & 0 & 0 & x_2^{(1)} & x_3^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 2x_2^{(1)} & 0 & 0 & x_0^{(1)} & 0 & 0 & 0 & x_3^{(1)} \\ 0 & 0 & 0 & 2x_3^{(1)} & 0 & 0 & x_0^{(1)} & 0 & 0 & x_2^{(1)} \\ 2x_0^{(2)} & 0 & 0 & 0 & x_1^{(2)} & 0 & x_3^{(2)} & 0 & 0 & 0 \\ 0 & 2x_1^{(2)} & 0 & 0 & x_0^{(2)} & 0 & 0 & 0 & x_3^{(2)} & 0 \\ 0 & 0 & 0 & 2x_3^{(2)} & 0 & 0 & x_0^{(2)} & 0 & x_1^{(2)} & 0 \end{array}\right).$$

If this matrix is of rank 9 then the quadratic equation defining S is, up to scaling, determined uniquely. In fact, this case is degenerate. There is a linear form in $x_0, \ldots, x_3$ only, vanishing on the three points given. The unique solution of the system corresponds to the square of this linear form.

Consequently, the rank is at most 8. The ten 9×9-minors must all vanish. These minors are polynomials in $x_0^{(0)}, \ldots, x_3^{(2)}$ having

$$(x_1^{(0)} x_2^{(1)} x_0^{(2)} - x_2^{(0)} x_0^{(1)} x_1^{(2)})$$

as their greatest common divisor. After division by this, we are left with ten sextics. It turns out that they are precisely the squares and pairwise products of the four cubics $x_1^{(0)} x_2^{(1)} x_0^{(2)} + x_2^{(0)} x_0^{(1)} x_1^{(2)}$, $x_1^{(0)} x_0^{(1)} x_3^{(2)} - x_1^{(0)} x_0^{(2)} x_3^{(1)} - x_0^{(1)} x_1^{(2)} x_3^{(0)}$, $x_1^{(0)} x_2^{(1)} x_3^{(2)} + x_2^{(0)} x_1^{(2)} x_3^{(1)} - x_2^{(1)} x_1^{(2)} x_3^{(0)}$, and $x_2^{(0)} x_0^{(1)} x_3^{(2)} - x_2^{(0)} x_0^{(2)} x_3^{(1)} + x_2^{(1)} x_0^{(2)} x_3^{(0)}$.

Second case. $L_4 = 0$ and $L_3 \neq 0$.

As the roles of the third and fourth coordinates may be interchanged, we have, as in the first case, $x_1^{(0)} x_2^{(1)} x_0^{(2)} - x_2^{(0)} x_0^{(1)} x_1^{(2)} = 0$ or

$$x_1^{(0)} x_2^{(1)} x_0^{(2)} + x_2^{(0)} x_0^{(1)} x_1^{(2)} = 0\,.$$

Suppose that the second variant is present. Then, the linear equation (4.1) implies that the vector $(x_3^{(0)}, x_3^{(1)}, x_3^{(2)})^t$ is linearly dependent of $(0, x_0^{(1)}, x_0^{(2)})^t$, $(x_1^{(0)}, 0, x_1^{(2)})^t$, and $(x_2^{(0)}, x_2^{(1)}, 0)^t$. For these vectors instead of $(x_3^{(0)}, x_3^{(1)}, x_3^{(2)})^t$, the three more relations asserted are clearly true.

Third case. $L_3 = L_4 = 0$.

In this situation, we may write the three points of tangency in the form $(0 : L_2 : (-L_1) : x_3^{(0)} : x_4^{(0)})$, $(L_2 : 0 : (-L_0) : x_3^{(1)} : x_4^{(1)})$, and $(L_1 : (-L_0) : 0 : x_3^{(2)} : x_4^{(2)})$. It turns out that the relation

$$x_1^{(0)} x_2^{(1)} x_0^{(2)} + x_2^{(0)} x_0^{(1)} x_1^{(2)} = 0$$

is automatically fulfilled. Further, $L_0, L_1, L_2 \neq 0$. Each of the three equations still to be proven reduces to $L_0 x_3^{(0)} - L_1 x_3^{(1)} + L_2 x_3^{(2)} = 0$.

We may use the linear equation (4.1) to eliminate x_0 from the quadratic equation. Write

$$\begin{aligned} & Q_0 x_1^2 + Q_1 x_2^2 + Q_2 x_3^2 + Q_3 x_4^2 + \\ & \qquad + Q_4 x_1 x_2 + Q_5 x_1 x_3 + Q_6 x_1 x_4 + Q_7 x_2 x_3 + Q_8 x_2 x_4 + Q_9 x_3 x_4 \;=\; 0\,. \end{aligned}$$

Tangency of H_0 at $(0 : L_2 : (-L_1) : x_3^{(0)} : x_4^{(0)})$ yields the linear relations

$$\begin{aligned} L_2(2L_2Q_0 - L_1Q_4 + x_3^{(0)}Q_5 + x_4^{(0)}Q_6) - \\ - L_1(-2L_1Q_1 + L_2Q_4 + x_3^{(0)}Q_7 + x_4^{(0)}Q_8) &= 0\,, \\ 2x_3^{(0)}Q_2 + x_1^{(0)}Q_5 + x_2^{(0)}Q_7 + x_4^{(0)}Q_9 &= 0\,, \\ 2x_4^{(0)}Q_3 + x_1^{(0)}Q_6 + x_2^{(0)}Q_8 + x_3^{(0)}Q_9 &= 0\,. \end{aligned}$$

Tangency of H_1 and H_2 leads to linear relations completely analogous to those given in (4.2). Altogether, we find the homogeneous linear system of equations associated with the 9×10-matrix

$$\begin{pmatrix} 2L_2^2 & 2L_1^2 & 0 & 0 & -2L_1L_2 & x_3^{(0)}L_2 & x_4^{(0)}L_2 & -x_3^{(0)}L_1 & -x_4^{(0)}L_1 & 0 \\ 0 & 0 & 2x_3^{(0)} & 0 & 0 & x_1^{(0)} & 0 & x_2^{(0)} & 0 & x_4^{(0)} \\ 0 & 0 & 0 & 2x_4^{(0)} & 0 & 0 & x_1^{(0)} & 0 & x_2^{(0)} & x_3^{(0)} \\ 0 & -2L_0 & 0 & 0 & 0 & 0 & 0 & x_3^{(1)} & x_4^{(1)} & 0 \\ 0 & 0 & 2x_3^{(1)} & 0 & 0 & 0 & 0 & -L_0 & 0 & x_4^{(1)} \\ 0 & 0 & 0 & 2x_4^{(1)} & 0 & 0 & 0 & 0 & -L_0 & x_3^{(1)} \\ -2L_0 & 0 & 0 & 0 & 0 & x_3^{(2)} & x_4^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 2x_3^{(2)} & 0 & 0 & -L_0 & 0 & 0 & 0 & x_4^{(2)} \\ 0 & 0 & 0 & 2x_4^{(2)} & 0 & 0 & -L_0 & 0 & 0 & x_3^{(2)} \end{pmatrix}.$$

If this matrix is of rank 9 then, again, we have a degenerate case. There is a linear form in $x_1, \ldots, x_4$ only, vanishing on the three points given. The unique solution of the system corresponds to the square of this linear form.

Consequently, all the ten 9×9-minors must vanish. Actually, when deleting the fourth column, the corresponding minor is

$$-16L_0^4L_1L_2(L_0x_3^{(0)} - L_1x_3^{(1)} + L_2x_3^{(2)})^2\,. \qquad \square$$

Remark 4.6 (Interpretation)**.** The relations established in Lemma 4.5 may be interpreted as follows. The coordinates of three points of tangency form a 3×5-matrix

$$\begin{pmatrix} 0 & x_1^{(0)} & x_2^{(0)} & x_3^{(0)} & x_4^{(0)} \\ x_0^{(1)} & 0 & x_2^{(1)} & x_3^{(1)} & x_4^{(1)} \\ x_0^{(2)} & x_1^{(2)} & 0 & x_3^{(2)} & x_4^{(2)} \end{pmatrix}.$$

We may scale such that $x_0^{(1)} = x_1^{(0)}$ and $x_0^{(2)} = x_2^{(0)}$.

i) Then, the leftmost 3×3-block is either symmetric, i.e., $x_1^{(2)} = x_2^{(1)}$, or symmetric up to sign. Then, $x_1^{(2)} = -x_2^{(1)}$.

ii) In the latter case, the column vector $(x_3^{(0)}, x_3^{(1)}, x_3^{(2)})^t$ is a linear combination of the column vectors $(0, x_0^{(1)}, x_0^{(2)})^t$, $(x_1^{(0)}, 0, x_1^{(2)})^t$, and $(x_2^{(0)}, x_2^{(1)}, 0)^t$.

Remarks 4.7. i) In the non-symmetric variant, $(x_4^{(0)}, x_4^{(1)}, x_4^{(2)})^t$ is a linear combination of the column vectors $(0, x_0^{(1)}, x_0^{(2)})^t$, $(x_1^{(0)}, 0, x_1^{(2)})^t$, and $(x_2^{(0)}, x_2^{(1)}, 0)^t$, too. The roles of the third and fourth coordinates may be interchanged.

ii) Actually, in this variant, linear dependence of the three vectors $(0, x_0^{(1)}, x_0^{(2)})^t$, $(x_1^{(0)}, 0, x_1^{(2)})^t$, and $(x_2^{(0)}, x_2^{(1)}, 0)^t$ is a non-trivial condition. Observe, they do not form a base of $\mathbb{R}^3$. In the symmetric variant, an analogous condition would be empty.

Remark 4.8. For each triple consisting of points of tangency of S with a coordinate hyperplane, relations of the same kind must be fulfilled.

Proof of Theorem 4.2. For each of the five points of tangency, we have at least two pairs $\{i, j\} \subset \{0, \dots, 4\}$ such that $x_i = x_j$. There are two cases.

First case. Each of the ten pairs of $\{0, \dots, 4\}$ appears exactly once.

Without restriction, the point of tangency to H_0 is $(0 : 1 : 1 : t : t)$. Again without loss of generality, $(1 : 0 : s : 1 : s)$ is the point of tangency to H_1. The structure of the remaining three points of tangency is then fixed. The five points form a matrix as follows,

$$\begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & s & 1 & s \\ 1 & r & 0 & r & 1 \\ t & q & t & 0 & q \\ t & t & p & p & 0 \end{pmatrix}.$$

Lemma 4.5 implies that $r = s$. Indeed, $r = -s$ would enforce that both $(t, 1, -s)^t$ and $(t, s, 1)^t$ are linearly dependent of $(0, 1, 1)^t$, $(1, 0, -s)^t$, and $(1, s, 0)^t$. This is a contradiction, since $(0, s-1, s+1)^t$ is not in the span of these three.

For the same reason, $p = q$. Further, we have $q = \pm 1$ and $s = \pm t$ such that we end up with four one-parameter families,

$$\begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & t & 1 & t \\ 1 & t & 0 & t & 1 \\ t & 1 & t & 0 & 1 \\ t & t & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & -t & 1 & -t \\ 1 & -t & 0 & -t & 1 \\ t & 1 & t & 0 & 1 \\ t & t & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & t & 1 & t \\ 1 & t & 0 & t & 1 \\ t & -1 & t & 0 & -1 \\ t & t & -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & -t & 1 & -t \\ 1 & -t & 0 & -t & 1 \\ t & -1 & t & 0 & -1 \\ t & t & -1 & -1 & 0 \end{pmatrix}.$$

The linear equation of S requires that the matrices considered are of rank at most 4. However, in the second and third families, the determinants $(t^2 - t - 1)(t^3 + 2t - 1)$ and $(t^2 + t - 1)(t^3 - 2t^2 - 1)$ have no rational zeroes. For the fourth family, we find $(t + 1)(t^2 - t + 1)(t^2 + 3t + 1)$ for the determinant. But, for $t = -1$, we had four equal coordinates in several of the points of tangency. Finally, for the first family, the determinant is $(t + 1)(t^2 - 3t + 1)^2$ and the value $t = -1$ could be possible.

The corresponding data lead to systems of equations that are uniquely solvable up to scaling. The resulting quadric surface is given by $l = q = 0$ for

$$\begin{aligned} l &:= x_0 + x_1 + x_2 + x_3 + x_4 \,, \\ q &:= x_0^2 + x_1^2 - x_2^2 - x_3^2 + 3x_0x_1 + x_0x_2 - x_0x_3 - x_1x_2 + x_1x_3 - 3x_2x_3 \,. \end{aligned}$$

This surface is indeed smooth and tangent to all five coordinate hyperplanes but the double covering $\pi \colon O \to \mathbf{P}^4_{\mathbb{Q}}$ does not split over it.

Second case. One of the ten pairs of $\{0, \dots, 4\}$ appears at least twice.

Without loss of generality, the points of tangency to H_0 and H_1, respectively, are $(0 : 1 : 1 : t : t)$ and $(1 : 0 : 1 : s : s)$. If the point of tangency to H_2 were $(1 : (-1) : 0 : 1 : (-1))$ then, by Lemma 4.5, both $(t, s, 1)^t$ and $(t, s, -1)^t$ had to be linear combinations of $(0, 1, 1)^t$, $(1, 0, -1)^t$, and $(1, 1, 0)^t$. This is a contradiction, since $(0, 0, 2)^t$ is not in the span of these three.

Consequently, the five points of tangency form a matrix as follows,

$$\begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & 1 & s & s \\ 1 & 1 & 0 & r & r \\ t & \pm s & \pm r & 0 & q \\ t & \pm s & \pm r & \pm q & 0 \end{pmatrix}.$$

Assume that one of the "r" or "s" actually carries a minus sign. Without restriction, there is "$-r$" in the fourth line. Then, Lemma 4.5 yields the contradiction that $(t, r, q)^t$ must be a linear combination of $(0, 1, t)^t$, $(1, 0, -r)^t$, and $(t, r, 0)^t$. Further, if there were a "$-q$" in the fifth line then $(1, r, r)^t$ had to be a linear combination of $(0, t, t)^t$, $(t, 0, -q)^t$, and $(t, q, 0)^t$, which is not the case, either.

Finally, in the fourth line, we must have two pairs of equal entries. Without restriction, suppose that $q = t$ and $r = s$. All in all, we find a matrix of the form

$$\begin{pmatrix} 0 & 1 & 1 & t & t \\ 1 & 0 & 1 & s & s \\ 1 & 1 & 0 & s & s \\ t & s & s & 0 & t \\ t & s & s & t & 0 \end{pmatrix}.$$

For the determinant, one calculates $2t^2(4s - t - 1)$. We may conclude that $s = \frac{t+1}{4}$.

For every $t \neq 0$, these data lead to systems of equations that are uniquely solvable up to scaling. The result is the one-parameter family S_t of quadric surfaces given by $l_t = q_t = 0$ for

$$\begin{aligned} l_t &:= (t-1)x_0 - 2tx_1 - 2tx_2 + 2x_3 + 2x_4\,, \\ q_t &:= (t+1)^2x_0^2 + 4(t+1)tx_1^2 + 4(t+1)tx_2^2 + 16x_3^2 - \\ &\qquad - 4(t+1)tx_0x_1 - 4(t+1)tx_0x_2 + 8(t-1)x_0x_3 + \\ &\qquad\qquad + 8(t-1)tx_1x_2 - 16tx_1x_3 - 16tx_2x_3\,. \end{aligned}$$

For each $t \neq 0$, the quadric surface S_t is indeed smooth and tangent to all five coordinate hyperplanes.

In order to check for which values of t the double covering $\pi\colon O \to \mathbf{P}^4_{\mathbb{Q}}$ splits over S_t, we first restrict to the intersection $C_t := S_t \cap$ "$x_1 = x_0 + x_2$". This is a smooth conic for each $t \neq 0$. A parametrization $\iota_t\colon \mathbf{P}^1 \to C_t$ is given by

$$\begin{aligned} &(u:v) \mapsto (16tu^2 : ((t^2+18t+1)u^2 + 8(t+1)tuv + 16t^2v^2) : \\ &\quad : ((t^2{+}2t{+}1)u^2 + 8(t{+}1)tuv + 16t^2v^2) : ((t^2{+}2t{+}1)tu^2 + 8(t{-}1)t^2uv + 16t^3v^2) : \\ &\qquad\qquad : ((t^2+10t+9)tu^2 + 8(t+3)t^2uv + 16t^3v^2))\,. \end{aligned}$$

The binary form $(-3)\Delta'(\iota_t(u,v))$ of degree 16 factors into $u^6((t+1)u + 4tv)^4$ and a form of degree six that is irreducible for general t. We ask for the values of t, for which this sextic is a perfect square. According to `magma`, its discriminant is equal to

$$C(t-3)(t-1)^6(3t-1)^6t^{83}(t^2+8t-1)^4(19t^3-82t^2+59t-16)^2$$

for C a 103-digit integer. Over S_1, the double covering $\pi\colon O \to \mathbf{P}^4_{\mathbb{Q}}$ does not split. The cases $t = 3$ and $t = \frac{1}{3}$ both yield the accumulating subvariety Q studied in subsection 3.3. They are equivalent to each other under the permutation $(0)(13)(24)$ of coordinates. $\square$

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