The Asymptotics of Points of Bounded Height on Diagonal Cubic and Quartic Threefolds

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Abstract. For the families $ax^3 = by^3 + z^3 + v^3 + w^3$, $a, b = 1, \ldots, 100$, and $ax^4 = by^4 + z^4 + v^4 + w^4$, $a, b = 1, \ldots, 100$, of projective algebraic threefolds, we test numerically the conjecture of Manin (in the refined form due to Peyre) about the asymptotics of points of bounded height on Fano varieties.

1 Introduction — Manin's Conjecture

Let V be a projective algebraic variety over $\mathbb{Q}$. We fix an embedding $\iota\colon V \to \mathbf{P}^n_{\mathbb{Q}}$. In this situation, there is the well-known *naive height* $\mathrm{H}_{\mathrm{naive}}\colon V(\mathbb{Q}) \to \mathbb{R}$ which is given by $\mathrm{H}_{\mathrm{naive}}(P) := \max_{i=0,\ldots,n} |x_i|$. Here, $(x_0 : \ldots : x_n) := \iota(P) \in \mathbf{P}^n(\mathbb{Q})$ where the projective coordinates are integers satisfying $\gcd(x_0, \ldots, x_n) = 1$.

It is of interest to ask for the asymptotics of the number of $\mathbb{Q}$-rational points on V of bounded naive height. This applies particularly to the case V is a Fano variety as those are expected to have many rational points (at least after a finite extension of the ground-field). Simplest examples of Fano varieties are complete intersections in $\mathbf{P}^n_{\mathbb{Q}}$ of a multidegree $(d_1, \ldots, d_r)$ such that $d_1 + \ldots + d_r \leq n$. In this case, Manin's conjecture reads as follows.

Conjecture 1. *Let $V \subseteq \mathbf{P}^n_{\mathbb{Q}}$ be a non-singular complete intersection of multidegree $(d_1, \ldots, d_r)$. Assume $\dim V \geq 3$ and $k := n + 1 - d_1 - \ldots - d_r > 0$. Then, there exists a Zariski open subset $V^\circ \subseteq V$ such that*

$$\#\{x \in V^\circ(\mathbb{Q}) \mid \mathrm{H}_{\mathrm{naive}}(x)^k < B\} \sim CB \tag{1}$$

for a constant C.

Example 2. Let $V \subset \mathbf{P}^4_{\mathbb{Q}}$ be a smooth hypersurface of degree 4. Conjecture 1 predicts $\sim CB$ rational points of height $< B$. However, the hypersurface

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$x^4 + y^4 = z^4 + v^4 + w^4$ contains the line given by $x = z$, $y = v$, and $w = 0$ on which there is quadratic growth, already. This explains the necessity of the restriction to a Zariski open subset $V^\circ \subseteq V$.

Remark 3. Conjecture 1 is proven for $\mathbf{P}^n$, linear subspaces, and quadrics. Further, it is established [Bi] in the case that the dimension of V is very large compared to $d_1, \ldots, d_r$. Generalizations are known to be true in a number of further particular cases. A complete list may be found in the survey article [Pe2, sec. 4].

In this note, we report numerical evidence for Conjecture 1 in the case of the varieties $V^e_{a,b}$ given by $ax^e = by^e + z^e + v^e + w^e$ in $\mathbf{P}^4_\mathbb{Q}$ for $e = 3$ and 4.

Remark 4. By the Noether-Lefschetz Theorem, the assumptions made on V imply that $\mathrm{Pic}(V_{\overline{\mathbb{Q}}}) \cong \mathbb{Z}$ [Ha1, Corollary IV.3.2]. This is no longer true in dimension two. See Remark 6.ii) for more details.

The constant. Conjecture 1 is compatible with results obtained by the classical circle method (e.g. [Bi]). Motivated by this, E. Peyre [Pe1] provided a description of the constant C expected in (1). In the situation considered here, Peyre's constant is equal to the Tamagawa-type number

$$\tau_\mathbf{H}(V) := \prod_{p \in \mathbb{P} \cup \{\infty\}} (1 - \tfrac{1}{p})\, \omega_{\mathbf{H},p}(V(\mathbb{Q}_p)).$$

In this formula, the Tamagawa measure $\omega_{\mathbf{H},p}$ is given in local p-adic analytic coordinates $x_1, \ldots, x_d$ by $\|\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_d}\|_p \, dx_1 \ldots dx_d$. Here, each dx_i denotes a Haar measure on $\mathbb{Q}_p$ which is normalized in the usual manner. $\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_d}$ is a section of $\mathscr{O}(-K) \cong \mathscr{O}(-k)$.

For p finite, one has a canonical model $\mathscr{V} \subseteq \mathbf{P}^n_{\mathbb{Z}_p}$ of V. This defines the norm $\|.\|_p$ on $\mathscr{O}(-k)$. It is almost immediate from the definition that

$$\omega_{\mathbf{H},p}(V(\mathbb{Q}_p)) = \lim_{n \to \infty} \frac{\# V(\mathbb{Z}/p^n\mathbb{Z})}{p^{dn}}.$$

The place at infinity. Here, $\|.\|_{\sup}$ on $\mathscr{O}(1)$ is the hermitian metric, in the sense of complex geometry, defined by $\|x_i\|_{\sup} := \inf\limits_{j=0,\ldots,n} |x_i/x_j|$. This induces the hermitian metric $\|.\|_\infty := \|.\|_{\sup}^{-k}$ on $\mathscr{O}(-k)$.

Lemma 5. *If $V \subset \mathbf{P}^n$ is a hypersurface defined by the equation $f = 0$ then*

$$\omega_\infty(V(\mathbb{R})) = \frac{1}{2} \int\limits_{\substack{f(x_0, \ldots, x_n) = 0 \\ |x_0|, \ldots, |x_n| \leq 1}} \omega_{\mathrm{Leray}}.$$

The Leray measure ω_{Leray} on $\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid f(x_0, \ldots, x_n) = 0\}$ is related to the usual hypersurface measure by the formula $\omega_{\mathrm{Leray}} = \frac{1}{\mathrm{grad}\, f}\, \omega_{\mathrm{hyp}}$. On the other hand, one may also write

$$\omega_{\mathrm{Leray}} = \frac{1}{|\frac{\partial f}{\partial x_i}(x_0, \ldots, x_n)|}\, dx_0 \ldots \widehat{dx_i} \ldots dx_n.$$

Proof. The equivalence of the two descriptions of the Leray measure is a standard calculation. The main assertion is a particular case of [Pe1, Lemma 5.4.7]. 2 is the number of roots of unity in $\mathbb{Q}$. □

Remark 6. There are several ways to generalize Conjecture 1.

i) One may consider more general heights corresponding to the tautological line bundle $\mathscr{O}(1)$. This includes to

a) replace the maximum norm by an arbitrary continuous hermitian metric on $\mathscr{O}(1)$. This would affect the domain of integration for the factor at infinity.

b) multiply $\mathrm{H}_{\mathrm{naive}}(x)$ with a function that depends on the reduction of x modulo some $N \in \mathbb{N}$. This augments Conjecture 1 by an equidistribution statement.

ii) Instead of complete intersections, one may consider arbitrary projective Fano varieties V. Then, $\mathrm{H}^k_{\mathrm{naive}}$ needs to be replaced by a height defined by the anticanonical bundle $\mathscr{O}(-K_V)$.

If $\mathrm{Pic}(V_{\overline{\mathbb{Q}}}) \ncong \mathbb{Z}$ then the description of the constant C gets more complicated in several ways. First, there is an additional factor $\beta := \#H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(V_{\overline{\mathbb{Q}}}))$. Further, instead of the factors $(1 - \frac{1}{p})$ one has to write $1/L_p(1, \mathrm{Pic}(V_{\overline{\mathbb{Q}}}))$, L_p being the local L-function corresponding to the Picard group. Finally, the Tamagawa measure has to be taken not of the full variety $V(\mathbb{A}_\mathbb{Q})$ but of the subset which is not affected by the Brauer-Manin obstruction.

If $\mathrm{Pic}(V) \ncong \mathbb{Z}$ already over $\mathbb{Q}$ then the right hand side of (1) has to be replaced by $CB\log^t B$. For the exponent of the log-term, there is the expectation that $t = \mathrm{rk}\,\mathrm{Pic}(V) - 1$. There are, however, examples [BT] in dimension three in which the exponent is larger. The constant C gets equipped with yet another additional factor α which depends on the structure of the effective cone in $\mathrm{Pic}(V)$ and on the position of $-K$ in it [Pe1, Définition 2.4].

iii) Finally, there is a generalization to arbitrary number fields [Pe1].

2 Computation of the Tamagawa number

Counting points over finite fields. We consider the projective varieties $V^e_{a,b}$ given by $ax^e = by^e + z^e + v^e + w^e$ in $\mathbf{P}^4_{\mathbb{F}_p}$. We assume $a, b \neq 0$ (and $p \nmid e$) in order to avoid singularities. Observe that, even for large p, these are at most e^2 varieties up to obvious $\mathbb{F}_p$-isomorphism as $\mathbb{F}_p^*$ consists of no more than e cosets modulo $(\mathbb{F}_p^*)^e$.

It follows from the Weil conjectures, proven by P. Deligne [De, Théorème (8.1)], that

$$\#V^e_{a,b}(\mathbb{F}_p) = p^3 + p^2 + p + 1 + E^e_{a,b}$$

where the error-term $E^e_{a,b}$ may be estimated by $|E^e_{a,b}| \leq C_e p^{3/2}$.

Here, $C_3 = 10$ and $C_4 = 60$ as $\dim H^3(\mathscr{V}^3, \mathbb{R}) = 10$ for every smooth cubic threefold and $\dim H^3(\mathscr{V}^4, \mathbb{R}) = 60$ for every smooth quartic threefold in $\mathbf{P}^4(\mathbb{C})$. These dimensions result from the Weak Lefschetz Theorem together with F. Hirzebruch's formula [Hi, Satz 2.4] for the Euler characteristic which actually works in much more generality.

Remark 7. Suppose $e = 3$ and $p \equiv 2 \pmod 3$. Then, $\#V^3_{a,b}(\mathbb{F}_p) = \#V^1_{a,b}(\mathbb{F}_p)$ as $\gcd(p-1,3) = 1$. Similarly, for $e = 4$ and $p \equiv 3 \pmod 4$, one has $\gcd(p-1,4) = 2$ and $\#V^4_{a,b}(\mathbb{F}_p) = \#V^2_{a,b}(\mathbb{F}_p)$. In these cases, the error term vanishes and $\#V^e_{a,b}(\mathbb{F}_p) = p^3 + p^2 + p + 1$.

In the remaining cases $p \equiv 1 \pmod 3$ for $e = 3$ and $p \equiv 1 \pmod 4$ for $e = 4$, our goal is to compute the number of $\mathbb{F}_p$-rational points on $V^e_{a,b}$. As $V^e_{a,b} \subseteq \mathbf{P}^4$, there would be an obvious $O(p^4)$-algorithm. We can do significantly better than that.

Definition 8. *Let K be a field and let $x \in K^n$ and $y \in K^m$ be two vectors. Then, their* convolution $z := x * y \in K^{n+m-1}$ *is defined to be* $z_k := \sum\limits_{i+j=k} x_i y_j$.

Theorem 9 (FFT convolution)**.** *Let $n = 2^l$ and K be a field which contains the $2n$-th roots of unity. Then, the convolution $x*y$ of two vectors x, y of length $\leq n$ can be computed in $O(n \log n)$ steps.*

Proof. The idea is to apply the Fast Fourier Transform (FFT) [Fo, Satz 20.3]. The connection to the convolution is shown in [Fo, Satz 20.2, or CLR, Theorem 32.8]. □

Theorem 9 is the basis for the following algorithm.

Algorithm 10 (FFT point counting on $V^e_{a,b}$)**.**

i) Initialize a vector $x[0 \ldots p]$ with zeroes.

ii) Let r run from 0 to $p-1$ and increase $x[r^e \bmod p]$ by 1.

iii) Calculate $\tilde{y} := x * x * x$ by FFT convolution.

iv) Normalize by putting $y[i] := \tilde{y}[i] + \tilde{y}[i+p] + \tilde{y}[i+2p]$ for $i = 0, \ldots, p-1$.

v) Initialize N as zero.

vi) (Counting points with first coordinate $\neq 0$)
Let j run from 0 to $p-1$ and increase N by $y[(a - bj^4) \bmod p]$.

vii) (Counting points with first coordinate 0 and second coordinate $\neq 0$)
Increase N by $y[(-b) \bmod p]$.

viii) (Counting points with first and second coordinate 0)
Increase N by $(y[0]-1)/(p-1)$.

ix) Return N as the number of all $\mathbb{F}_p$-valued points on $V^e_{a,b}$.

Remark 11. For the running-time, step iii) is dominant. Therefore, the running-time of Algorithm 10 is $O(p \log p)$.

To count, for fixed e and p, $\mathbb{F}_p$-rational points on $V^e_{a,b}$ with varying a and b, one needs to execute the first four steps only once. Afterwards, one may perform steps v) through ix) for all pairs (a,b) of elements from a system of representatives for $\mathbb{F}_p^*/(\mathbb{F}_p^*)^e$. Note that steps v) through ix) alone are of complexity $O(p)$.

We ran this algorithm for all primes $p \leq 10^6$ and stored the cardinalities in a file. This took several days of CPU time.

Remark 12. There is a formula for $\#V^e_{a,b}(\mathbb{F}_p)$ in terms of Jacobi sums. A skilful manipulation of these sums should lead to another efficient algorithm which serves the same purpose as Algorithm 10.

The local factors at finite places. We are interested in the Euler product

$$\tau^e_{a,b,\mathrm{fin}} := \prod_{p\in\mathbb{P}} \left(1-\frac{1}{p}\right) \lim_{n\to\infty} \frac{\#V^e_{a,b}(\mathbb{Z}/p^n\mathbb{Z})}{p^{3n}}.$$

Lemma 13. a) (Good reduction)
If $p \nmid abe$ then the sequence $(\#V^e_{a,b}(\mathbb{Z}/p^n\mathbb{Z})/p^{3n})_{n\in\mathbb{N}}$ is constant.

b) (Bad reduction)

i) *If p divides ab but not e then the sequence $(\#V^e_{a,b}(\mathbb{Z}/p^n\mathbb{Z})/p^{3n})_{n\in\mathbb{N}}$ becomes stationary as soon as p^n divides neither a nor b.*

ii) *If $p = 2$ and $e = 4$ then the sequence $(\#V^e_{a,b}(\mathbb{Z}/p^n\mathbb{Z})/p^{3n})_{n\in\mathbb{N}}$ becomes stationary as soon as 2^n does not divide $8a$ or $8b$.*

iii) *If $p = 3$ and $e = 3$ then the sequence $(\#V^e_{a,b}(\mathbb{Z}/p^n\mathbb{Z})/p^{3n})_{n\in\mathbb{N}}$ becomes stationary as soon as 3^n divides neither $3a$ nor $3b$.*

Theorem 14. *For every pair (a, b) of integers such that $a, b \neq 0$, the Euler product $\tau^e_{a,b,\mathrm{fin}}$ is convergent.*

Proof. Let p be a prime bigger than $|a|$, $|b|$, and e. Then, the factor at p is $\tau_p := (1-\frac{1}{p})(1+p+p^2+p^3+D_p p^{3/2})/p^3$ where $|D_p| \leq C_e$ for $C_3 = 10$ and $C_4 = 60$, respectively.

Taking the logarithm, we consider $\sum_p \log \tau_p$. In the case $e = 3$, the sum is effectively over the primes $p = 1 \pmod 3$. If $e = 4$ then summation extends over all primes $p = 1 \pmod 4$. In either case, we take a sum over one-half of all primes. This leads to the following estimate,

$$\sum_{p\geq N} |\log \tau_p| \leq \sum_{p\geq N} \left[\frac{C_e}{p^{3/2}} + O(p^{-5/2})\right] \sim \frac{C_e}{2} \int_N^\infty \frac{1}{t^{3/2}\log t}\, dt$$
$$\leq \frac{C_e}{2\log N} \int_N^\infty t^{-3/2} dt = \frac{C_e}{\sqrt{N}\log N}. \quad \square$$

Remark 15. We are interested in an explicit upper bound for $|\sum\limits_{p\geq 10^6} \log \tau_p|$. Using Taylor's formula, one gets

$$\left| \sum_{p\geq 10^6} \log \tau_p - \sum_{p\geq 10^6} \frac{D_p}{p^{3/2}} \right| \leq 10^{-8}.$$

Since $\frac{D_p}{p^{3/2}}$ is zero for $p \equiv 3 \pmod 4$ (or $p \equiv 2 \pmod 3$), the sum should be compared with $\log(\zeta_K(3/2))$. Here, ζ_K is the Dedekind Zeta function of $K = \mathbb{Q}(i)$ or $\mathbb{Q}(\zeta_3)$, respectively. This yields

$$\sum_{\substack{p\geq 10^6 \\ p\equiv 1 \pmod 4}} \frac{1}{p^{3/2}} \leq \log \frac{\sqrt{\zeta_{\mathbb{Q}(i)}(3/2)}}{(1-2^{-3/2})^{-1/2} \cdot \prod\limits_{p\equiv 3 \pmod 4} (1-p^{-3})^{-1/2} \cdot \prod\limits_{\substack{p<10^6 \\ p\equiv 1 \pmod 4}} (1-p^{-3/2})^{-1}}$$

and, for the other case, a similar estimate containing $\zeta_{\mathbb{Q}(\zeta_3)}(3/2)$. Note that the infinite product in the denominator converges a lot faster than the left hand side.

Using `Pari`, we evaluated the right hand side numerically. We found 0.39% for the quartic and 0.065% for the cubic as upper bounds for the error of approximation.

Remark 16. In practice, the error of the approximation is much smaller. The main reason is that the error-term D_p may have a positive or a negative sign. Some cancellations happen during summation. The assumption of a random distribution would result in a higher order of convergence. In fact, we observed this effect numerically.

Approximation of the Euler product. Lemma 13 allows us to determine each factor of the Euler product exactly. As we need to know the numerical value of $\tau^e_{a,b,\mathrm{fin}}$, we approximate it by a finite product.

Observe that the factor at a good prime p is simply $(1-1/p)\cdot\#V^e_{a,b}(\mathbb{F}_p)/p^3$. In particular, for this factor there are only e^2 values possible. Even more, these numbers had been precomputed using FFT point counting (Algorithm 10). The algorithm below is based on the fact that the vast majority of the factors actually do not need to be computed. They are available from a list.

Algorithm 17 **(**Compute an approximate value for $\tau^3_{a,b,\mathrm{fin}}$ $(\tau^4_{a,b,\mathrm{fin}})$**).**

i) Let p run over all prime numbers such that $p \equiv 2 \pmod 3$ ($p \equiv 3 \pmod 4$) and $p \leq N$ and calculate the product of all values of $(1-1/p^4)$.

ii) Compute the factor corresponding to $p = 3$ ($p = 2$) by Lemma 13.b).

iii) Let p run over all prime numbers such that $p \equiv 1 \pmod 3$ ($p \equiv 1 \pmod 4$) and $p \leq N$. Calculate the product of the factors described below.

If $p|ab$ then the corresponding factor is given by Lemma 13.b). Otherwise, compute the e-th power residue-symbols of a and b and look up the precomputed factor for this $\mathbb{F}_p$-isomorphism class of varieties in the list.

iv) Multiply the two products from steps i) and iii) and the factor from step ii) with each other. Correct the product by taking the bad primes $p \equiv 2 \pmod 3$ ($p \equiv 3 \pmod 4$) into consideration.

Remark 18. When we meet a bad prime p, we have to count $\mathbb{Z}/p^n\mathbb{Z}$-valued points on $V^e_{a,b}$. This is done by an algorithm which is very similar to Algorithm 10.

We used Algorithm 17 to compute the Euler products $\tau^3_{a,b,\mathrm{fin}}$ and $\tau^4_{a,b,\mathrm{fin}}$ for $a, b = 1, \ldots, 100$. We did all calculations for $N = 10^6$. Note that step i) had to be done only once for $e = 3$ and once for $e = 4$. The running-time was a quarter of an hour for either exponent.

The factor at the infinite place. For the quartic $V^4_{a,b}$, we have the integral

$$\omega_{\mathbf{H},\infty}(V^4_{a,b}(\mathbb{R})) = \frac{1}{4\sqrt[4]{a}} \iiiint\limits_R \frac{1}{(by^4+z^4+v^4+w^4)^{3/4}}\, dy\, dz\, dv\, dw$$

over $R := \{(y,z,v,w) \in \mathbb{R}^4 \mid |y|,|z|,|v|,|w| \leq 1 \text{ and } |by^4+z^4+v^4+w^4| \leq a\}$. The integrand is singular in one point. We used a simple substitution to make it sufficiently smooth for numerical integration.

On the other hand, for the cubic $V^3_{a,b}$, we have to consider

$$\omega_{\mathbf{H},\infty}(V^3_{a,b}(\mathbb{R})) = \frac{1}{6\sqrt[3]{a}} \iiiint\limits_R \frac{1}{(by^3+z^3+v^3+w^3)^{2/3}}\, dy\, dz\, dv\, dw$$

for $R := \{(y,z,v,w) \in \mathbb{R}^4 \mid |y|,|z|,|v|,|w| \leq 1 \text{ and } |by^3+z^3+v^3+w^3| \leq a\}$. The difficulty here is the handling of the singularity of the integrand. It is located in the zero set of $by^3+z^3+v^3+w^3$ in R.

Since $(by^3+z^3+v^3+w^3)^{-2/3}$ is a homogeneous function, it is enough to integrate over the boundary of R. This reduces the problem to several three-dimensional integrals of functions having a two-dimensional singular locus. If $a \geq b+3$ then R is a cube and the boundary of R is easy to describe. We restricted our attention to this case. We smoothed the singularities by separation of Puiseux expansions and substitutions. The resulting integrals were treated by the Gauß-Legendre formula [Kr].

3 On the geometry of diagonal cubic threefolds

Lemma 19. *Let $V \subset \mathbf{P}^4$ be any smooth hypersurface. Then, every (reduced but possibly singular) surface $S \subset V$ is a complete intersection $V \cap H_d$ with a hypersurface $H_d \subset \mathbf{P}^4$.*

Proof. By the Noether-Lefschetz Theorem, we have $\mathrm{Pic}(V) \cong \mathbb{Z}$. The surface S is a Weil divisor on V. Hence, $\mathscr{O}(S) = \mathscr{O}(d) \in \mathrm{Pic}(V)$ for a certain $d > 0$. The restriction $\Gamma(\mathbf{P}^4, \mathscr{O}(d)) \to \Gamma(V, \mathscr{O}(d))$ is surjective as $H^1(\mathbf{P}^4, \mathscr{O}_V(d - \deg V)) = 0$ [Ha2, Theorem III.5.1.b)]. □

Elliptic Cones. Let $V \subset \mathbf{P}^4(\mathbb{C})$ be the diagonal cubic threefold given by the equation $x^3+y^3+z^3+v^3+w^3=0$. Fix $\zeta \in \mathbb{C}$ such that $\zeta^3 = 1$. Then, for every point $(x_0 : y_0 : z_0)$ on the elliptic curve $F\colon x^3+y^3+z^3=0$, the line given by $(x:y:z) = (x_0:y_0:z_0)$ and $v = -\zeta w$ is contained in V. All these lines together form a cone C_F over F the cusp of which is $(0:0:0:-\zeta:1)$. C_F is a singular model of a ruled surface over an elliptic curve. This shows, there are no other rational curves contained in C_F.

By permuting coordinates, one finds a total of thirty elliptic cones of that type within V. The cusps of these cones are usually named *Eckardt points* [Mu,CG]. We call the lines contained in one of these cones the *obvious lines* lying on V. It is clear that there are an infinite number of lines on V running through each of the thirty Eckardt points.

Proposition 20 (cf. [Mu, Lemma 1.18]). *Let $V \subset \mathbf{P}^4$ be the diagonal cubic threefold given by the equation $x^3+y^3+z^3+v^3+w^3 = 0$. Then, through each point $p \in V$ different from the thirty Eckardt points there are precisely six lines on V.*

Proof. Let $P = (x_0 : y_0 : z_0 : v_0 : w_0)$. A line l through P and another point $Q = (x : y : z : v : w)$ is parametrized by $(s : t) \mapsto ((sx_0 + tx) : \ldots : (sw_0 + tw))$. Comparing coefficients at s^2t, st^2, and t^3, we see that the condition that l lies on V may be expressed by the three equations below.

$$x_0^2 x + y_0^2 y + z_0^2 z + v_0^2 v + w_0^2 w = 0 \tag{2}$$

$$x_0 x^2 + y_0 y^2 + z_0 z^2 + v_0 v^2 + w_0 w^2 = 0 \tag{3}$$

$$x^3 + y^3 + z^3 + v^3 + w^3 = 0 \tag{4}$$

The first equation means that Q lies on the tangent hyperplane H_P at P while equation (4) just encodes that $Q \in V$. By [Za, Corollary 1.15.b)], $H_P \cap V$ is an irreducible cubic surface.

On the other hand, the quadratic form q on the left hand side of equation (3) is of rank at least 3 as P is not an Eckardt point. Therefore, q is not just the product of two linear forms. In particular, $q|_{H_P} \not\equiv 0$.

As $H_P \cap V$ is irreducible, $Z(q|_{H_P})$ and $H_P \cap V$ do not have a component in common. By Bezout's theorem, their intersection in H_P is a curve of degree 6. □

Remark 21. It may happen that some of the six lines coincide. Actually, it turns out that a line appears with multiplicity > 1 if and only if it is obvious [Mu, Lemma 1.19]. In particular, for a general point P the six lines through it are different from each other.

Under certain exceptional circumstances it is possible to write down all six lines explicitly. For example, if $P = (\sqrt[3]{-4} : 1 : 1 : 1 : 1)$ then the line $(\sqrt[3]{-4}t : (t+s) : (t+is) : (t-s) : (t-is))$ through P lies on V. Permuting the three rightmost coordinates yields all six lines.

4 Detection of accumulating subvarieties

The detection of $\mathbb{Q}$-rational lines on the cubics. On a cubic threefold $V^3_{a,b}$, quadratic growth is predicted for the number of $\mathbb{Q}$-rational points of bounded height. Lines are the only curves with such a growth rate.

The moduli space of the lines on a cubic threefold is well-understood. It is a surface of general type [CG, Lemma 10.13]. Nevertheless, we do not know of a method to find all $\mathbb{Q}$-rational lines on a given cubic threefold, explicitly. For that reason, we use the algorithm below which is an irrationality test for the six lines through a given point $(x_0 : y_0 : z_0 : v_0 : w_0) \in V^3_{a,b}(\mathbb{Q})$.

Algorithm 22 (Test the six lines through a given point for irrationality).

i) Let p run through the primes from 3 to N.

For each p, solve the system of equations (2), (3), (4) (adapted to $V^3_{a,b}$) in $\mathbb{F}_p^5$. If the multiples of $(x_0, y_0, z_0, v_0, w_0)$ are the only solutions then output that there is no $\mathbb{Q}$-rational line through $(x_0 : y_0 : z_0 : v_0 : w_0)$ and terminate prematurely.

ii) If the loop comes to its regular end then output that the point is *suspicious*. It could possibly lie on a $\mathbb{Q}$-rational line.

Remark 23. We use a very naive $O(p)$-algorithm to solve the system of equations over $\mathbb{F}_p$. If, say, $x_0 \neq 0$ then it is sufficient to consider quintuples such that $x = 0$. We parametrize the projective plane given by (2). Then, we compute all points on the conic given by (2) and (3). For each such point, we compute the cubic form on the left hand side of (4).

We carried out the irrationality test on every $\mathbb{Q}$-rational point found on any of the cubics except for the points lying on an obvious line. We worked with $N = 600$. It turned out that suspicious points are rare and that, at least in our sample, each of them actually lies on a $\mathbb{Q}$-rational line.

We found only 42 non-obvious $\mathbb{Q}$-rational lines on all of the cubics $V^3_{a,b}$ for $100 \geq a \geq b \geq 1$ together. Among them, there are only five essentially different ones. We present them in the table below. The list might be enlarged by two, as $V^3_{21,6}$ and $V^3_{22,5}$ may be transformed into $V^3_{48,21}$ and $V^3_{40,22}$, respectively, by an automorphism of $\mathbf{P}^4$. Further, each line has six pairwise different images under the obvious operation of the group S_3.

Table 1. Sporadic lines on the cubic threefolds

a	b	Smallest point	Point s.t. $x = 0$
19	18	(1 : 2 : 3 : -3 : -5)	(0 : 7 : 1 : -7 : -18)
21	6	(1 : 2 : 3 : -3 : -3)	(0 : 9 : 1 : -10 : -15)
22	5	(1 : -1 : 3 : 3 : -3)	(0 : 27 : -4 : -60 : 49)
45	18	(1 : 1 : 3 : 3 : -3)	(0 : 3 : -1 : 3 : -8)
73	17	(1 : 5 : -2 : 11 : -15)	(0 : 27 : -40 : 85 : -96)

Remark 24. It is a priori unnecessary to search for accumulating surfaces, at least if we assume some conjectures.

First of all, only rational surfaces are supposed to accumulate that many rational points that it could be seen through our asymptotics. Indeed, a surface which is abelian or bielliptic may not have more than $O(\log^t B)$ points of height $< B$. Non-rational ruled surfaces accumulate points in curves, anyway. Further, it is expected [Pe2, Conjecture 3.6] that $K3$ surfaces, Enriques surfaces, and surfaces of Kodaira dimension one may have no more than $O(B^\varepsilon)$ points of height $< B$ outside a finite union of rational curves. For surfaces of general type, finally, expectations are even stronger (Lang's conjecture).

A rational surface S is, up to exceptional curves, the image of a rational map $\varphi\colon \mathbf{P}^2 \dashrightarrow V \subset \mathbf{P}^4$. There is a birational morphism $\varepsilon\colon P \to \mathbf{P}^2$ such that $\overline{\varphi} := \varphi \circ \varepsilon$ is a morphism of schemes. ε is given by a sequence of blowing-ups [Bv, Theorem II.11]. $\overline{\varphi}$ is defined by the linear system $|dH - E|$ where $d := \deg \varphi$, H is a hyperplane section, and E is the exceptional divisor. On the other hand, $K := K_P = -3H + E$. Therefore, if $d \geq 3$ then

$$\mathrm{H}_{\mathrm{naive}}(\overline{\varphi}(p)) = \mathrm{H}_{dH-E}(p) = \mathrm{H}_{3H-\frac{3}{d}E}(p)^{d/3} \geq c \cdot \mathrm{H}_{-K}(p)^{d/3}$$

for $p \notin \mathrm{supp}(E)$. Manin's conjecture implies there are $O((B \log^t B)^{\frac{3}{d}}) = o(B^2)$ points of height $< B$ on the Zariski-dense subset $\overline{\varphi}(P \setminus \mathrm{supp}(E)) \subseteq S$.

It remains to show that there are no rational maps $\varphi\colon \mathbf{P}^2 \dashrightarrow V$ of degree $d \leq 2$. Indeed, under this assumption, $\deg \varphi(\mathbf{P}^2) \leq 4$. This implies, by virtue of Lemma 19, that $\varphi(\mathbf{P}^2)$ is necessarily a hyperplane section $V \cap H$. Zak's theorem [Za, Corollary 1.8] shows that $V \cap H$ contains only finitely many singular points. It is, however, well known that cubic surfaces in 3-space which are the image of $\mathbf{P}^2$ under a quadratic map have a singular line [Bv, Corollary IV.8].

The detection of $\mathbb{Q}$-rational conics on the quartics. On a quartic threefold, linear growth is predicted for the number of $\mathbb{Q}$-rational points of bounded height. The assumption $b > 0$ ensures that there are no $\mathbb{R}$-rational lines contained in $V^4_{a,b}$. The only other curves with at least linear growth one could think about are conics.

We were not able to create an efficient routine to test whether there is a $\mathbb{Q}$-rational conic through a given point. The resulting system of equations seems to be too complicated to handle.

Conics through two points. A conic Q through $(x_0 : y_0 : z_0 : v_0 : w_0)$ and $(x_1 : y_1 : z_1 : v_1 : w_1)$ may be parametrized in the form

$$(s:t) \mapsto ((\lambda x_0 s^2 + \mu x_1 t^2 + xst) : \ \ldots \ : (\lambda w_0 s^2 + \mu w_1 t^2 + wst))$$

for some $x, y, z, v, w, \lambda, \mu \in \mathbb{Z}$. The condition that Q is contained in $V^4_{a,b}$ leads to a system G of seven equations in x, y, z, v, w, and $\lambda\mu$. The phenomenon that λ and μ do not occur individually is explained by the fact that they are not invariant under the automorphisms of $\mathbf{P}^1$ which fix 0 and ∞.

Algorithm 25 (Test for conic through two points)**.**

i) Let p run through the primes from 3 to N.

In the exceptional case that G could allow a solution such that $p | x, y, z, v, w$ but $p^2 \nmid \lambda\mu$, do nothing. Otherwise, solve G in $\mathbb{F}_p^6$. If $(0, 0, 0, 0, 0, 0)$ is the only solution then output that there is no $\mathbb{Q}$-rational conic through $(x_0 : y_0 : z_0 : v_0 : w_0)$ and $(x_1 : y_1 : z_1 : v_1 : w_1)$ and terminate prematurely.

ii) If the loop comes to its regular end then output that the pair is *suspicious.* It could possibly lie on a $\mathbb{Q}$-rational conic.

To solve the system G in $\mathbb{F}_p^6$, we use an $O(p)$-algorithm. Actually, comparison of coefficients at s^7t and st^7 yields two linear equations in x, y, z, v, and w. We parametrize the projective plane I given by them. Comparison of coefficients at s^6t^2 and s^2t^6 leads to a quadric O and an equation $\lambda\mu = q(x, y, z, v, w)/M$ with a quadratic form q over $\mathbb{Z}$ and an integer $M \neq 0$. The case $p | M$ sends us to the next prime immediately. Otherwise, we compute all points on the conic $I \cap O$. For each of them, we test the three remaining equations.

Conics through three points. Three points P_1, P_2, and P_3 on $V^4_{a,b}$ define a projective plane $\mathbf{P}$. The points together with the two tangent lines $\mathbf{P} \cap T_{P_1}$ and $\mathbf{P} \cap T_{P_2}$ determine a conic Q, uniquely. It is easy to transform this geometric insight into a formula for a parametrization of Q. We then need a test whether a conic given in parametrized form is contained in $V^4_{a,b}$. This part is algorithmically simple but requires the use of multiprecision integers.

Detecting conics. For each quartic $V^4_{a,b}$, we tested every pair of $\mathbb{Q}$-rational points of height $< 10\,000$ for a conic through them. The existence of a conic through (P, Q) is equivalent to the existence of a conic through (gP, gQ) for $g \in (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_3 \subseteq \mathrm{Aut}(V^4_{a,b})$. This reduces the running time by a factor of about 96. Further, pairs already known to lie on the same conic were excluded from the test.

For each pair (P, Q) found suspicious, we tested the triples (P, Q, R) for R running through the $\mathbb{Q}$-rational points of height $< 10\,000$, until a conic was found. Due to the symmetries, one finds several conics at once. For each conic detected, all points on it were marked as lying on this conic.

Actually, there were a few pairs found suspicious through which no conic could be found. In any of these cases, it was easy to prove by hand that there is actually no $\mathbb{R}$-rational conic passing through the two points. This means, we detected every conic which meets at least two of the rational points of height $< 10\,000$.

The conics found. Up to symmetry, we found a total of 1 533 $\mathbb{Q}$-rational conics on all of the quartics $V^4_{a,b}$ for $1 \leq a, b \leq 100$ together.

Among them, 1 410 are contained in a plane of type $z = v + w$ and $Yx - Xy = 0$ for (X, Y, t) a rational point on the genus one curve $aX^4 - bY^4 = 2t^2$. Further, there are 90 conics which are slight modifications of the above with y interchanged with z, v, or w. This is possible if b is a fourth power.

There is a geometric explanation for the occurrence of these conics. The hyperplane given by $z = v + w$ intersects $V^4_{a,b}$ in a surface S with the two singular points $(0 : 0 : -1 : e^{\pm 2\pi i/3} : e^{\mp 2\pi i/3})$. The linear projection $\pi\colon S \dashrightarrow \mathbf{P}^1$ to the first two coordinates is undefined only in these two points. Its fibers are plane quartics which split into two conics as $(v+w)^4 + v^4 + w^4 = 2(v^2 + vw + w^2)^2$. After resolution of singularities, the two conics become disjoint. $\widetilde{S}$ is a ruled surface over a twofold cover of $\mathbf{P}^1$ ramified in the four points such that $ax^4 - by^4 = 0$, i.e. over a curve of genus one.

In the case a is twice a square, a different sort of conics comes from the equations $v = z + Dy$ and $w = Ly$ when (L, D) is a point on the affine genus three curve $C_b\colon L^4 + b = D^4$. We found 28 conics of this type. C_b has a $\mathbb{Q}$-rational point for $b = 5$, 15, 34, 39, 65, 80, and 84. The conics actually admit a $\mathbb{Q}$-rational point for $a = 2$, 18, 32, and 98.

The remaining five conics are given as follows. For $a = 3$, 12, 27, or 48 and $b = 10$, intersect with the plane given by $v = y + z$ and $w = 2y + z$. For $a = 17$ and $b = 30$, put $v = 2x + y$ and $w = x + 3y + z$.

Remark 26. Again, it is not necessary to search for accumulating surfaces. Here, rational maps $\varphi\colon \mathbf{P}^2 \dashrightarrow V \subset \mathbf{P}^4$ such that $\deg\varphi \leq 3$ need to be taken into consideration. We claim, such a map is impossible.

If $\deg\varphi = 3$ then we had $\varphi\colon (\lambda : \mu : \nu) \mapsto (K_0(\lambda,\mu,\nu) : \ldots : K_4(\lambda,\mu,\nu))$ where $K_0, \ldots, K_4$ are cubic forms defined over $\mathbb{Q}$. $K_0 = 0$ defines a plane cubic which has infinitely many real points, automatically. As the image of φ is assumed to be contained in $V^4_{a,b}$, we have that $K_0(\lambda,\mu,\nu) = 0$ implies $K_1(\lambda,\mu,\nu) = \ldots = K_4(\lambda,\mu,\nu) = 0$ for $\lambda,\mu,\nu \in \mathbb{R}$. By consequence, $K_1, \ldots, K_4$ are divisible by K_0 (or by a linear factor of K_0 in the case it is reducible) and φ is not of degree three.

For $\deg\varphi \leq 2$, we had $\deg\overline{\varphi(\mathbf{P}^2)} \leq 4$ such that $\overline{\varphi(\mathbf{P}^2)} = V \cap H$ is a hyperplane section. Zak's theorem [Za, Corollary 1.8] shows it has at most finitely many singular points. On the other hand, a quartic in $\mathbf{P}^3$ which is the image of a quadratic map from $\mathbf{P}^2$ is a Steiner surface. It is known [Ap, p. 40] to have one, two, or (in generic case) three singular lines.

5 The final results

A technology to find solutions of Diophantine equations. In [EJ1] and [EJ2], we described a modification of D. Bernstein's [Be] method to search efficiently for all solutions of naive height $< B$ of a Diophantine equation of the particular form $f(x_1, \ldots, x_n) = g(y_1, \ldots, y_m)$. The expected running-time of our algorithm is $O(B^{\max\{n,m\}})$. Its basic idea is as follows.

Algorithm 27 (Search for solutions of a Diophantine equation)**.**

i) (Writing)
Evaluate f on all points of the cube $\{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid |x_i| < B\}$ of dimension n. Store the values within a hash table H.

ii) (Reading)
Evaluate g on all points of the cube $\{(y_1, \ldots, y_m) \in \mathbb{Z}^m \mid |y_i| < B\}$. For each value, start a search in order to find out whether it occurs in H. When a coincidence is detected, reconstruct the corresponding values of $x_1, \ldots, x_n$ and output the solution.

Remark 28. In the case of a variety $V^e_{a,b}$, the running-time is obviously $O(B^3)$. We decided to store the values of $z^e + v^e + w^e$ into the hash table. Afterwards, we have to look up the values of $ax^e - by^e$.

In this form, the algorithm would lead to a program in which almost the entire running-time is consumed by the writing part. Observe, however, the following particularity of our method. When we search on up to $O(B)$ threefolds, differing only by the values of a and b, simultaneously, then the running-time is still $O(B^3)$.

We worked with $B = 5\,000$ for the cubics and $B = 10\,000$ for the quartics. In either case, we dealt with all threefolds arising for $a, b = 1, \ldots, 100$, simultaneously. For the quartics, the running-time was around four days of CPU time.

This is approximately only three times longer than searching on a single threefold had lasted. For the cubics, a program with integrated line detection took us approximately ten days.

The result for the cubics. We counted all $\mathbb{Q}$-rational points of height less than 5 000 on the threefolds $V^3_{a,b}$ where $a, b = 1, \dots, 100$ and $b \leq a$. Note that $V^3_{a,b} \cong V^3_{b,a}$. Points lying on one of the elliptic cones or on a sporadic $\mathbb{Q}$-rational line in $V_{a,b}$ were excluded from the count. The smallest number of points found is 3 930 278 for $(a, b) = (98, 95)$. The largest numbers of points are 332 137 752 for $(a, b) = (7, 1)$ and 355 689 300 in the case that $a = 1$ and $b = 1$.

On the other hand, for each threefold $V^3_{a,b}$ whereas $a, b = 1, \dots, 100$ and $b + 3 \leq a$, we calculated the expected number of points and the quotients

$$\#\{\text{points of height} < B \text{ found}\} \,/\, \#\{\text{points of height} < B \text{ expected}\}.$$

Let us visualize the quotients by two histograms.

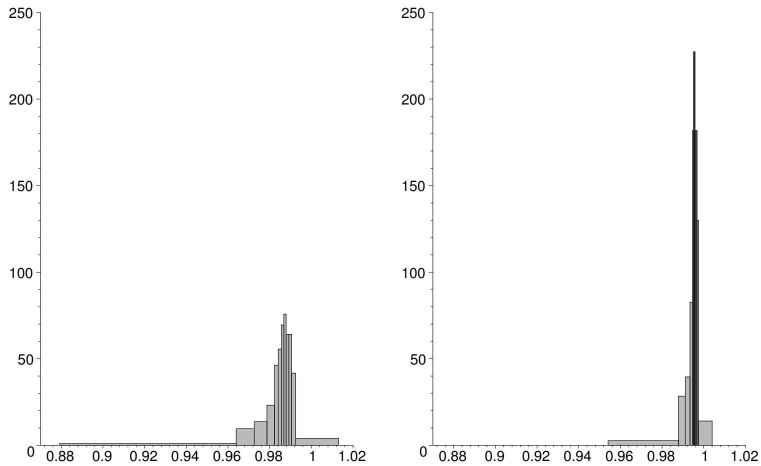

Fig. 1. Distribution of the quotients for $B = 1\,000$ and $B = 5\,000$.

The statistical parameters are listed in the table below.

Table 2. Parameters of the distribution in the cubic case

	$B = 1\,000$	$B = 2\,000$	$B = 5\,000$
mean value	0.981 79	0.988 54	0.993 83
standard deviation	0.012 74	0.008 23	0.004 55

The results for the quartics. We counted all $\mathbb{Q}$-rational points of height less than 10 000 on the threefolds $V^4_{a,b}$ where $a, b = 1, \ldots, 100$. It turns out that on 5 015 of these varieties, there are no $\mathbb{Q}$-rational points occurring at all as the equation is unsolvable in $\mathbb{Q}_p$ for some small p. In this situation, Manin's conjecture is true, trivially.

Further, there is the case $(a, b) = (58, 87)$ in which the smallest 96 solutions are the images of $(6\,465 : 637 : 4\,321 : 6\,989 : 17\,719)$ under the obvious operation of the group $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_3$. Here, $\tau_{\mathbf{H}}(V^4_{58,87}) \approx 0.002\,722$.

For the remaining varieties, the points lying on a known $\mathbb{Q}$-rational conic in $V_{a,b}$ were excluded from the count. Table 3 shows the quartics sorted by the numbers of points remaining.

Table 3. Numbers of points of height < 10 000 on the quartics.

a	b	#points	# not on conic	# expected
29	29	2	2	13.5
58	58	2	2	38.8
51	71	96	96	319.8
87	87	98	98	35.7
⋮	⋮	⋮	⋮	⋮
34	1	995 808	569 088	567 300
17	64	581 640	581 640	564 300
1	14	682 830	598 038	648 300
3	1	1 262 048	739 008	752 600

We see that the variation of the quotients is higher than in the cubic case.

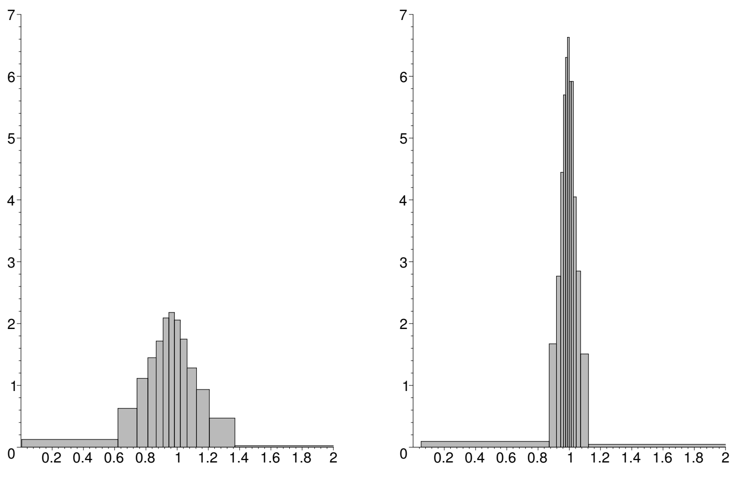

Fig. 2. Distribution of the quotients for $B = 1\,000$ and $B = 10\,000$.

The statistical parameters are listed in the table below.

Table 4. Parameters of the distribution in the quartic case

	$B = 1\,000$	$B = 10\,000$
mean value	0.9853	0.9957
standard deviation	0.3159	0.1130

Interpretation of the result. The results suggest that Manin's conjecture should be true for the two families of threefolds considered. In the cubic case, the standard deviation is by far smaller than in the case of the quartics. This, however, is not very surprising as on a cubic there tend to be much more rational points than on a quartic. This makes the sample more reliable.

Remark 29. The data we collected might be used to test the sharpening of the asymptotic formula (1) suggested by Sir P. Swinnerton-Dyer [S-D].

Question 30. Our calculations seem to indicate that the number of rational points often approaches its expected value from below. Is that more than an accidental effect?

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