Rational points on diagonal quartic surfaces

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Abstract

We searched up to height 10^7 for rational points on diagonal quartic surfaces. The computations fill several gaps in earlier lists computed by Pinch, Swinnerton-Dyer, and Bright.

1 Introduction

The set of rational points on a variety is one of the central objects in arithmetic geometry. For some classes of varieties, one has at least a precise idea how it should look like.

In the case of Fano varieties, many rational points are expected. This expectation is described by the famous conjecture of Manin [FMT]. The case of a variety of general type is described by the Lang conjecture. It claims that the Zariski-closure of the set of rational points has strictly smaller dimension than the variety considerd. Both conjectures are proven only in a few special cases. But in the intermediate case (i.e., varieties with are neither Fano nor of general type) it is not clear how a general conjecture should look like.

In this note, we inspect diagonal quartic surfaces. These are special K3 surfaces and form one of the most famous examples for varieties of intermediate type. More precisely, we focus on surfaces of the form

$$ax^4 + by^4 = cz^4 + dw^4$$

with coefficients $a, b, c, d \in \mathbb{Z}$ and $1 \leq a, b, c, d \leq 15$. We test local solvability and search for rational points.

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1.1. Remark. —— The only known technique to prove that there are no rational points on a K3 surface which has local points everywere is given by the Brauer-Manin obstruction. The algebraic part of the obstruction was intensively studied by Martin Bright in his PhD thesis [Br1]. As explained in [Br2], for many diagonal quartic surfaces the algebraic and the transcendental Brauer-Manin obstruction does not imply the absence of rational points. See [ISZ] for details on the transcendental part.

2 Point search algorithms

The main idea to search for points of absolute height at most B on varieties of the form $f(x, y) = g(z, w)$ is as follows.

Compute the two sets $\{f(x, y) \mid |x|, |y| \leq B\}$ and $\{g(z, w) \mid |z|, |w| \leq B\}$. Each solution of the equation leads to an element in the intersection. On the other hand, one can find the solutions when one knows the intersection. If one can handle sets fast enough, an $O(B^2)$-algorithm results.

But as these sets tend to be very big, a more sophisticated approach has to be used. One way is given in [Be] for functions f and g being sums of two univariate functions. Then, one can enumerate the two sets above in sorted form. From this, the intersection can be formed easily.

A second approach is given in [EJ1] and [EJ2]. There, the sets are implemented using hash tables. To reduce the size of the sets, a page prime p_p is introduced. Then, the p_p independent problems $f(x, y) = g(z, w)$ and $f(x, y) \equiv a \pmod{p_p}$ are solved for each $a \in [0, p_p - 1]$. Assuming equidistribution, this reduced the size of the sets approximately by a factor of p_p.

As we focus on diagonal quartic surfaces, some additional optimizations can be done. First, one can restrict to non-negative values of x, y, z, w. In about one half of the cases, one can find at least one variable which must be divisible by 5. Further, in many cases, the parity of some or all variables for a primitive solution can be determined.

Usually, many other modules lead to congruences with could be used for a speed up if one knows how to handle them fast on a computer. See [EJ3] for an analysis in a particular example.

Details of the program written

The point search was done using the hashing approach. In an initialization step, congruences modulo 5 and powers of 2 were checked to get congruence conditions for primitive solutions. The page prime 500083 was chosen and the hash table had 134217728 entries. To speed up the modular arithmetic, a table containing fourth roots and multiplicative inverse elements modulo the page prime was build up in the initialization part. Note that the page prime is congruent to 3 modulo 4 and, thus, the fourth root is unique up to sign.

To avoid multiprecision computations, the computations were done modulo 2^{64}. We found less than 100000 coincidences modulo the page prime and modulo 2^{64}. Only these were checked by multiprecision computations.

The running time depends highly on the congruences found. Searching on one surface for points up to height 10^7 took between 12 and 86 days of CPU time on a 2.27GHz Xeon processor. In total, 13 years of CPU time were used.

3 Results

In total, there are 7194 quadruples (a, b, c, d) with $a, b, c, d \in \{1, \ldots, 15\}$, $a \leq b$, $a \leq c$, and $c \leq d$ and $\gcd(a, b, c, d) = 1$. Testing for local solvability excludes 3904 of the corresponding equations $ax^4 + by^4 = cz^4 + dw^4$.

A point search with height-bound 10 solves 3009 cases. Increasing the bound to 100 leads to solutions for 52 of the 281 remaining equations. Further, 31 equations have a first solution of height at most 1000. The remaining 198 equations are the entries of the list [E2]. In 21 cases, a solution with height between 10^3 and 10^4 was found by Martin Bright.

In 18 cases a new solution with height between 10^4 and 10^5 was found. Further, in 14 cases, there is a first solution with height between 10^5 and 10^6. Finally, in 15 cases a first solution with height between 10^6 and 10^7 was detected. In 130 cases, still, no solution is known. A list of them is available at `http://www.staff.uni-bayreuth.de/~btm216/pinch_list_cases_2010.txt`.

As several equations have a first solution with height above 10^6, one can not expect the unsolved examples to be unsolvable. However, $x^4 + y^4 = 6z^4 + 12w^4$ is known to be unsolvable. See [Br1] for details. Note that this surface is isomorphic to $3x^4 + 6y^4 = 8z^4 + 8w^4$. This is the only pair of isomorphic surfaces in the list.

It would be nice if one could make a complete list of solvable and unsolvable cases as done in [CKS] for diagonal cubic surfaces.

a	b	c	d	x	y	z	w	ρ
1	15	7	11	2903019	391311	1780640	549424	1
2	10	7	11	5742991	2277664	4262801	1865875	1
4	11	7	13	873483	1115876	1281143	448499	1
4	12	11	14	3902789	1356045	1015370	2875318	1
4	5	11	14	394427	1355547	1112545	308333	1
5	11	6	7	1545359	3316097	187414	3732530	1
7	9	11	13	3094925	7817089	6049224	6224852	1
2	9	12	15	3625719	1832215	1639331	2213957	1
2	11	7	9	2957980	1748992	468557	2308737	1
5	14	7	9	1943732	493862	984595	1643257	1
4	4	11	13	1668661	1272265	324881	1335627	1
2	3	8	11	1216988	924293	384555	873425	1
2	8	5	11	1315404	988742	1272177	470035	1
1	8	4	13	3730667	1735542	2189289	1913815	1
3	7	12	14	1116485	269121	345539	754095	2

Table 1: Surfaces with smallest solutions of height above 10^6 found.

3.1. Remark. —— Some people tend to believe that the arithmetic Picard rank ρ of a K3-surface has a strong influence on the set of rational points. We have no unsolved case with Picard rank greater than 2. The unsolved cases with rank equal to 2 are $[1, 1, 6, 12]$, $[2, 4, 9, 9]$, $[2, 4, 11, 11]$, $[2, 9, 6, 12]$, $[3, 6, 8, 8]$, $[3, 6, 11, 11]$, $[4, 9, 8, 8]$, and $[6, 12, 11, 11]$. On the other hand, the table above contains one equation with Picard rank 2 and a smallest solution of height 1116485.

Comparing the rank 1 and the rank 2 cases in the sample, one does not find a great difference for the proportion of unknown cases.

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