RATIONAL POINTS ON SOME FANO QUADRATIC BUNDLES

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Abstract. We study the number of rational points of bounded height on a certain threefold. The accumulating subvarieties are Zariski-dense in this example. The computations support an extension of a conjecture of Manin to this situation.

1. Introduction

The set of rational points on varieties is one of the central objects in arithmetic geometry. For Fano varieties, many rational points are expected (at least after an extension of the ground field). So one could ask for the number of such points of bounded height. This leads to the famous conjecture of Y. I. Manin [FMT].

Conjecture 1. *(Manin) Let V be an arbitrary Fano variety and H be the anticanonical height. Then, there exists a dense, Zariski open subset $V^\circ \subset V$ such that for each number-field K*

$$\#\{x \in V^\circ(K) \mid H(x) < B\} \sim C \cdot B \log^r B$$

holds. r is expected to be $\operatorname{rk}\operatorname{Pic}(V) - 1$.

To illustrate the choice of the Zariski open subset $V^\circ \subset V$ we give an example.

Example 2. A smooth cubic surface with 27 rational lines has Picard rank 7. So we expect $\sim C \cdot B \log^6 B$ rational points of height at most B. But every line has $\sim CB^2$ rational points. In this case one chooses V° as the complement of the lines.

Examples such that the rational points in the exceptional set have the same magnitude as the rational points in V° are given in [EJ2] and [Ho].

The Manin conjecture is proven in a number of special cases. In the case of high dimension and low degree, this can be done by using the circle-method [Bi]. Beyond this, there are a lot of (theoretical and numerical) results supporting this conjecture. On the other hand, we have the following counterexample [BT]:

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Example 3. Let $V \subset \mathbf{P}^3 \times \mathbf{P}^3$ be the Fano variety

$$\{([a:b:c:d],[x:y:z:w]) \mid ax^3+by^3+cz^3+dw^3=0\}.$$

The number of rational points over $\mathbb{Q}(\zeta_3)$ grows at least as fast as $B\log^3 B$ on each non-empty Zariski-open subset but the conjecture predicts $\sim C\,B\log B$.

The point is that the fibers with a,b,c,d cubes are cubic surfaces with Picard rank 7. On these fibers $C\,B\log^6 B$ rational points are expected. The lower bound $B\log^3 B$ is proven. As these fibers form a Zariski-dense subset we get a counterexample to the Manin conjecture.

At this point at least the following possibilities arise:

- The class of all Fano varieties is too big for a uniform conjecture.
- The value of the constant r has to be modified.
- The requirement of V° to be Zariski open is too strong.

Remark 4. Aside from this counterexample several people have extended the conjecture. Most important is a conjectural value of the constant C. This is expected to be a Tamagawa-type number introduced by E. Peyre [Pe1]. (See below for more details.)

Furthermore, people have tried to construct a more precise asymptotic formula for the number of rational points [SD]. It is given by $BP(\log B)+O(B^{\frac{1}{2}+\varepsilon})$. Here, P is a polynomial of degree $\operatorname{rk}\operatorname{Pic}(V)-1$ with leading coefficient C and ε is some value in $(0,\frac{1}{2})$. See [BBD] for a proof in the case of a special singular cubic surface.

In this note we focus on varieties of the form

$$ax^2+by^2+l_1(a,b)z^2+l_2(a,b)w^2=0\,. \tag{1}$$

in $\mathbf{P}^1 \times \mathbf{P}^3$. Here l_1 and l_2 are two linear forms. This example was suggested by Emmanuel Peyre and Yuri Tschinkel during the arithmetic and algebraic geometry conference of higher-dimensional varieties (Bristol 2009). After a study of the geometry of these varieties we will search for rational points and compare their number with the predicted value.

For a numerical check of the Manin-Conjecture in simpler cases the reader might consult [PT], [EJ2] or [EJ3]. Arguments and computations carried out in great detail there are only sketched in this note.

2. The Tamagawa number

For an arbitrary Fano variety, a conjectural value of C was introduced by E. Peyre. It is an infinite product of Tamagawa type. We will recall this constant in a special case, which is yet general enough for our situation.

More precisely, we will work over $\mathbb{Q}$ and we assume that the Galois action on the Picard group of the variety is trivial. The latter implies that we do not have to care about the Brauer-Manin-obstruction. Furthermore, we will restrict to varieties given by a single equation in a product of projective spaces. Then, C is given as the following product

$$C = \alpha \prod_{p \in \mathbb{P} \cup \{\infty\}} \tau_p \, .$$

For a prime $p \in \mathbb{P}$ the local factor τ_p is given by

$$\tau_p = \left(1 - \frac{1}{p}\right)^{\operatorname{rk}\operatorname{Pic}(V)} \cdot \lim_{k \to \infty} \frac{\#V(\mathbb{Z}/p^k\mathbb{Z})}{p^{k \dim V}} \, .$$

At the infinite place we have the following formula

$$\tau_\infty = \frac{(n-d+1)(m-e+1)}{4} \int_{CU \cap N} \omega_{\mathrm{Leray}}$$

for a variety in $\mathbf{P}^n \times \mathbf{P}^m$ given by a polynomial of bi-degree (d, e). Here, CU is the affine cone given by the equation and N is $[-1, 1]^{n+m+2}$. Finally, α is given by

$$\alpha = \operatorname{rk}\operatorname{Pic}(V) \cdot \operatorname{vol}\{x \in \Lambda_{\mathrm{eff}}^\vee \mid \langle x \mid -K \rangle \leq 1\} \, .$$

Here $\Lambda_{\mathrm{eff}}^\vee$ is the dual cone of the cone of effective divisors $\Lambda_{\mathrm{eff}}(V) \subset \operatorname{Pic}(V)^\vee \otimes \mathbb{R} = \mathbb{R}^{\operatorname{rk}\operatorname{Pic}(V)}$. Here, vol denotes the Lebesgue measure on $\operatorname{Pic}(V)^\vee \otimes \mathbb{R}$, normalized such that the primitive cell of the lattice $\operatorname{Pic}(V)^\vee$ is of measure 1.

3. Computation of the Tamagawa number

Now, we consider a variety of the form (1). Note that the Picard group is isomorphic to $\mathbb{Z}^2$.

One local factor. At a place of good reduction, the local factor is

$$\tau_p = \left(1 - \frac{1}{p}\right)^2 \frac{\#V(\mathbb{F}_p)}{p^3} \, .$$

At a place of bad reduction, the sequence in the definition becomes stationary after a finite number of steps.

At a place of good reduction, $\#V(\mathbb{F}_p)$ can be computed as follows. The variety V is given by the equation $ax^2 + by^2 + l_1(a,b)z^2 + l_2(a,b)w^2 = 0$. For a fixed point $[a : b] \in \mathbf{P}^1$ we get a quadric in $\mathbf{P}^3$. As we are at a place of good reduction only the following three cases are possible.

i) The quadric is smooth and the discriminant is not a square. Then, it has $p^2 + 1$ points.

ii) The quadric is smooth and the discriminant is a square. Then, it has $p^2 + 2p + 1$ points.

iii) The quadric is singular. Exactly one coefficient is zero. In this case, we get $(p+1)p+1$ points. This happens exactly 4 times.

Summarizing, we get $p^2+p+1+\left(\frac{F(a,b)}{p}\right)p$ points on a fiber. Here, we set $F(a,b) := ab\, l_1(a,b)\, l_2(a,b)$.

We introduce the elliptic curve

$$E\colon u^2 = F(a,b)$$

as a double cover of $\mathbf{P}^1$. Points on E with $u \neq 0$ correspond to split quadrics. Thus we get the following formula

$$\begin{aligned}\#V(\mathbb{F}_p) &= 4(p^2+p+1) + (p^2+1)(p+1-4) + (\#E(\mathbb{F}_p)-4)p \\ &= p^3+2p^2+2p+1-pT_p(E)\,.\end{aligned}$$

Here, $T_p(E)$ denotes the trace of Frobenius on E at p. Note that $|T_p(E)| < 2\sqrt{p}$ by Hasse's theorem. This leads to

$$\begin{aligned}\tau_p &= \left(1-\frac{2}{p}+\frac{1}{p^2}\right)\frac{p^3+2p^2+2p+1-pT_p(E)}{p^3} \\ &= 1-\frac{1}{p^2}-\frac{1}{p^3}+\frac{1}{p^5}-T_p(E)\left(\frac{1}{p^2}-\frac{2}{p^3}+\frac{1}{p^4}\right)\,.\end{aligned}$$

The infinite product. The infinite product of the local factors calculated above is absolutely convergent. The local factors have many similarities with local factors of the L-series L of E and the Riemann zeta function ζ. The local factor $L_p(s)$ is given by $(1-T_p(E)p^{-s}+p^{1-2s})^{-1}$ at places of good reduction and $\zeta_p(s)$ is simply $(1-p^{-s})^{-1}$. Thus we have $\zeta(s) = \prod_p \zeta_p(s)$ and $L(s) = \prod_p L_p(s)$. Extracting such factors improves the convergence properties. We get

$$\prod_p \tau_p = \frac{L(3)^2}{\zeta(2)\zeta(3)^2 L(2)L(4)^2}\prod_p \frac{\zeta_p(2)\zeta_p(3)^2 L_p(2)L_p(4)^2}{L_p(3)^2}\tau_p\,.$$

The factors of the new product are $1+\frac{2T_p(E)^2}{p^5}+O(p^{-9/2})$.

In this way, we can evaluate the Euler product with a precision of 14 digits within a few minutes. Most of the time is used for the evaluation of the L-series. Here we use `magma`. See [Dok] for details of the algorithm.

The infinite place. The factor at the infinite place is a 5-dimensional integral on a compact domain. By using homogeneity we can reduce to a or b equal 1 and $1 \in \{x,y,z,w\}$. This is a sum of eight 3-dimensional integrals. The summand for $a = x = 1$ is given by

$$\iiint \frac{1}{|y^2+l_1(0,1)z^2+l_2(0,1)w^2|}\,dy\,dz\,dw\,.$$

The other seven cases are similar. The domain of integration for the case $a = x = 1$ is given by

$$\left\{(y, z, w) \in [0, 1]^3 \mid \left|\frac{1 + l_1(1, 0)z^2 + l_2(1, 0)w^2}{y^2 + l_1(0, 1)z^2 + l_2(0, 1)w^2}\right| \leq 1\right\} .$$

The innermost integral can be evaluated by hand. The remaining 2-dimensional integral can be evaluated by using standard methods from numerical analysis. Here, we apply an adaptive version of the iterated Gauss-Legendre method. We expect a precision of at least six decimal places.

4. Some subvarieties of the threefold

The quadric fibration. First recall the fibration into quadrics. A fixed point $[a : b] \in \mathbf{P}^1$ leads to the quadric $V_{a,b}$ in $\mathbf{P}^3$. This quadric is singular if and only if one of $a, b, l_1(a, b), l_2(a, b)$ is zero.

The quadric is a split quadric (i.e., has Picard rank 2 over $\mathbb{Q}$) if and only if its discriminant is a square and it has a rational point.

Thus, a necessary condition for a quadric to be split is that it corresponds to a point on the elliptic curve E introduced above by the equation $u^2 = ab\, l_1(a, b)\, l_2(a, b)$. Furthermore note that the map

$$[a : b\, ;\, u] \mapsto \left(\frac{a}{b}, \frac{l_1(a, b)}{b}, \frac{l_2(a, b)}{b}\right) \in \left(\mathbb{Q}/\mathbb{Q}^2\right)^3$$

for $[a : b\, ;\, u] \in E(\mathbb{Q})$ with $u \neq 0$ extends to a homomorphism $\phi\colon E(\mathbb{Q}) \to \left(\mathbb{Q}^*/\mathbb{Q}^{*2}\right)^3$ with kernel $2E(\mathbb{Q})$ [Si, Chap. X, Prop. 1.4].

This shows that if $[a : b\, ;\, u]$ leads to a split quadric, than all points in the coset $[a : b\, ;\, u] + 2E(\mathbb{Q})$ lead to singular or split quadrics.

If E has positive rank and at least one split quadric exists then all split quadrics form a Zariski-dense subset of the threefold. We will give examples for this below.

An elliptic cylinder. Another interesting subvariety of V is the following. Rewrite the equation as $aq_1(x, y, z, w) + bq_2(x, y, z, w) = 0$. From this we get the subvariety

$$\mathbf{P}^1 \times \{[x : y : z : w] \in \mathbf{P}^3 \mid q_1(x, y, z, w) = q_2(x, y, z, w) = 0\} .$$

This is the product of $\mathbf{P}^1$ and a genus 1 curve.

One could count the points on $V(\mathbb{F}_p)$ by checking whether $[x : y : z : w] \in \mathbf{P}^3$ leads to a point or a line on V. This leads to the observation that this genus 1 curve has the same Frobenius trace as the elliptic curve E given above. Further one can check that both curves have the same j-invariant. But it may happen that only one of the curves has a rational point.

5. Counting rational points

The point counting algorithm. Our variety V is given by the equation

$$ax^2 + by^2 + l_1(a,b)z^2 + l_2(a,b)w^2 = 0$$

in $\mathbf{P}^1 \times \mathbf{P}^3$. Because of symmetry we restrict to non-negative values for x, y, z, w. Points with one zero are counted with weight $\frac{1}{2}$. Points with two or three zeros are counted with weight $\frac{1}{4}$ or $\frac{1}{8}$.

Recall that the anticanonical height is given by the formula

$$H([a:b\,;\,x:y:z:w]) = H_{\text{naive}}([a:b])H_{\text{naive}}([x:y:z:w])^2\,.$$

For a point of bounded height, at least one of the factors is small. This observation leads to the following splitting. Choose a search bound B and an auxiliary bound A.

For all $[a:b] \in \mathbf{P}^1$ with $H_{\text{naive}}([a:b]) < A$, search for rational points on the quadric $C_{a,b}$ of naive height at most $\sqrt{\frac{B}{H([a:b])}}$.

For all points $[x:y:z:w] \in \mathbf{P}^3$ with $H_{\text{naive}}([x:y:z:w]) < \sqrt{\frac{B}{A}}$, solve the linear Diophantine equation for $[a:b]$. If this is the zero equation we get a line on the elliptic cylinder. If it is not the zero equation we get a point. We take it if the height is below $\frac{B}{H([x:y:z:w])^2}$.

The optimal value of the auxiliary bound A depends on details of the implementation and the machine. We took $A = 400$ for $B = 10^8$. Searching for points on $C_{a,b}$ is fast and easy. Just apply the ideas of [EJ1], [EJ2] and [EJ4]. Some optimizations are possible by using congruences.

Rational points on subvarieties. Recall that the Manin conjecture is proven for degree 2 surfaces. It leads to $\sim CB$ points of height at most B on non-split and $\sim CB\log B$ points on split quadrics.

The singular fibers lead to $\sim CB\log B$ points of anticanonical height at most B on V.

Counting points on the elliptic cylinder means counting points on the elliptic curve itself. As the heights of rational points on elliptic curves grow fast we get $\sim CB^2$ rational points on the elliptic cylinder with anticanonical height at most B on V.

Numerical results. To get an overview of the different phenomena we list all the data for three examples. Choosing examples is always more or less random. Here we take examples with a considerable proportion of points on the split and the singular quadrics.

$$V_1 : ax^2 + by^2 + (a - b)z^2 + (a - 2b)w^2 = 0$$
$$V_2 : ax^2 + by^2 + (2a - b)z^2 + (-6a + b)w^2 = 0$$
$$V_3 : ax^2 + by^2 + (-91a - 92b)z^2 + (99a + 100b)w^2 = 0$$

We apply the point search algorithm described above. Points on the elliptic cylinder were excluded from the count. We counted the points in each height-interval of length 10^5 up to 10^8. Some of the values are listed in Tables 1 and 2.

Number of points	V_1	V_2	V_3
at all	593147.75	910514	313622
on singular quadrics	262910.75	276816	111
on split quadrics	0	121182.5	263390
on non-split quadrics	330237	512515.5	50121

Table 1: Number of rational points of anticanonical height below 10^5

Number of points	V_1	V_2	V_3
at all	923032815.25	1545094531.75	468691343.75
on singular quadrics	377837495.25	404003678	173805
on split quadrics	0	183375308	381163936.5
on non-split quadrics	545195320	957715545.75	87353602.25

Table 2: Number of rational points of anticanonical height below 10^8

We approximate the number of rational points by functions of the form $c_1 B \log(B) + c_2 B$ using the least-square method. The RMS-error of the approximations is always below 0.03%. The results are listed in table 3.

Set of points	Number of points of height below B
on V_1 at all	$0.477469B\log(B) + 0.436195B$
without singular quadrics	$0.312111B\log(B) - 0.297472B$
on non-split quadrics	$0.312111B\log(B) - 0.297472B$
on V_2 at all	$0.925062B\log(B) - 1.588194B$
without singular quadrics	$0.742798B\log(B) - 2.272602B$
on non-split quadrics	$0.652422B\log(B) - 2.441540B$
on split quadrics	$0.090376B\log(B) + 0.168938B$
on V_3 at all	$0.225190B\log(B) + 0.538876B$
without singular quadrics	$0.225028B\log(B) + 0.540088B$
on non-split quadrics	$0.054220B\log(B) - 0.125521B$
on split quadrics	$0.170809B\log(B) + 0.665609B$

Table 3: Experimental formulas for the number of rational points

The values of the Tamagawa numbers are listed in table 4.

	V_1	V_2	V_3
α	0.25	0.25	0.25
τ_∞	2.8331245	3.2014086	0.34700116
$\prod_p \tau_p$	0.441828484431	0.819836740267	0.627048795633
C_{Peyre}	0.3129388	0.6561581	0.05439666

Table 4: Approximated values of Tamagawa numbers

Conclusion. We observe a good coincidence of Peyre's constant and the leading coefficient for the approximated formula of the number of rational points outside the elliptic cylinder on smooth non-split fibers.

The split-quadrics. Now we take a closer look on the split quadrics. The split quadrics for V_2 are given by

$$\begin{aligned}(a,b) \in \{\ & (1,-2),(1,-48),(1,-6),(1,3),(1,4),(4,-1),(9,50),\\ & (25,54),(49,-1058),(169,867),(289,676),(529,-294),\\ & (3600,-36481),(36481,-43200),(43681,116162),\\ & (58081,262086),(8958049,52911076),\\ & (13227769,26874147),(17497489,-837218),\dots\}\,.\end{aligned}$$

The split quadrics for V_3 are given by

$$\begin{aligned}(a,b) \in \{\ & (1,-1),(1,-25),(4900,-8019),(96100,-107811),\\ & (101761,-198025),(261121,-259081),\\ & (504100,-527571),(2989441,-2961841),\dots\}\,.\end{aligned}$$

As the coefficients of these quadrics grow fast, the density of rational points decreases fast. In the search range only the first two split-quadrics on V_3 have rational points.

We calculate $\sum \frac{\tau(C_{a,b})}{H_{\text{naive}}([a:b])}$. I.e., we compute the sum of the Tamagawa-numbers of the split-quadrics. The convergence of the series is fast. The smallest contribution of a quadric listed above is of magnitude 10^{-16}. The sum over these quadrics is very close to the limit. For V_2 (resp. V_3) this sum is approximately 0.0903666 (resp. 0.170788).

Conclusion. We observe a good coincidence of the sum of the Tamagawa numbers of the split quadrics and the leading coefficient of the experimental formula for the number of rational points in these fibers.

6. Discussion

The computations show that the Manin conjecture as presented in the introduction does not hold for quadratic bundles, as it allows only to exclude Zariski-closed sets like the the singular fibers and the elliptic cylinder. For this the smooth split fibers contain to many points and the predicted number of points is to small.

The examples suggest the following possibilities:

- Modify the formula for the expected number of rational points. I. e., add a term for the contribution of the split-fibers.
- Allow the exceptional set to be a *thin* Zariski-dense part of the variety. For examples fibers given by rational points on an elliptic curve.

For a theoretic treatment, compare E. Peyre's talk *Freedom and goodness*, given at the the conference *Arithmetic and algebraic geometry of higher-dimensional varieties, Bristol, September 2009.*

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