# Estimates for Tamagawa numbers of diagonal cubic surfaces

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**Abstract**

For diagonal cubic surfaces, we give an upper bound for E. Peyre's Tamagawa type number in terms of the coefficients of the defining equation.

# 1 Introduction

**1.1.** — A conjecture, due to Yu. I. Manin, asserts that the number of $\mathbb{Q}$-rational points of anticanonical height $< B$ on a Fano variety $S$ is asymptotically equal to $\tau B \log^{\operatorname{rk}\operatorname{Pic}(S)-1} B$, for $B \to \infty$. Further, the coefficient $\tau \in \mathbb{R}$ is conjectured to be the Tamagawa-type number $\tau(S)$ introduced by E. Peyre in [Pe]. In the particular case of a cubic surface, the anticanonical height is the same as the naive height.

**1.2. *E. Peyre's constant.*** — E. Peyre's Tamagawa-type number is defined in [PT, Definition 2.4] as

$$\tau(S) := \alpha(S)\cdot\beta(S)\cdot\lim_{s\to 1}(s-1)^t L(s,\chi_{\operatorname{Pic}(S_{\overline{\mathbb{Q}}})})\cdot\tau_H\big(S(\mathbb{A}_\mathbb{Q})^{\operatorname{Br}}\big)$$

for $t = \operatorname{rk}\operatorname{Pic}(S)$.

Here, the factor $\beta(S)$ is simply defined as

$$\beta(S) := \#H^1\big(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\operatorname{Pic}(S_{\overline{\mathbb{Q}}})\big).$$

$\alpha(S)$ is given as follows [Pe, Définition 2.4]. Let $\Lambda_{\text{eff}}(S) \subset \operatorname{Pic}(S)\otimes_\mathbb{Z}\mathbb{R}$ be the cone generated by the effective divisors. Identify $\operatorname{Pic}(S)\otimes_\mathbb{Z}\mathbb{R}$ with $\mathbb{R}^t$ via a mapping induced by an isomorphism $\operatorname{Pic}(S)\xrightarrow{\cong}\mathbb{Z}^t$. Consider the dual cone $\Lambda_{\text{eff}}^\vee(S)\subset(\mathbb{R}^t)^\vee$. Then,

$$\alpha(S) := t\cdot\operatorname{vol}\{\,x\in\Lambda_{\text{eff}}^\vee \mid \langle x,-K\rangle\leq 1\,\}\,.$$

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$L(\,\cdot\,, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})})$ denotes the Artin $L$-function of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation $\mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{C}$ which contains the trivial representation $t$ times as a direct summand. Therefore, $L(s, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})}) = \zeta(s)^t \cdot L(s, \chi_P)$ and

$$\lim_{s \to 1} (s-1)^t L(s, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})}) = L(1, \chi_P)$$

where $\zeta$ denotes the Riemann zeta function and $P$ is a representation which does not contain trivial components. [Mu, Corollary 11.5 and Corollary 11.4] show that $L(s, \chi_P)$ has neither a pole nor a zero at $s = 1$.

Finally, $\tau_H$ is the *Tamagawa measure* on the set $S(\mathbb{A}_{\mathbb{Q}})$ of adelic points on $S$ and $S(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \subseteq S(\mathbb{A}_{\mathbb{Q}})$ denotes the part which is not affected by the Brauer-Manin obstruction.

**1.3.** — As $S$ is projective, we have

$$S(\mathbb{A}_{\mathbb{Q}}) = \prod_{\nu \in \mathrm{Val}(\mathbb{Q})} S(\mathbb{Q}_\nu).$$

$\tau_H$ is defined to be a product measure $\tau_H := \prod_{\nu \in \mathrm{Val}(\mathbb{Q})} \tau_\nu$.

For a prime number $p$, the local measure $\tau_p$ is given as follows. Let $a \in S(\mathbb{Z}/p^k\mathbb{Z})$ and put $\mathfrak{U}_a^{(k)} := \{\, x \in S(\mathbb{Q}_p) \mid x \equiv a \pmod{p^k} \,\}$. Then,

$$\tau_p(\mathfrak{U}_a^{(k)}) := \det(1 - p^{-1} \operatorname{Frob}_p \mid \mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}) \cdot \lim_{m \to \infty} \frac{\#\{\, y \in S(\mathbb{Z}/p^m\mathbb{Z}) \mid y \equiv a \pmod{p^k} \,\}}{p^{m \dim S}}.$$

Here, $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}$ denotes the fixed module under the inertia group.

$\tau_\infty$ is described in [Pe, Lemme 5.4.7]. In the case of a hypersurface of degree $d$ in $\mathbf{P}^n$, defined by the equation $f = 0$, this yields

$$\tau_\infty(U) = \frac{n+1-d}{2} \int\limits_{\substack{CU \\ |x_0|, \ldots, |x_n| \leq 1}} \omega_{\mathrm{Leray}}$$

for $U \subset S(\mathbb{R})$. Here, $\omega_{\mathrm{Leray}}$ is the *Leray measure* on the cone $CS(\mathbb{R})$ associated to the equation $f = 0$. Note that for a cubic surface, one has $n + 1 - d = 1$.

The Leray measure is related to the usual hypersurface measure by the formula $\omega_{\mathrm{Leray}} = \frac{1}{\|\operatorname{grad} f\|} \omega_{\mathrm{hyp}}$. Observe, $\frac{1}{\|\operatorname{grad} f\|}$ is an integrable function on the whole of $CS(\mathbb{R})$ since $\deg f \leq n$.

**1.4. *The main result.*** — For diagonal cubic surfaces, there is an estimate for $\tau(S)$ in terms of the coefficients of the defining equation. More precisely, we will prove the following theorem.

**Theorem.** *Let* $\mathfrak{a} = (a_0, \ldots, a_3) \in (\mathbb{Z}\backslash\{0\})^4$ *be a vector. Denote by* $S^{\mathfrak{a}}$ *the cubic surface in* $\mathbf{P}^3_{\mathbb{Q}}$ *given by* $a_0x_0^3 + \ldots + a_3x_3^3 = 0$*. Then, for each* $\varepsilon > 0$ *there exists a constant* $C(\varepsilon) > 0$ *such that*

$$\frac{1}{\tau(S^{\mathfrak{a}})} \geq C(\varepsilon) \cdot \mathrm{H}_{\text{naive}}\big(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_3}\big)^{\frac{1}{3}-\varepsilon}.$$

**Corollary** (Fundamental finiteness)**.** *For each* $T > 0$*, there are only finitely many diagonal cubic surfaces* $S^{\mathfrak{a}}\colon a_0x_0^3 + \ldots + a_3x_3^3 = 0$ *in* $\mathbf{P}^3_{\mathbb{Q}}$ *such that* $\tau(V^{\mathfrak{a}}) > T$*.*

**1.5. *Application: The height of the smallest point.*** — We denote by $\mathrm{m}(S)$ the smallest naive height of a $\mathbb{Q}$-rational point on $S$, or $\infty$ if there are no $\mathbb{Q}$-rational points. The main result implies that there is an estimate for $\mathrm{m}(S)$ in terms of $\tau(S)$.

**Corollary** (An inefficient search bound)**.** *There exists a monotonically decreasing function* $F\colon (0,\infty) \to [0,\infty)$*, the* search bound*, satisfying the following condition. Let* $S^{\mathfrak{a}}$ *be the cubic surface given by the equation* $a_0x_0^3 + \ldots + a_3x_3^3 = 0$*. Assume* $S^{\mathfrak{a}}(\mathbb{Q}) \neq \emptyset$*. Then,* $S^{\mathfrak{a}}$ *admits a* $\mathbb{Q}$*-rational point of height* $\leq F(\tau(S^{\mathfrak{a}}))$*.*

**Proof.** One may simply put $F(t) := \max\limits_{\substack{\tau(S^{\mathfrak{a}})\geq t \\ S^{\mathfrak{a}}(\mathbb{Q})\neq\emptyset}} \min\limits_{P\in S^{\mathfrak{a}}(\mathbb{Q})} \mathrm{H}_{\text{naive}}(P)$. □

In other words, we have $\mathrm{m}(S^{\mathfrak{a}}) \leq F(\tau(S^{\mathfrak{a}}))$ as soon as $S^{\mathfrak{a}}(\mathbb{Q}) \neq \emptyset$.

**1.6. Remark.** — For diagonal quartic threefolds, these results were known before [EJ]. The case of the classical cubic surfaces is, however, more complicated.

The reason is that quartic threefolds are of geometric Picard rank one. Therefore, the factors $\alpha$ and $\beta$ are always the same and could essentially be ignored. Further, the $L$-factor is equal to 1 as the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation considered is trivial. In the situation of diagonal cubic surfaces, all these factors need to be considered.

# 2 Estimates for Peyre's constant

Consider a general diagonal cubic surface $S^{(a_0,\ldots,a_3)} \subset \mathbf{P}^3_{\mathbb{Q}}$ given by

$$a_0x_0^3 + \ldots + a_3x_3^3 = 0\,.$$

Our goal is to establish the estimate for $\tau^{(a_0,\ldots,a_3)} := \tau(S^{(a_0,\ldots,a_3)})$ formulated in Theorem 1.4. For this, in the subsections below, we will give an individual estimate for each of the factors occurring in the definition of $\tau(S^{(a_0,\ldots,a_3)})$.

## 2.1 Estimates for $\alpha$ and $\beta$

**2.1.** — Recall that on a smooth cubic surface $\mathscr{S}$ over an algebraically closed field, there are exactly 27 lines. For the Picard group, which is isomorphic to $\mathbb{Z}^7$, the classes of these lines form a system of generators.

**2.2. Notation.** — i) The set $\mathscr{L}$ of the 27 lines is equipped with the intersection product $\langle\ ,\ \rangle\colon \mathscr{L}\times\mathscr{L} \to \{-1,0,1\}$. The pair $(\mathscr{L}, \langle\ ,\ \rangle)$ is the same for all smooth cubic surfaces. It is well known [Ma, Theorem 23.9.ii] that the group of permutations of $\mathscr{L}$ respecting $\langle\ ,\ \rangle$ is isomorphic to $W(E_6)$. We fix such an isomorphism.

Denote by $F \subset \mathrm{Div}(\mathscr{S})$ the group generated by the 27 lines and by $F_0 \subset F$ the subgroup of principal divisors. Then, $F$ is equipped with an operation of $W(E_6)$ such that $F_0$ is a $W(E_6)$-submodule. We have $\mathrm{Pic}(\mathscr{S}) \cong F/F_0$.

ii) If $S$ is a smooth cubic surface over $\mathbb{Q}$ then $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates canonically on the set $\mathscr{L}_S$ of the 27 lines on $S_{\overline{\mathbb{Q}}}$. Fix a bijection $i_S\colon \mathscr{L}_S \xrightarrow{\cong} \mathscr{L}$ respecting the intersection pairing. This induces a group homomorphism $\iota_S\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to W(E_6)$. We denote its image by $G \subset W(E_6)$.

**2.3. Lemma.** — *There is a constant $c$ such that, for all smooth cubic surfaces $S$ over $\mathbb{Q}$,*

$$1 \leq \beta(S) \leq c\,.$$

**Proof.** By definition, $\beta(S) = \#H^1\big(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})\big)$. Using the notation just introduced, we may write $H^1\big(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})\big) = H^1(G, F/F_0)$.

Note that this cohomology group is always finite. Indeed, since $G$ is a finite group and $F/F_0$ is a finite $\mathbb{Z}[G]$-module, the description via the standard complex shows it is finitely generated. Further, it is annihilated by $\#G$.

$H^1(G, F/F_0)$ depends only on the subgroup $G \subset W(E_6)$ occurring. For that, there are finitely many possibilities. This implies the claim. □

**2.4. Remark.** — A more precise consideration [Ma, Proposition 31.3] yields a canonical isomorphism $H^1\big(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})\big) \cong \mathrm{Hom}\,\big((NF \cap F_0)/NF_0, \mathbb{Q}/\mathbb{Z}\big)$. Here, $N$ is the norm map under the operation of $G$.

As an application of this, one may inspect the 350 conjugacy classes of subgroups of $W(E_6)$ using `GAP`. The calculations show that the lemma is actually true for $c = 9$.

**2.5. Lemma.** — *There are positive constants $c_1$ and $c_2$ such that, for all smooth cubic surfaces $S$ over $\mathbb{Q}$ satisfying $S(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$,*

$$c_1 \leq \alpha(S) \leq c_2.$$

**Proof.** Again, we claim that $\alpha(S)$ is completely determined by the group $G \subset W(E_6)$. Thus, suppose that we do not have the full information available about what surface $S$ is but are given the group $G$ only.

The assumption $S(\mathbb{A}_\mathbb{Q}) \neq \emptyset$ makes sure that $\text{Pic}(S) \cong \text{Pic}(S_{\overline{\mathbb{Q}}})^G$ [KT, Remark 3.2.ii)]. We may therefore write $\text{Pic}(S) \cong (F/F_0)^G$. The effective cone $\Lambda_{\text{eff}}(S) \subset \text{Pic}(S) \otimes_\mathbb{Z} \mathbb{C} \cong (F/F_0)^G \otimes_\mathbb{Z} \mathbb{C}$ is generated by the symmetrizations of the classes $\ell_1, \ldots, \ell_{27}$ of the 27 lines in $F$. In particular, it is determined by $G$, completely. Further, we have $K = -\frac{1}{9}(\ell_1 + \ldots + \ell_{27})$. These data are sufficient to compute $\alpha(S)$ according to its very definition. □

**2.6. Remark.** —— Here, we do not know the optimal values of $c_1$ and $c_2$ in explicit form. $\alpha(S)$ has not yet been computed in all cases.

## 2.2 An estimate for the *L*-factor

**2.2.1.** —— In the case of the diagonal cubic surface $S^{(a_0,\ldots,a_3)} \subset \mathbf{P}^3_\mathbb{Q}$, given by $a_0x_0^3 + \ldots + a_3x_3^3 = 0$ for $a_0, \ldots, a_3 \in \mathbb{Z}\backslash\{0\}$, the 27 lines on $S^{(a_0,\ldots,a_3)}$ may easily be written down explicitly. Indeed, for each pair $(i,j) \in (\mathbb{Z}/3\mathbb{Z})^2$, the system

$$\begin{aligned}\sqrt[3]{a_0}\, x_0 + \zeta_3^i \sqrt[3]{a_1}\, x_1 &= 0\\ \sqrt[3]{a_2}\, x_2 + \zeta_3^j \sqrt[3]{a_3}\, x_3 &= 0\end{aligned}$$

of equations defines a line on $S^{(a_0,\ldots,a_3)}$. Decomposing the index set $\{0,\ldots,3\}$ differently into two subsets of two elements each yields all the lines. In particular, we see that the 27 lines may be defined over $K = \mathbb{Q}\big(\zeta_3, \sqrt[3]{a_1/a_0}, \sqrt[3]{a_2/a_0}, \sqrt[3]{a_3/a_0}\,\big)$.

**2.2.2.** —— This is an abelian extension of $\mathbb{Q}(\zeta_3)$. Therefore, the irreducible representations of $\text{Gal}(K/\mathbb{Q})$ are at most two-dimensional. Besides the trivial representation, there is the non-trivial Dirichlet character $\lambda$ of $\mathbb{Q}(\zeta_3)/\mathbb{Q}$. The two-dimensional irreducible representations are actually representations of a factor group of the form $\text{Gal}\big(\mathbb{Q}(\zeta_3, \sqrt[3]{a_0^{e_0} \cdot \ldots \cdot a_3^{e_3}})/\mathbb{Q}\big) \cong S_3$ for $e_0, \ldots, e_3 \in \{0,1,2\}$.

**2.2.3. Lemma.** —— *Let $a$ and $b$ be integers different from zero. Then,*

$$\big|\,\text{Disc}\,\big(\mathbb{Q}(\zeta_3, \sqrt[3]{ab^2})/\mathbb{Q}\big)\big| \leq 3^9 a^4 b^4.$$

**Proof.** We have, at first,

$$\begin{aligned}\big|\,\text{Disc}\,\big(\mathbb{Q}(\zeta_3, \sqrt[3]{ab^2})/\mathbb{Q}\big)\big| &\leq \big|\,\text{Disc}\,\big(\mathbb{Q}(\zeta_3)/\mathbb{Q}\big)\big|^3 \cdot \text{Disc}\,\big(\mathbb{Q}(\sqrt[3]{ab^2})/\mathbb{Q}\big)^2\\ &= 27 \cdot \text{Disc}\,\big(\mathbb{Q}(\sqrt[3]{ab^2})/\mathbb{Q}\big)^2.\end{aligned}$$

Further, by [Mc, Chapter 2, Exercise 41], we know $\big|\,\text{Disc}\,\big(\mathbb{Q}(\sqrt[3]{ab^2})/\mathbb{Q}\big)\big| \leq 3^3 a^2 b^2$. This shows $\big|\,\text{Disc}\,\big(\mathbb{Q}(\zeta_3, \sqrt[3]{ab^2})/\mathbb{Q}\big)\big| \leq 3^9 a^4 b^4$. □

**2.2.4. Proposition.** —— *For each $\varepsilon > 0$, there exist positive constants $c_1$ and $c_2$ such that*

$$c_1 \cdot |a_0 \cdot \ldots \cdot a_3|^{-\varepsilon} < \lim_{s \to 1} (s-1)^t L\big(s, \chi_{\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}})}\big) < c_2 \cdot |a_0 \cdot \ldots \cdot a_3|^{\varepsilon}$$

*for all $(a_0, \ldots, a_3) \in (\mathbb{Z} \backslash \{0\})^4$. Here, $t = \operatorname{rk} \operatorname{Pic}(S)$.*

**Proof.** The Galois representation $\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{C}$ contains the trivial representation $t$ times as a direct summand. Therefore,

$$L\big(s, \chi_{\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}})}\big) = \zeta(s)^t \cdot L(s, \chi_P)$$

where $\zeta$ denotes the Riemann zeta function and $P$ is a representation which does not contain trivial components. All we need to show is

$$c_1 \cdot |a_0 \cdot \ldots \cdot a_3|^{-\varepsilon} < L(1, \chi_P) < c_2 \cdot |a_0 \cdot \ldots \cdot a_3|^{\varepsilon}.$$

$L(\,\cdot\,, \chi_P)$ is the product [Ne, Chapter VII, Theorem (10.4).ii)] of not more than six factors of the form $L(\,\cdot\,, \lambda)$ for $\lambda$ the non-trivial Dirichlet character of $\mathbb{Q}(\zeta_3)/\mathbb{Q}$ and at most three factors which are Artin-$L$-functions $L(\,\cdot\,, \nu^K)$ for two-dimensional irreducible representations.

Here, $K = \mathbb{Q}(\zeta_3, \sqrt[3]{a_0^{e_0} \cdot \ldots \cdot a_3^{e_3}})$ for certain $e_0, \ldots, e_3 \in \{0, 1, 2\}$. As $L(1, \lambda)$ does not depend on $a_0, \ldots, a_3$, at all, it will suffice to show

$$c_1(\varepsilon) \cdot |a_0 \cdot \ldots \cdot a_3|^{-\varepsilon} < L(1, \nu^K) < c_2(\varepsilon) \cdot |a_0 \cdot \ldots \cdot a_3|^{\varepsilon}$$

for each $\varepsilon > 0$.

$\nu^K$ is the only irreducible two-dimensional character of $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$. For that reason, by virtue of [Ne, Chapter VII, Corollary (10.5)], we have

$$\begin{aligned}\zeta_K(s) &= \zeta_{\mathbb{Q}}(s) \cdot L(s, \lambda) \cdot L(s, \nu^K)^2 \\ &= \zeta_{\mathbb{Q}(\zeta_3)}(s) \cdot L(s, \nu^K)^2\end{aligned}$$

for a complex variable $s$. It, therefore, suffices in our particular situation to estimate the residue $\mathrm{res}_{s=1}\, \zeta_K(s)$ of the Dedekind zeta function of $K$.

An estimate from above has been given by C. L. Siegel. In view of the analytic class number formula, his [Si, Satz 1] gives

$$\begin{aligned}\operatorname*{res}_{s=1} \zeta_K(s) &< C[\log \mathrm{Disc}(K/\mathbb{Q})]^5 \\ &\leq C[\log(3^9 a_0^4 a_1^4 a_2^4 a_3^4)]^5 \\ &= C[4 \log |a_0 \cdot \ldots \cdot a_3| + 9 \log 3]^5\end{aligned}$$

for a certain constant $C$. The final term is less than $c_2(\varepsilon) \cdot |a_0 \cdot \ldots \cdot a_3|^{\varepsilon}$ for every $\varepsilon > 0$.

On the other hand, H. M. Stark [St, formula (1)] shows

$$\operatorname*{res}_{s=1} \zeta_K(s) > C(\varepsilon) \cdot \mathrm{Disc}(K/\mathbb{Q})^{-\varepsilon/4}$$

for every $\varepsilon > 0$ which implies $\operatorname*{res}_{s=1} \zeta_K(s) > c_1(\varepsilon) \cdot |a_0 \cdot \ldots \cdot a_3|^{-\varepsilon}$. □

## 2.3 An estimate for the factors at the finite places

**2.3.1. Notation.** —— i) For a prime number $p$ and an integer $x \neq 0$, we put $x^{(p)} := p^{\nu_p(x)}$. Note $x^{(p)} = 1/\|x\|_p$ for the normalized $p$-adic valuation.

ii) For integers $x_1, \ldots, x_n$, not all equal to zero, we write

$$\gcd_p(x_1, \ldots, x_n) := [\gcd(x_1, \ldots, x_n)]^{(p)}.$$

Observe, if $x_1, \ldots, x_n \neq 0$ then we have $\gcd_p(x_1, \ldots, x_n) = \gcd(x_1^{(p)}, \ldots, x_n^{(p)})$.

iii) By putting $\nu(x) := \min\limits_{\substack{\xi \in \mathbb{Z}_p \\ x = (\xi \bmod p^r)}} \nu(\xi)$, we carry the *p-adic valuation* from $\mathbb{Z}_p$ over to $\mathbb{Z}/p^r\mathbb{Z}$.

Note that any $0 \neq x \in \mathbb{Z}/p^r\mathbb{Z}$ has the form $x = \varepsilon{\cdot}p^{\nu(x)}$ where $\varepsilon \in (\mathbb{Z}/p^r\mathbb{Z})^*$ is a unit. Clearly, $\varepsilon$ is unique only in the case $\nu(x) = 0$.

**2.3.2. Definition.** —— For $(a_0, \ldots, a_3) \in \mathbb{Z}^4$, $r \in \mathbb{N}$, and $\nu_0, \ldots, \nu_3 \leq r$, put

$$N^{(r)}_{\nu_0,\ldots,\nu_3;a_0,\ldots,a_3} := \{\, (x_0, \ldots, x_3) \in (\mathbb{Z}/p^r\mathbb{Z})^4 \mid \nu(x_0) = \nu_0, \ldots, \nu(x_3) = \nu_3;\ a_0x_0^3 + \ldots + a_3x_3^3 = 0 \in \mathbb{Z}/p^r\mathbb{Z} \,\}.$$

For the particular case $\nu_0 = \ldots = \nu_3 = 0$, we will write $Z^{(r)}_{a_0,\ldots,a_3} := N^{(r)}_{0,\ldots,0;a_0,\ldots,a_3}$. I.e.,

$$Z^{(r)}_{a_0,\ldots,a_3} = \{\, (x_0, \ldots, x_3) \in [(\mathbb{Z}/p^r\mathbb{Z})^*]^4 \mid a_0x_0^3 + \ldots + a_3x_3^3 = 0 \in \mathbb{Z}/p^r\mathbb{Z} \,\}.$$

We will use the notation $z^{(r)}_{a_0,\ldots,a_3} := \#Z^{(r)}_{a_0,\ldots,a_3}$.

**2.3.3. Sublemma.** —— *If $p^k | a_0, \ldots, a_3$ and $r > k$ then we have*

$$z^{(r)}_{a_0,\ldots,a_3} = p^{4k} \cdot z^{(r-k)}_{a_0/p^k,\ldots,a_3/p^k}.$$

**Proof.** Since $a_0x_0^3 + \ldots + a_3x_3^3 = p^k(a_0/p^k \cdot x_0^3 + \ldots + a_3/p^k \cdot x_3^3)$, there is a surjection

$$\iota\colon Z^{(r)}_{a_0,\ldots,a_3} \longrightarrow Z^{(r-k)}_{a_0/p^k,\ldots,a_3/p^k},$$

given by $(x_0, \ldots, x_3) \mapsto \big((x_0 \bmod p^{r-k}), \ldots, (x_3 \bmod p^{r-k})\big)$. The kernel of the homomorphism of modules underlying $\iota$ is $(p^{r-k}\mathbb{Z}/p^r\mathbb{Z})^4$. $\square$

**2.3.4. Lemma.** —— *Assume $\gcd_p(a_0, \ldots, a_4) = p^k$. Then, there is an estimate*

$$z^{(r)}_{a_0,\ldots,a_4} \leq 3p^{3r+k}.$$

**Proof.** Suppose first that $k = 0$. This means, one of the coefficients is prime to $p$. Without restriction, assume $p \nmid a_0$.

For any $(x_1, x_2, x_3) \in (\mathbb{Z}/p^r\mathbb{Z})^3$, there appears an equation of the form $a_0x_0^3 = c$. It cannot have more than three solutions in $(\mathbb{Z}/p^r\mathbb{Z})^*$. Indeed, for $p$ odd, this follows directly from the fact that $(\mathbb{Z}/p^r\mathbb{Z})^*$ is a cyclic group. On the other hand, in the case $p = 2$, we have $(\mathbb{Z}/2^r\mathbb{Z})^* \cong \mathbb{Z}/2^{r-2}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Again, there are only up to three solutions possible.

The general case may now easily be deduced from Sublemma 2.3.3. Indeed, if $k < r$ then

$$z^{(r)}_{a_0,\ldots,a_3} = p^{4k} \cdot z^{(r-k)}_{a_0/p^k,\ldots,a_3/p^k} \leq p^{4k} \cdot 3p^{3(r-k)} = 3p^{3r+k}.$$

On the other hand, if $k \geq r$ then the assertion is completely trivial since

$$z^{(r)}_{a_0,\ldots,a_3} = \#Z^{(r)}_{a_0,\ldots,a_3} < p^{4r} \leq p^{3r+k} < 3p^{3r+k}.$$ □

**2.3.5. Remark.** —— The proof shows that in the case $p \equiv 2 \pmod 3$ one could reduce the coefficient to 1. Unfortunately, this observation does not lead to a substantial improvement of our final result.

**2.3.6. Lemma.** —— *Let $r \in \mathbb{N}$ and $\nu_0, \ldots, \nu_3 \leq r$. Then,*

$$\#N^{(r)}_{\nu_0,\ldots,\nu_3;a_0,\ldots,a_3} = \frac{z^{(r)}_{p^{3\nu_0}a_0,\,\ldots\,,p^{3\nu_3}a_3} \cdot \varphi(p^{r-\nu_0}) \cdot \ldots \cdot \varphi(p^{r-\nu_3})}{\varphi(p^r)^4}.$$

**Proof.** As $p^{3\nu_0}a_0x_0^3 + \ldots + p^{3\nu_3}a_3x_3^3 = a_0(p^{\nu_0}x_0)^3 + \ldots + a_3(p^{\nu_3}x_3)^3$, we have a surjection

$$\pi\colon Z^{(r)}_{p^{3\nu_0}a_0,\ldots,p^{3\nu_3}a_3} \longrightarrow N^{(r)}_{\nu_0,\ldots,\nu_3;a_0,\ldots,a_3},$$

given by $(x_0, \ldots, x_3) \mapsto (p^{\nu_0}x_0, \ldots, p^{\nu_3}x_3)$.

For $i = 0, \ldots, 3$, consider the mapping $\iota\colon \mathbb{Z}/p^r\mathbb{Z} \to \mathbb{Z}/p^r\mathbb{Z}$, $x \mapsto p^{\nu_i}x$. If $\nu_i = r$ then $\iota$ is the zero map. All $\varphi(p^r) = (p-1)p^{r-1}$ units are mapped to zero. Otherwise, observe that $\iota$ is $p^{\nu_i} : 1$ on its image. Further, $\nu(\iota(x)) = \nu_i$ if and only if $x$ is a unit. By consequence, $\pi$ is $(K^{(\nu_0)} \cdot \ldots \cdot K^{(\nu_3)}) : 1$ when we put $K^{(\nu)} := p^\nu$ for $\nu < r$ and $K^{(r)} := (p-1)p^{r-1}$. Summarizing, we could have written $K^{(\nu)} := \varphi(p^r)/\varphi(p^{r-\nu})$. The assertion follows. □

**2.3.7. Corollary.** —— *Let $(a_0, \ldots, a_3) \in (\mathbb{Z}\backslash\{0\})^4$. Then, for the local factor $\tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big)$, one has*

$$\tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) = \det\big(1 - p^{-1}\,\mathrm{Frob}_p \mid \mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}\big) \cdot \lim_{r\to\infty} \sum_{\nu_0,\ldots,\nu_3=0}^{r} \frac{z^{(r)}_{p^{3\nu_0}a_0,\,\ldots\,,p^{3\nu_3}a_3} \cdot \varphi(p^{r-\nu_0}) \cdot \ldots \cdot \varphi(p^{r-\nu_3})}{p^{3r} \cdot \varphi(p^r)^4}.$$

**Proof.** [PT, Corollary 3.5] implies that

$$\tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) = \det\big(1 - p^{-1}\,\mathrm{Frob}_p \mid \mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}\big) \cdot \lim_{r\to\infty} \sum_{\nu_0,\ldots,\nu_3=0}^{r} \frac{\#N^{(r)}_{\nu_0,\ldots,\nu_3;a_0,\ldots,a_3}}{p^{3r}}.$$

Lemma 2.3.6 yields the assertion. □

**2.3.8. Proposition.** — *Let* $(a_0,\ldots,a_3) \in (\mathbb{Z}\setminus\{0\})^4$. *Then, for each* $\varepsilon$ *such that* $0<\varepsilon<\frac{1}{3}$, *one has*

$$\tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) \leq \Big(1+\frac{1}{p}\Big)^7 \cdot 3\Big(\frac{1}{1-\frac{1}{p^{1-3\varepsilon}}}\Big)\Big(\frac{1}{1-\frac{1}{p^{\varepsilon}}}\Big)^3 \cdot \big(a_0^{(p)}a_1^{(p)}a_2^{(p)}\big)^{\frac{1-\varepsilon}{3}}\big(a_3^{(p)}\big)^{\varepsilon}.$$

**Proof.** We use the formula from Corollary 2.3.7. The eigenvalues of the Frobenius on $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}$ are all roots of unity. Therefore, the first factor is at most $(1+1/p)^7$. Further, by Lemma 2.3.4,

$$\begin{aligned} z^{(r)}_{p^{3\nu_0}a_0,\ldots,p^{3\nu_3}a_3}/p^{3r} &\leq 3\gcd_p(p^{3\nu_0}a_0,\ \ldots,p^{3\nu_3}a_3) \\ &= 3\gcd\big(p^{3\nu_0}a_0^{(p)},\ \ldots,p^{3\nu_3}a_3^{(p)}\big). \end{aligned}$$

Writing $k_i := \nu_p(a_i) = \nu_p(a_i^{(p)})$, we see

$$\begin{aligned} z^{(r)}_{p^{3\nu_0}a_0,\ldots,p^{3\nu_3}a_3}/p^{3r} &\leq 3\gcd(p^{3\nu_0+k_0},\ \ldots,p^{3\nu_3+k_3}) \\ &= 3p^{\min\{3\nu_0+k_0,\ \ldots,3\nu_3+k_3\}}. \end{aligned}$$

We estimate the minimum by a weighted arithmetic mean with weights $\frac{1-\varepsilon}{3}$, $\frac{1-\varepsilon}{3}$, $\frac{1-\varepsilon}{3}$, and $\varepsilon$,

$$\begin{aligned} \min\{3\nu_0+k_0,\ \ldots,3\nu_3+k_3\} &\leq \frac{1-\varepsilon}{3}\cdot(3\nu_0+k_0)+\frac{1-\varepsilon}{3}\cdot(3\nu_1+k_1) \\ &\qquad + \frac{1-\varepsilon}{3}\cdot(3\nu_2+k_2)+\varepsilon(3\nu_3+k_3) \\ &= (1-\varepsilon)(\nu_0+\nu_1+\nu_2)+3\varepsilon\nu_3 \\ &\qquad + \frac{1-\varepsilon}{3}(k_0+k_1+k_2)+\varepsilon k_3\,. \end{aligned}$$

This shows

$$\begin{aligned} z^{(r)}_{p^{3\nu_0}a_0,\ldots,p^{3\nu_3}a_3}/p^{3r} &\leq 3p^{(1-\varepsilon)(\nu_0+\nu_1+\nu_2)+3\varepsilon\nu_3+\frac{1-\varepsilon}{3}(k_0+k_1+k_2)+\varepsilon k_3} \\ &= 3p^{(1-\varepsilon)(\nu_0+\nu_1+\nu_2)+3\varepsilon\nu_3}\cdot\big(a_0^{(p)}a_1^{(p)}a_2^{(p)}\big)^{\frac{1-\varepsilon}{3}}\big(a_3^{(p)}\big)^{\varepsilon}. \end{aligned}$$

We may therefore write

$$\begin{aligned} \tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) &\leq \Big(1+\frac{1}{p}\Big)^7\cdot 3\big(a_0^{(p)}a_1^{(p)}a_2^{(p)}\big)^{\frac{1-\varepsilon}{3}}\big(a_3^{(p)}\big)^{\varepsilon} \\ &\qquad \cdot\lim_{r\to\infty}\sum_{\nu_0,\ldots,\nu_3=0}^{r}\frac{p^{(1-\varepsilon)(\nu_0+\nu_1+\nu_2)+3\varepsilon\nu_3}\cdot\varphi(p^{r-\nu_0})\cdot\ldots\cdot\varphi(p^{r-\nu_3})}{\varphi(p^r)^4}\,. \end{aligned}$$

Here, the term under the limit is precisely the product of three copies of the finite sum

$$\sum_{\nu=0}^{r}\frac{p^{(1-\varepsilon)\nu}\cdot\varphi(p^{r-\nu})}{\varphi(p^r)} = \sum_{\nu=0}^{r-1}\frac{1}{(p^{\varepsilon})^{\nu}}+\frac{p}{p-1}\frac{1}{(p^{\varepsilon})^{r}}$$

and one copy of the finite sum

$$\sum_{\nu=0}^{r} \frac{p^{3\varepsilon\nu} \cdot \varphi(p^{r-\nu})}{\varphi(p^r)} = \sum_{\nu=0}^{r-1} \frac{1}{(p^{1-3\varepsilon})^\nu} + \frac{p}{p-1} \frac{1}{(p^{1-3\varepsilon})^r} \, .$$

For $r \to \infty$, geometric series do appear while the additional summands tend to zero. □

**2.3.9. Remark.** —— Unfortunately, the constants

$$C_p^{(\varepsilon)} := \Big(1+\frac{1}{p}\Big)^7 \cdot 3 \Big(\frac{1}{1-\frac{1}{p^{1-3\varepsilon}}}\Big) \Big(\frac{1}{1-\frac{1}{p^\varepsilon}}\Big)^3$$

have the property that the product $\prod_p C_p^{(\varepsilon)}$ diverges. On the other hand, we have at least that $C_p^{(\varepsilon)}$ is bounded for $p \to \infty$, say $C_p^{(\varepsilon)} \leq C^{(\varepsilon)}$.

**2.3.10. Lemma.** —— *Let $C > 1$ be any constant. Then, for each $\varepsilon > 0$, one has*

$$\prod_{\substack{p \,\text{prime} \\ p|x}} C \leq c \cdot x^\varepsilon$$

*for a suitable constant $c$ (depending on $\varepsilon$).*

**Proof.** This follows directly from [Na, Theorem 7.2] together with [Na, Section 7.1, Exercise 7]. □

**2.3.11. Proposition.** —— *For each $\varepsilon$ such that $0 < \varepsilon < \frac{1}{3}$, there exists a constant $c$ such that*

$$\prod_{p \,\text{prime}} \tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) \leq c \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{1}{3}-\frac{\varepsilon}{8}} \cdot \prod_{p \,\text{prime}} \min_{i=0,\ldots,3} \|a_i\|_p^{\frac{1}{3}-\varepsilon}$$

*for all $(a_0, \ldots, a_3) \in (\mathbb{Z}\backslash\{0\})^4$.*

**Proof.** The product over all primes of good reduction is bounded by virtue of Sublemma 2.3.12 below. It, therefore, remains to show that

$$\prod_{\substack{p \,\text{prime} \\ p|3a_0\cdots a_3}} \tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) \leq c \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{1}{3}-\frac{\varepsilon}{8}} \cdot \prod_{p \,\text{prime}} \min_{i=0,\ldots,3} \|a_i\|_p^{\frac{1}{3}-\varepsilon}.$$

For this, by Proposition 2.3.8, we have at first

$$\begin{aligned} \tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) &\leq C_p^{(\varepsilon)} \cdot \big(a_0^{(p)} a_1^{(p)} a_2^{(p)}\big)^{\frac{1}{3}-\frac{\varepsilon}{4}} \cdot (a_3^{(p)})^{\frac{3}{4}\varepsilon} \\ &= C_p^{(\varepsilon)} \cdot \big(a_0^{(p)} a_1^{(p)} a_2^{(p)} a_3^{(p)}\big)^{\frac{1}{3}-\frac{\varepsilon}{4}} \cdot (a_3^{(p)})^{-\frac{1}{3}+\varepsilon}. \end{aligned}$$

Here, the indices 0, … , 3 are interchangeable. Hence, it is even allowed to write

$$\tau_p\big(S^{(a_0,\dots,a_3)}(\mathbb{Q}_p)\big) \leq C_p^{(\varepsilon)} \cdot \big(a_0^{(p)}a_1^{(p)}a_2^{(p)}a_3^{(p)}\big)^{\frac{1}{3}-\frac{\varepsilon}{4}} \cdot \big(\max_i a_i^{(p)}\big)^{-\frac{1}{3}+\varepsilon}$$
$$= C_p^{(\varepsilon)} \cdot \big(a_0^{(p)}a_1^{(p)}a_2^{(p)}a_3^{(p)}\big)^{\frac{1}{3}-\frac{\varepsilon}{4}} \cdot \min_i \|a_i\|_p^{\frac{1}{3}-\varepsilon}.$$

Now, we multiply over all prime divisors of $a_0 \cdot \ldots \cdot a_3$. Thereby, on the right hand side, we may twice write the product over all primes since the two rightmost factors are equal to one for $p \nmid 3a_0 \cdot \ldots \cdot a_3$, anyway.

$$\prod_{\substack{p \text{ prime} \\ p | 3a_0 \cdots a_3}} \tau_p\big(S^{(a_0,\dots,a_3)}(\mathbb{Q}_p)\big) \leq \prod_{\substack{p \text{ prime} \\ p | 3a_0 \cdots a_3}} C_p^{(\varepsilon)} \cdot \prod_{p \text{ prime}} \big(a_0^{(p)}a_1^{(p)}a_2^{(p)}a_3^{(p)}\big)^{\frac{1}{3}-\frac{\varepsilon}{4}} \cdot \prod_{p \text{ prime}} \min_{i=0,\dots,3} \|a_i\|_p^{\frac{1}{3}-\varepsilon}$$
$$= \prod_{\substack{p \text{ prime} \\ p | 3a_0 \cdots a_3}} C_p^{(\varepsilon)} \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{1}{3}-\frac{\varepsilon}{4}} \cdot \prod_{p \text{ prime}} \min_{i=0,\dots,3} \|a_i\|_p^{\frac{1}{3}-\varepsilon}$$

when we observe that $\prod_p a^{(p)} = |a|$. Further, we have $C_p^{(\varepsilon)} \leq C^{(\varepsilon)}$ and, by Lemma 2.3.10,

$$\prod_{\substack{p \text{ prime} \\ p | 3a_0 \cdots a_3}} C^{(\varepsilon)} \leq c \cdot |3a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{8}}.$$

We finally estimate $3^{\frac{\varepsilon}{8}}$ by a constant. The assertion follows. □

**2.3.12. Sublemma.** —— *There are two positive constants $c_1$ and $c_2$ such that, for all $a_0, \dots, a_3 \in \mathbb{Z}\backslash\{0\}$,*

$$c_1 < \prod_{\substack{p \text{ prime} \\ p \nmid 3a_0 \cdot \ldots \cdot a_3}} \tau_p\big(S^{(a_0,\dots,a_3)}(\mathbb{Q}_p)\big) < c_2\,.$$

**Proof.** For a prime $p$ of good reduction, Hensel's Lemma implies

$$\tau_p\big(S^{(a_0,\dots,a_3)}(\mathbb{Q}_p)\big) = \det\big(1 - p^{-1}\operatorname{Frob}_p \mid \operatorname{Pic}(S_{\overline{\mathbb{Q}}})\big) \cdot \frac{\#S^{(a_0,\dots,a_3)}(\mathbb{F}_p)}{p^2}.$$

Further, for the number of points on a non-singular cubic surface over a finite field, the Lefschetz trace formula can be made completely explicit [Ma, Theorem 27.1]. It shows $\#S^{(a_0,\dots,a_3)}(\mathbb{F}_p) = p^2 + p \cdot \operatorname{tr}\big(\operatorname{Frob}_p \mid \operatorname{Pic}(S_{\overline{\mathbb{Q}}})\big) + 1$.

Denoting the eigenvalues of the Frobenius on $\operatorname{Pic}(S_{\overline{\mathbb{Q}}})$ by $\lambda_1, \dots, \lambda_7$, we find

$$\tau_p\big(S^{(a_0,\dots,a_3)}(\mathbb{Q}_p)\big) = (1-\lambda_1 p^{-1})(1-\lambda_2 p^{-1}) \cdot \ldots \cdot (1-\lambda_7 p^{-1}) \cdot [1 + (\lambda_1 + \cdots + \lambda_7)p^{-1} + p^{-2}]$$
$$= (1 - \sigma_1 p^{-1} + \sigma_2 p^{-2} \mp \ldots - \sigma_7 p^{-7})(1 + \sigma_1 p^{-1} + p^{-2})$$
$$= 1 + (1 - \sigma_1^2 + \sigma_2)p^{-2} - (\sigma_1 - \sigma_1\sigma_2 + \sigma_3)p^{-3} \pm$$
$$\pm \ldots - (\sigma_5 - \sigma_1\sigma_6 + \sigma_7)p^{-7} + (\sigma_6 - \sigma_1\sigma_7)p^{-8} - \sigma_7 p^{-9}$$

where $\sigma_i$ denote the elementary symmetric functions in $\lambda_1, \ldots, \lambda_7$.

We know $|\lambda_i| = 1$ for all $i$. Estimating very roughly, we have $|\sigma_j| \leq \binom{7}{j} \leq 7^j$ and see

$$1 - 99p^{-2} - 7{\cdot}99p^{-3} - \ldots - 7^7{\cdot}99p^{-9} \leq \tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) \leq$$
$$\leq 1 + 99p^{-2} + 7{\cdot}99p^{-3} + \ldots + 7^7{\cdot}99p^{-9}\,.$$

I.e., $1 - 99p^{-2}\frac{1}{1-7/p} < \tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) < 1 + 99p^{-2}\frac{1}{1-7/p}$. The infinite product over all $1 - 99p^{-2}\frac{1}{1-7/p}$ (respectively $1 + 99p^{-2}\frac{1}{1-7/p}$) is convergent.

The left hand side is positive for $p > 13$. For the small primes remaining, we need a better lower bound. For this, note that a cubic surface over a finite field $\mathbb{F}_p$ always has at least one $\mathbb{F}_p$-rational point. This yields $\tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) \geq (1-1/p)^7/p^2 > 0$. $\square$

## 2.4 An estimate for the factor at the infinite place

**2.4.1. Fact.** —— *Let $U \subset \mathbb{R}^{n+1}$ be an open subset and $X \subset U$ be a hypersurface defined by the equation $f = 0$. Assume that $\frac{\partial f}{\partial x_0} \neq 0$ outside a zero set of $X$. Then, on $X$, $\omega_{\text{Leray}}$ is given by the differential form*

$$\frac{1}{|\frac{\partial f}{\partial x_0}|}\, dx_1 \wedge \ldots \wedge dx_n\,.$$

**Proof.** Let $x \in X$ be a point such that $\frac{\partial f}{\partial x_0}(x) \neq 0$. The theorem on implicit functions yields an open neighbourhood $O$ of $x$ and a function $g\colon O \to \mathbb{R}$ such that $f(g(x_1, \ldots, x_n), x_1, \ldots, x_n) = 0$. This means, near $x$, $X$ is given by the parametrization $i\colon (x_1, \ldots, x_n) \mapsto (g(x_1, \ldots, x_n), x_1, \ldots, x_n)$. We immediately see $\partial g/\partial x_i = -\frac{\partial f}{\partial x_i}/\frac{\partial f}{\partial x_0}$.

The hypersurface measure on the image of $i$ is classically given by

$$\omega_{\text{hyp}} = \Big[\sqrt{1 + (\partial g/\partial x_1)^2 + \ldots + (\partial g/\partial x_n)^2}\, dx_1 \wedge \ldots \wedge dx_n\Big]$$

which may be rewritten in the form $\omega_{\text{hyp}} = \big[\frac{|\operatorname{grad} f|}{|\frac{\partial f}{\partial x_0}|}\, dx_1 \wedge \ldots \wedge dx_n\big]$. Recall that the Leray measure is defined by $\omega_{\text{Leray}} = \frac{1}{|\operatorname{grad} f|}\,\omega_{\text{hyp}}$. $\square$

**2.4.2. Corollary.** —— *Let $a_0, \ldots, a_3 \in \mathbb{R}\backslash\{0\}$. Then,*

$$\omega_{\text{Leray}}^{CS^{(a_0,\ldots,a_3)}(\mathbb{R})} = \Big[\frac{1}{3|a_0|x_0^2}\, dx_1 \wedge dx_2 \wedge dx_3\Big].$$

**Proof.** We apply Fact 2.4.1 to $U = \mathbb{R}^4$ and $f(x_0, \ldots, x_3) := a_0x_0^3 + \ldots + a_3x_3^3$. Note that $\{(x_0, \ldots, x_3) \in CS^{(a_0,\ldots,a_3)}(\mathbb{R}) \mid x_0 = 0\}$ is a zero set according to the Leray measure as it is for the hypersurface measure. $\square$

**2.4.3. Lemma.** — *Let $a_0, \ldots, a_3 \in \mathbb{R}\backslash\{0\}$. Then,*

$$\tau_\infty\big(S^{(a_0,\ldots,a_3)}(\mathbb{R})\big) = \frac{1}{2\sqrt[3]{|a_0 \cdot \ldots \cdot a_3|}} \int\limits_{\substack{CS^{(1,\ldots,1)}(\mathbb{R}) \\ |x_0|\leq \sqrt[3]{|a_0|}, \ldots, |x_3|\leq \sqrt[3]{|a_3|}}} \omega_{\text{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})}.$$

**Proof.** According to the definition of $\tau_\infty\big(S^{(a_0,\ldots,a_3)}(\mathbb{R})\big)$ and the corollary above, we need to show

$$\frac{1}{6\,|a_0|} \int\limits_{\substack{CS^{(a_0,\ldots,a_3)}(\mathbb{R}) \\ |x_0|\leq 1, \ldots, |x_3|\leq 1}} \frac{1}{x_0^2}\, dx_1 \wedge dx_2 \wedge dx_3 = \frac{1}{6\sqrt[3]{|a_0 \cdot \ldots \cdot a_3|}} \int\limits_{\substack{CS^{(1,\ldots,1)}(\mathbb{R}) \\ |X_0|\leq \sqrt[3]{|a_0|}, \ldots, |X_3|\leq \sqrt[3]{|a_3|}}} \frac{1}{X_0^2}\, dX_1 \wedge dX_2 \wedge dX_3 \,.$$

For that, consider the linear mapping $l\colon CS^{(a_0,\ldots,a_3)}(\mathbb{R}) \to CS^{(1,\ldots,1)}(\mathbb{R})$ given by $(x_0, \ldots, x_3) \mapsto (\sqrt[3]{a_0}x_0, \ldots, \sqrt[3]{a_3}x_3)$. Then,

$$l^*\Big(\frac{1}{X_0^2}\, dX_1 \wedge dX_2 \wedge dX_3\Big) = \frac{\sqrt[3]{a_1a_2a_3}}{a_0^{2/3}} \frac{1}{x_0^2}\, dx_1 \wedge dx_2 \wedge dx_3.$$

This immediately yields the assertion when we take into consideration that orientations are chosen in such a way that both integrals are positive. □

**2.4.4. Proposition.** — *For real numbers $0 < b_0 \leq b_1 \leq b_2 \leq b_3$, we have*

$$\int\limits_{\substack{CS^{(1,\ldots,1)(\mathbb{R})} \\ |x_0|\leq b_0, \ldots, |x_3|\leq b_3}} \omega_{\text{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})} \leq \Big(64 + \frac{64}{3}\log 3 + \frac{1}{3}\sqrt[3]{3}\,\omega_2\Big)b_0 + 64b_0 \log\frac{b_1}{b_0}$$

*where $\omega_2$ is the two-dimensional hypersurface measure of the $l_3$-unit sphere*

$$S^2 := \{\, (x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x_1|^3 + |x_2|^3 + |x_3|^3 = 1 \,\}\,.$$

**Proof.** *First step.* We cover the domain of integration by 25 sets as follows. We put $R_0 := [-b_0, b_0]^4 \cap CS^{(1,\ldots,1)}(\mathbb{R})$. Further, for each $\sigma \in S_4$, we set

$$R_\sigma := \{\, (x_0, \ldots, x_3) \in \mathbb{R}^4 \mid |x_{\sigma(0)}| \leq \cdots \leq |x_{\sigma(3)}|,\ |x_i| \leq b_i,\ \text{and } b_0 \leq |x_{\sigma(3)}| \,\} \cap CS^{(1,\ldots,1)}(\mathbb{R})\,.$$

*Second step.* One has $\int_{R_\sigma} \omega_{\text{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})} \leq \int_{R_{\text{id}}} \omega_{\text{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})}$ for every $\sigma \in S_4$.
Consider the map $i_\sigma\colon \mathbb{R}^4 \to \mathbb{R}^4$ given by $(x_0, \ldots, x_3) \mapsto (x_{\sigma(0)}, \ldots, x_{\sigma(3)})$. Since $CS^{(1,\ldots,1)}(\mathbb{R})$ is defined by a symmetric cubic form, it is invariant under $i_\sigma$. We claim that

$$i_\sigma(R_\sigma) \subseteq R_{\text{id}}\,.$$

Indeed, let $(x_0, \ldots, x_3) \in R_\sigma$. Then, $i_\sigma(x_0, \ldots, x_3) = (x_{\sigma(0)}, \ldots, x_{\sigma(3)})$ has the properties $|x_{\sigma(0)}| \leq \ldots \leq |x_{\sigma(3)}|$ and $b_0 \leq |x_{\sigma(3)}|$. In order to show $i_\sigma(x_0, \ldots, x_3) \in R_{\mathrm{id}}$, all we need to verify is $|x_{\sigma(i)}| \leq b_i$ for $i = 0, \ldots, 3$.

For this, we use that the $b_i$ are sorted. We have $|x_{\sigma(3)}| \leq b_{\sigma(3)} \leq b_3$. Further, $|x_{\sigma(2)}| \leq b_{\sigma(2)}$ and $|x_{\sigma(2)}| \leq |x_{\sigma(3)}| \leq b_{\sigma(3)}$ one of which is at most equal to $b_2$. Similarly, $|x_{\sigma(1)}| \leq b_{\sigma(1)}$, $|x_{\sigma(1)}| \leq |x_{\sigma(2)}| \leq b_{\sigma(2)}$, and $|x_{\sigma(1)}| \leq |x_{\sigma(3)}| \leq b_{\sigma(3)}$, the smallest of which is not larger than $b_1$. Finally, $|x_{\sigma(0)}| \leq b_{\sigma(0)}$, $|x_{\sigma(0)}| \leq |x_{\sigma(1)}| \leq b_{\sigma(1)}$, $|x_{\sigma(0)}| \leq |x_{\sigma(2)}| \leq b_{\sigma(2)}$, and $|x_{\sigma(0)}| \leq |x_{\sigma(3)}| \leq b_{\sigma(3)}$. This shows $|x_{\sigma(0)}| \leq b_0$.

Since $x_0^3 + \ldots + x_3^3$ is a symmetric form, the Leray measure on $CS^{(1,\ldots,1)}(\mathbb{R})$ is invariant under the canonical operation of $S_4$ on $CS^{(1,\ldots,1)}(\mathbb{R}) \subset \mathbb{R}^4$. Therefore, we have $(i_\sigma)_* \omega_{\mathrm{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})} = \omega_{\mathrm{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})}$ for each $\sigma \in S_4$.

Altogether,

$$\int\limits_{R_\sigma} \omega_{\mathrm{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})} \leq \int\limits_{i_\sigma^{-1}(R_{\mathrm{id}})} \omega_{\mathrm{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})} = \int\limits_{R_{\mathrm{id}}} (i_\sigma)_* \, \omega_{\mathrm{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})} = \int\limits_{R_{\mathrm{id}}} \omega_{\mathrm{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})}.$$

*Third step.* We have $\int_{R_0} \omega_{\mathrm{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})} \leq \frac{1}{3}\sqrt[3]{3}\,\omega_2 b_0$.

By virtue of Corollary 2.4.2, we have

$$\begin{aligned} \int_{R_0} \omega_{\mathrm{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})} &= \frac{1}{3} \int_{R_0} \frac{1}{x_3^2} \, dx_0 \wedge dx_1 \wedge dx_2 \\ &= \frac{1}{3} \iiint_{\pi(R_0)} \frac{1}{(x_0^3 + x_1^3 + x_2^3)^{2/3}} \, dx_0 \, dx_1 \, dx_2 \end{aligned}$$

where $\pi \colon CS^{(1,\ldots,1)}(\mathbb{R}) \to \mathbb{R}^3$, $(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2)$, denotes the projection to the first three coordinates.

We enlarge the domain of integration to

$$R' := \{\, (x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x_0|^3 + |x_1|^3 + |x_2|^3 \leq 3b_0^3 \,\} \,.$$

Then, by homogeneity, we see

$$\iiint_{R'} \frac{1}{(x_0^3 + x_1^3 + x_2^3)^{2/3}} \, dx_0 \, dx_1 \, dx_2 = \omega_2 \cdot \int\limits_0^{\sqrt[3]{3}b_0} \frac{1}{r^2} \cdot r^2 \, dr = \omega_2 \cdot \sqrt[3]{3} b_0 \,.$$

*Fourth step.* We have $\int_{R_{\mathrm{id}}} \omega_{\mathrm{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})} \leq (\frac{8}{3} + \frac{8}{9}\log 3) b_0 + \frac{8}{3} b_0 \log \frac{b_1}{b_0}$.

Observe $|x_3| = \big|\sqrt[3]{x_0^3 + x_1^3 + x_2^3}\,\big| \leq \sqrt[3]{|x_0|^3 + |x_1|^3 + |x_2|^3}$. For $(x_0, \ldots, x_3) \in R_{\mathrm{id}}$, this implies $|x_3| \leq \sqrt[3]{3}\,|x_2|$ and $|x_2| \geq b_0/\sqrt[3]{3}$. We find

$$\begin{aligned}
\int_{R_{\rm id}} \omega_{\rm Leray}^{CS^{(1,\dots,1)}(\mathbb{R})} &= \frac{1}{3}\int_{R_{\rm id}} \frac{1}{x_3^2}\, dx_0 \wedge dx_1 \wedge dx_2 \\
&\leq \frac{1}{3}\int_{R_{\rm id}} \frac{1}{x_2^2}\, dx_0 \wedge dx_1 \wedge dx_2 \\
&< \frac{1}{3}\int_{-b_0}^{b_0} \int_{|x_1|\in[|x_0|,b_1]} \int_{\substack{|x_2|\geq b_0/\sqrt[3]{3} \\ |x_2|\geq |x_1|}} \frac{1}{x_2^2}\, dx_2\, dx_1\, dx_0 \\
&\leq \frac{1}{3}\int_{-b_0}^{b_0} \int_{|x_1|\in[|x_0|,b_1]} \frac{2}{\max\{b_0/\sqrt[3]{3}, |x_1|\}}\, dx_1\, dx_0 \\
&\leq \frac{2}{3}\left[\int_{-b_0}^{b_0} \int_{|x_1|\in[|x_0|,b_0/\sqrt[3]{3}]} \frac{\sqrt[3]{3}}{b_0}\, dx_1\, dx_0 + \int_{-b_0}^{b_0} \int_{|x_1|\in[b_0/\sqrt[3]{3},b_1]} \frac{1}{|x_1|}\, dx_1\, dx_0\right] \\
&\leq \frac{2}{3}\cdot\frac{4b_0^2}{\sqrt[3]{3}}\cdot\frac{\sqrt[3]{3}}{b_0} + \frac{2}{3}\int_{-b_0}^{b_0} 2\log\frac{\sqrt[3]{3}b_1}{b_0}\, dx_0 \\
&= \frac{8}{3}b_0 + \frac{8}{3}b_0\log\frac{\sqrt[3]{3}b_1}{b_0} \\
&= \Big(\frac{8}{3} + \frac{8}{9}\log 3\Big)b_0 + \frac{8}{3}b_0\log\frac{b_1}{b_0}. \qquad \square
\end{aligned}$$

**2.4.5. Corollary.** — *For every $\varepsilon > 0$, there exists a constant $c$ such that*

$$\tau_\infty\big(S^{(a_0,\dots,a_3)}(\mathbb{R})\big) \leq c\cdot|a_0\cdot\ldots\cdot a_3|^{-\frac{1}{3}+\varepsilon}\cdot\min_{i=0,\dots,3}\|a_i\|_\infty^{\frac{1}{3}}$$

*for each $(a_0,\dots,a_3)\in(\mathbb{Z}\backslash\{0\})^4$.*

**Proof.** We assume without restriction that $|a_0|\leq\ldots\leq|a_3|$. Then, Lemma 2.4.3 and Proposition 2.4.4 together show that, for certain explicit positive constants $c_1$ and $c_2$,

$$\begin{aligned}
\tau_\infty\big(S^{(a_0,\dots,a_3)}(\mathbb{R})\big) &\leq |a_0\cdot\ldots\cdot a_3|^{-\frac{1}{3}}\cdot\left(c_1|a_0|^{\frac{1}{3}} + c_2|a_0|^{\frac{1}{3}}\log\sqrt[3]{\frac{|a_1|}{|a_0|}}\right) \\
&= |a_0\cdot\ldots\cdot a_3|^{-\frac{1}{3}}\cdot|a_0|^{\frac{1}{3}}\left(c_1 + \frac{1}{3}c_2\log\frac{|a_1|}{|a_0|}\right) \\
&\leq |a_0\cdot\ldots\cdot a_3|^{-\frac{1}{3}}\cdot\min_{i=0,\dots,3}\|a_i\|_\infty^{\frac{1}{3}}\cdot\Big(c_1 + \frac{1}{3}c_2\log|a_0\cdot\ldots\cdot a_3|\Big).
\end{aligned}$$

There is a constant $c$ such that $c_1 + \frac{1}{3}c_2\log|a_0\cdot\ldots\cdot a_3| \leq c|a_0\cdot\ldots\cdot a_3|^\varepsilon$ for every $(a_0,\dots,a_3)\in(\mathbb{Z}\backslash\{0\})^4$. $\square$

## 2.5 The Tamagawa number

**2.5.1. Proposition.** —— *For every $\varepsilon > 0$, there exists a constant $C > 0$ such that*

$$\frac{1}{\tau^{(a_0,\ldots,a_3)}} \geq C \cdot \frac{\mathrm{H}_{\mathrm{naive}}\big(\frac{1}{a_0} : \ldots : \frac{1}{a_3}\big)^{\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\varepsilon}}$$

*for each $(a_0, \ldots, a_3) \in (\mathbb{Z}\backslash\{0\})^4$.*

**Proof.** We may assume that $\varepsilon$ is small, say $\varepsilon < \frac{2}{3}$. Then, immediately from the definition of $\tau^{(a_0,\ldots,a_3)}$, we have

$$\begin{aligned}
&\tau^{(a_0,\ldots,a_3)} \\
&= \alpha\big(S^{(a_0,\ldots,a_3)}\big)\cdot\beta\big(S^{(a_0,\ldots,a_3)}\big) \cdot \lim_{s\to 1}\,(s-1)^t L\big(s, \chi_{\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}})}\big) \cdot \tau_H\big(S^{(a_0,\ldots,a_3)}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}\big) \\
&\leq \alpha\big(S^{(a_0,\ldots,a_3)}\big)\cdot\beta\big(S^{(a_0,\ldots,a_3)}\big) \cdot \lim_{s\to 1}\,(s-1)^t L\big(s, \chi_{\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}})}\big) \cdot \tau_H\big(S^{(a_0,\ldots,a_3)}(\mathbb{A}_{\mathbb{Q}})\big) \\
&= \alpha\big(S^{(a_0,\ldots,a_3)}\big)\cdot\beta\big(S^{(a_0,\ldots,a_3)}\big) \cdot \lim_{s\to 1}\,(s-1)^t L\big(s, \chi_{\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}})}\big) \cdot \prod_{\nu\in\mathrm{Val}(\mathbb{Q})} \tau_\nu\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_\nu)\big).
\end{aligned}$$

Let us collect estimates for the factors. First, by Proposition 2.2.4, we have

$$\lim_{s\to 1}\,(s-1)^t L\big(s, \chi_{\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}})}\big) < c_1 \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{16}}$$

for a certain constant $c_1$. Further, Proposition 2.3.11 yields

$$\prod_{p\,\mathrm{prime}} \tau_p\big(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)\big) \leq c_2 \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{1}{3}-\frac{\varepsilon}{16}} \cdot \prod_{p\,\mathrm{prime}} \min_{i=0,\ldots,3} \|a_i\|_p^{\frac{1}{3}-\frac{\varepsilon}{2}}.$$

Finally, Corollary 2.4.5 shows

$$\tau_\infty\big(S^{(a_0,\ldots,a_3)}(\mathbb{R})\big) \leq c \cdot |a_0 \cdot \ldots \cdot a_3|^{-\frac{1}{3}+\frac{\varepsilon}{2}} \cdot \min_{i=0,\ldots,3} \|a_i\|_\infty^{\frac{1}{3}}.$$

We assert that the three inequalities together imply the following estimate for Peyre's constant $\tau^{(a_0,\ldots,a_3)} = \tau\big(S^{(a_0,\ldots,a_3)}\big)$,

$$\tau^{(a_0,\ldots,a_3)} \leq c_3 \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{2}} \cdot \prod_{p\,\mathrm{prime}} \min_{i=0,\ldots,3} \|a_i\|_p^{\frac{1}{3}} \cdot \min_{i=0,\ldots,3} \|a_i\|_\infty^{\frac{1}{3}} \cdot \prod_{p\,\mathrm{prime}} \Big[\min_{i=0,\ldots,3} \|a_i\|_p\Big]^{-\frac{\varepsilon}{2}}.$$

Indeed, this is trivial in the case $\tau^{(a_0,\ldots,a_3)} = 0$. Otherwise, $S^{(a_0,\ldots,a_3)}$ has an adelic point. Lemmas 2.5 and 2.3 show that we may estimate the factors $\alpha$ and $\beta$

by constants. By consequence,

$$\begin{aligned}
\frac{1}{\tau(a_0,\ldots,a_3)} &\geq \frac{1}{c_3} \cdot \frac{\prod\limits_{p\,\mathrm{prime}} \Big[\min\limits_{i=0,\ldots,3} \|a_i\|_p\Big]^{-\frac{1}{3}} \cdot \Big[\min\limits_{i=0,\ldots,3} \|a_i\|_\infty\Big]^{-\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{2}} \cdot \prod\limits_{p\,\mathrm{prime}} \Big[\min\limits_{i=0,\ldots,3} \|a_i\|_p\Big]^{-\frac{\varepsilon}{2}}} \\
&= \frac{1}{c_3} \cdot \frac{\prod\limits_{p\,\mathrm{prime}} \max\limits_{i=0,\ldots,3} \Big\|\frac{1}{a_i}\Big\|_p^{\frac{1}{3}} \cdot \max\limits_{i=0,\ldots,3} \Big\|\frac{1}{a_i}\Big\|_\infty^{\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{2}} \cdot \prod\limits_{p\,\mathrm{prime}} \Big[\max\limits_{i=0,\ldots,3} a_i^{(p)}\Big]^{\frac{\varepsilon}{2}}} \\
&= \frac{1}{c_3} \cdot \frac{\mathrm{H}_{\mathrm{naive}}\big(\frac{1}{a_0} : \ldots : \frac{1}{a_3}\big)^{\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{2}} \cdot \prod\limits_{p\,\mathrm{prime}} \Big[\max\limits_{i=0,\ldots,3} a_i^{(p)}\Big]^{\frac{\varepsilon}{2}}} \, .
\end{aligned}$$

It is obvious that $\max\limits_{i=0,\ldots,3} a_i^{(p)} \leq |a_0^{(p)} \cdot \ldots \cdot a_3^{(p)}|$ and $\prod\limits_{p\,\mathrm{prime}} |a_0^{(p)} \cdot \ldots \cdot a_3^{(p)}| = |a_0 \cdot \ldots \cdot a_3|$. This shows

$$\begin{aligned}
\frac{1}{\tau(a_0,\ldots,a_3)} &\geq \frac{1}{c_3} \cdot \frac{\mathrm{H}_{\mathrm{naive}}\big(\frac{1}{a_0} : \ldots : \frac{1}{a_3}\big)^{\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{2}} \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{2}}} \\
&= \frac{1}{c_3} \cdot \frac{\mathrm{H}_{\mathrm{naive}}\big(\frac{1}{a_0} : \ldots : \frac{1}{a_3}\big)^{\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\varepsilon}} \, .
\end{aligned}$$

□

**2.5.2. Lemma.** — *Let* $(a_0 : \ldots : a_3) \in \mathbf{P}^3(\mathbb{Q})$ *be any point such that* $a_0 \neq 0, \ldots, a_3 \neq 0$. *Then,*

$$\mathrm{H}_{\mathrm{naive}}(a_0 : \ldots : a_3) \leq \mathrm{H}_{\mathrm{naive}}(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_3})^3 \, .$$

**Proof.** First, observe that $(a_0 : \ldots : a_3) \mapsto \big(\frac{1}{a_0} : \ldots : \frac{1}{a_3}\big)$ is a well-defined map. Hence, we may assume without restriction that $a_0, \ldots, a_3 \in \mathbb{Z}$ and $\gcd(a_0, \ldots, a_3) = 1$. This yields $\mathrm{H}_{\mathrm{naive}}(a_0 : \ldots : a_3) = \max\limits_{i=0,\ldots,3} |a_i|$.

On the other hand, $(\frac{1}{a_0} : \ldots : \frac{1}{a_3}) = (a_1a_2a_3 : \ldots : a_0a_1a_2)$. Consequently,

$$\mathrm{H}_{\mathrm{naive}}\left(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_3}\right) \leq [\max_{i=0,\ldots,3} |a_i|]^3 = \mathrm{H}_{\mathrm{naive}}(a_0 : \ldots : a_3)^3 \, .$$

From this, the asserted inequality emerges when the roles of $a_i$ and $\frac{1}{a_i}$ are interchanged. □

**2.5.3. Corollary.** — *Let* $a_0, \ldots, a_3 \in \mathbb{Z}$ *such that* $\gcd(a_0, \ldots, a_3) = 1$. *Then,*

$$|a_0 \cdot \ldots \cdot a_3| \leq \mathrm{H}_{\mathrm{naive}}\left(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_3}\right)^{12}.$$

**Proof.** Observe that $|a_0 \cdot \ldots \cdot a_3| \leq \max\limits_{i=0,\ldots,3} |a_i|^4 = \mathrm{H}_{\mathrm{naive}}(a_0 : \ldots : a_3)^4$ and apply Lemma 2.5.2. □

**2.5.4. Theorem.** —— *For each $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that, for all $(a_0, \ldots, a_3) \in (\mathbb{Z}\backslash\{0\})^4$,*

$$\frac{1}{\tau^{(a_0,\ldots,a_3)}} \geq C(\varepsilon) \cdot \mathrm{H}_{\mathrm{naive}}\big(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_3}\big)^{\frac{1}{3}-\varepsilon}.$$

**Proof.** We may assume that $\gcd(a_0, \ldots, a_3) = 1$. Then, by Proposition 2.5.1,

$$\frac{1}{\tau^{(a_0,\ldots,a_3)}} \geq C(\varepsilon) \cdot \frac{\mathrm{H}_{\mathrm{naive}}\big(\frac{1}{a_0} : \ldots : \frac{1}{a_3}\big)^{\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{12}}}.$$

Corollary 2.5.3 yields $|a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{12}} \leq \mathrm{H}_{\mathrm{naive}}\big(\frac{1}{a_0} : \ldots : \frac{1}{a_3}\big)^{\varepsilon}$. □

**2.5.5. Corollary (**Fundamental finiteness**).** —— *For each $T > 0$, there are only finitely many diagonal cubic surfaces $S^{(a_0,\ldots,a_3)}\colon a_0x_0^3 + \ldots + a_3x_3^3 = 0$ in $\mathbf{P}^3_{\mathbb{Q}}$ such that $\tau^{(a_0,\ldots,a_3)} > T$.*

**Proof.** This is an immediate consequence of the comparison to the naive height established in Theorem 2.5.4. □

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