Explicit Galois realization of transitive groups of degree up to 15

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We describe methods for the construction of polynomials with certain types of Galois groups. As an application we deduce that all transitive groups G up to degree 15 occur as Galois groups of regular extensions of $\mathbb{Q}(t)$, and in each case compute a polynomial $f \in \mathbb{Q}[x]$ with $\mathrm{Gal}(f) = G$.

1. Introduction

Until now, the inverse problem of Galois theory, i.e., the question whether every finite group occurs as the Galois group of a field extension of $\mathbb{Q}$, has not been solved. Even less is known in the direction of explicit results. Complete results for permutation groups of small degree were previously only published for degrees up to eleven (see Eichenlaub (1996), Malle and Matzat (1999)). More then ten years ago, Malle (1987) completed the explicit realization of primitive non-solvable permutation groups of degree $d \leq 15$ as Galois groups over $\mathbb{Q}$. The purpose of this paper is to extend this result to cover all transitive permutation groups up to this degree. Thus we give polynomials for the 477 transitive groups of degree between 12 and 15. Note that there exist 1954 distinct transitive groups of degree 16. In fact we verify the following stronger result:

THEOREM 1.1. *The transitive groups of degree $d \leq 15$ have regular Galois realizations over $\mathbb{Q}(t)$.*

The methods presented here would in principle allow to explicitly construct such regular field extensions in all cases.

We encounter two types of problems. First, as mentioned above, not all the groups in the range were even theoretically known to occur as Galois groups over $\mathbb{Q}$. Secondly, there arises the practical problem how to come from theoretical existence results to explicit polynomials. We recall some of the approaches to the first problem and give algorithmic solutions for the second. The algorithms are not restricted to the ground field $\mathbb{Q}$ but work, for example, for any Hilbertian field.

An important tool in the constructions is a Galois group program which for a polynomial $f \in \mathbb{Z}[x]$ gives a permutation representation of $\mathrm{Gal}(f)$ on the set of complex (approximations to the) roots of f, as provided by the computer algebra system Kant

(Daberkow *et al.*, 1997). We are grateful to the Kant-team for obtaining access to this facility prior to its official release.

Conversely, the polynomials constructed here can be (and have been) used as comprehensive test input for Galois group programs.

2. General methods

As noted in the introduction, it is not known whether every finite group is a Galois group over $\mathbb{Q}$. In particular, there does not (yet) exist a general method which would allow, starting from some finite group G, to construct an extension $N/\mathbb{Q}$ with Galois group G. Nevertheless, there exist approaches allowing to handle certain subclasses of groups. In this section we recall and describe some of these methods for the (explicit) construction of field extensions with given Galois group. Very roughly, these divide into the rigidity method on the one hand, which first constructs regular extensions over some rational function field and then invokes Hilbert's irreducibility theorem, and into applications of class field theory on the other. But we will stress another subdivision, which relates to the structure of groups to be considered. First, in order to get started, one needs to realize simple groups as Galois groups. As a second step, field extensions with composite groups can be obtained by solving embedding problems. Let's first recall the approaches used to tackle simple groups.

2.1. Construction of extensions with simple Galois group

The most successful method so far for realizing finite simple groups as Galois groups is via rigidity. For cyclic groups of prime order, Kummer theory provides an alternative method. We first review a very special case of rigidity and then explain how in some cases a descent to subgroups is possible; for details and proofs we refer the reader to Malle and Matzat (1999), Chap. 1.

2.1.1. The rigidity method

Let G be a finite group, and assume that the center $Z(G)$ has a complement in G. A triple $\mathbf{C} := (C_1, C_2, C_3)$ of conjugacy classes of G is called *rigid*, if the set

$$\{(\sigma_1, \sigma_2, \sigma_3) \mid \sigma_i \in C_i,\ \sigma_1\sigma_2\sigma_3 = 1,\ \langle\sigma_1, \sigma_2\rangle = G\}$$

is non-empty and a single orbit under G-conjugation. To any conjugacy class C we attach the cyclotomic field extension of $\mathbb{Q}$ generated by the values of the irreducible characters of G on C. Let $\mathbb{Q}_\mathbf{C}$ be the composite of these extensions for the three classes in $\mathbf{C}$. The class vector $\mathbf{C}$ is called *rationally rigid* if $\mathbb{Q}_\mathbf{C} = \mathbb{Q}$.

Assume that $\mathbf{C}$ is rigid. Then the Rigidity Theorem states that there exists a regular Galois extension $N/\mathbb{Q}_\mathbf{C}(t)$ with group G, i.e., an extension such that $\mathbb{Q}_\mathbf{C}$ is algebraically closed in N. Moreover, $N/\mathbb{Q}_\mathbf{C}(t)$ is ramified in precisely three points, and the ramification is described in terms of $\mathbf{C}$. This allows to explicitly construct a generating polynomial for $N/\mathbb{Q}_\mathbf{C}(t)$ by solving a system of non-linear equations which can be derived solely from the knowledge of $\mathbf{C}$ (see Malle (1987), Malle and Matzat (1999) for worked out examples).

By Hilbert's irreducibility theorem, for any such extension there exist infinitely many specializations of t such that the specialized extension of $\mathbb{Q}_\mathbf{C}$ is Galois with group G. In particular, a rationally rigid class vector leads to infinitely many Galois realizations of G over $\mathbb{Q}$ which can be constructed explicitly.

It has turned out that this criterion is particularly well-suited for an application to almost simple groups, i.e., to groups lying between a non-abelian simple group and its automorphism group (see Malle and Matzat (1999)). A variant, which takes into account the action of the cyclotomic character, can also be applied to cyclic groups (which have rigid, but not rationally rigid class vectors) and gives regular Galois realizations over $\mathbb{Q}(t)$ for all cyclic groups (see Malle and Matzat (1999), Chap. 3.4).

2.1.2. Descent to subgroups

The Galois extensions $N/\mathbb{Q}(t)$ constructed by the rigidity method lend themselves to a descent trick which under favorable circumstances allows the realization of subgroups as well. So assume that $N/\mathbb{Q}(t)$ is a regular Galois extension with group G ramified in three prime divisors of degree 1 of $\mathbb{Q}(t)$. Let $H \leq G$ be a subgroup and $K := N^H$ its fixed field, so N/K is Galois with group H. If it can be shown that K is again a rational function field $K = \mathbb{Q}(u)$, then this yields a regular Galois realization of H over $\mathbb{Q}(t)$, which by the Hilbert irreducibility theorem has infinitely many specializations to Galois extensions of $\mathbb{Q}$ with group H. This descent procedure always works if H is of index 2 in G. Indeed, in this case H is normal, the Galois extension $K/\mathbb{Q}(t)$ has Galois group Z_2 and is ramified in precisely two prime divisors. The Hurwitz relative genus formula then shows that K has genus 0, and moreover any of the ramified prime divisors has degree 1, so that K is a rational function field.

2.2. Construction of extensions with composite Galois group

We now describe some methods to obtain new Galois extensions from given ones. The ground field k is assumed to be Hilbertian, that is, for every irreducible polynomial $f(t, X) \in k(t)[X]$ there exist infinitely many $t_0 \in k$ with $f(t_0, X)$ irreducible.

2.2.1. Factor groups

Some of the constructions to be presented first yield a Galois extension N/k whose Galois group $G = \mathrm{Gal}(N/k)$ has the desired group G' as a factor. By the main theorem of Galois theory, if H denotes the kernel of the canonical epimorphism $G \to G'$, then $G' = G/H$ occurs as the Galois group $\mathrm{Gal}(N^H/k)$ of the fixed field of H, so theoretically the problem is solved. For the explicit construction using resolvents see Section 3.3.

2.2.2. Direct products

Let G_1, G_2 be two groups which are known to occur as Galois groups over k. If at least for one of the two, say G_2, we have a regular realization over $k(t)$, then also the direct product $G_1 \times G_2$ occurs over k. Indeed, by the Hilbert irreducibility theorem, for any given G_1-extension N/k there exist infinitely many specializations of the regular G_2-extension linearly disjoint from N/k. The composite of any such specialization with N/k then gives a Galois extension with group $G_1 \times G_2$.

2.2.3. Wreath products

Let H, G be two finite groups such that G is realized as the Galois group of N/k and H is realized as a regular extension of $k(t)$. We want to construct a field extension N'/k with Galois group $G' = H \wr G$ the wreath product of H with G (with respect to some faithful transitive permutation representation of G). Let $f(x) \in k[x]$ be a polynomial with group G such that the induced permutation representation of G on the roots of f is the desired one. Let $g(t,x) \in k(t)[x]$ be a generating polynomial for a regular H-extension. Denote by $\alpha_1, \ldots, \alpha_m$ the ramification points of its splitting field in some algebraic closure of k. Let β be a primitive element of a stem field of N/k, with Galois conjugates $\beta_1 = \beta, \ldots, \beta_n$. Since k is Hilbertian, it is infinite and there exists

$$\gamma \in k \setminus \left\{ \frac{\alpha_{i_1} - \alpha_{i_2}}{\beta_{j_1} - \beta_{j_2}} \mid 1 \le i_1, i_2 \le m,\ 1 \le j_1 \ne j_2 \le n \right\}.$$

Then the splitting fields of the polynomials $g(t - \gamma\beta_j, x)$, $1 \le j \le n$, are pairwise linearly disjoint over $k(t)$ since they are ramified in disjoint sets. Thus the splitting field of $h(t,x) := \prod_{j=1}^n g(t - \gamma\beta_j, x)$ has as Galois group the wreath product $H \wr G$. Since k is Hilbertian, there exist infinitely many specializations of t to $t_0 \in k$ such that the splitting field of $h(t_0, x)$ has the same Galois group.

2.2.4. Split extensions with abelian kernel

Let G' be a finite group with an abelian normal subgroup H having a complement G in G'. Then by Suzuki (1982), Th. 10.10 the group G' is a factor group of the regular wreath product $H \wr_r G$. In particular, if G is realized as the Galois group of an extension N/k then G' can be realized by combining the methods for wreath products and factor groups. One class of groups for which this method is particularly suited are the *semi-abelian groups*. They are defined inductively as follows. All finite abelian groups are semi-abelian, and a finite group G is called semi-abelian if it can be obtained as a quotient of a semi-direct product with abelian kernel and strictly smaller semi-abelian complement.

2.2.5. Subdirect products

Let K_i/k, $i = 1, 2$, be two Galois extensions inside some algebraic closure of k with groups G_i. Let G denote the Galois group of the composite of K_1 and K_2 over k and $\epsilon_i : G_i \to H$ the epimorphisms onto $H := \mathrm{Gal}((K_1 \cap K_2)/k)$ induced by restriction to $K_1 \cap K_2$. Then G is the subdirect product $G_1 \times_H G_2$, i.e., the subgroup $\{(g_1, g_2) \mid g_i \in G_i,\ \epsilon_1(g_1) = \epsilon_2(g_2)\}$ of the direct product.

Thus, conversely, we can realize the subdirect product $G_1 \times_H G_2$ provided we have Galois extensions with groups G_1, G_2 whose intersection has group H and such that the restriction maps are correct.

3. Algorithms

Having reviewed the theoretical solutions, we now turn to the algorithmic and practical point of view. The setup will always be as follows. A Galois extension N/k (with group G) is given by the minimal polynomial $f \in k[x]$ of a primitive element α of a stem field M of N/k, i.e., of an intermediate field $k \le M \le N$ whose Galois closure over k equals

N. Thus $G = \mathrm{Gal}(N/k)$ is realized as a transitive permutation group on the set of roots of f, with point stabilizer equal to the fix group of M. To construct a Galois extension N'/k with group G' from N/k will then mean to give an algorithm how to compute the minimal polynomial of a primitive element of a stem field of N'/k from f.

We first describe how the realization of the Galois group as permutation group on the roots can be obtained in practice. Then we address the different cases described in the previous section. Finally, we remark about how to reduce the size of the coefficients of the constructed polynomials.

3.1. The computation of Galois groups

Let $f \in \mathbb{Z}[x]$ be a monic irreducible polynomial. In this section we describe how to identify the Galois group G of f as a permutation group on (complex approximations of) the roots of f. For our purposes it is very important not only to know the abstract group G, but also its action on the roots of f. This has been implemented in KASH (Daberkow *et al.*, 1997) for polynomials of degree up to 15. The algorithm is based on Stauduhar's method and described for degree up to 12 in Geißler (1997). The result of this algorithm is the name of the group and an ordering of the roots such that the action of the Galois group on this root ordering is equivalent to the computed transitive permutation group as classified in GAP (Schönert *et al.*, 1997).

In the special situation that $K/\mathbb{Q}$ is an abelian number field, Acciaro and Klüners (1999) present an algorithm to compute the automorphism group. It is feasible to compute the automorphism group for abelian fields with degree bigger than 100. Klüners (1997) has extended this algorithm to normal number fields $K/\mathbb{Q}$ and to relative extensions L/K, where L/K is abelian.

3.2. The computation of H-invariant G-relative polynomials

Definition 3.1. *Let $H \leq G$ be permutation groups on $\{x_1, \ldots, x_n\}$. We call $F \in \mathbb{Z}[x_1, \ldots, x_n]$ an H-invariant G-relative polynomial if*

1 $F^\sigma = F$ for all $\sigma \in H$,
2 $F^\sigma \neq F$ for all $\sigma \in G \setminus H$.

In this case

$$R_{G,H,F} := \prod_{\sigma \in G//H} (X - F^\sigma) \in \mathbb{Z}[x_1, \ldots, x_n, X]$$

is called a resolvent, where $G//H$ denotes a full system of representatives of (right cosets of) G/H.

It is well known that H-invariant G-relative polynomials always exist:

Lemma 3.2. *For $H \leq \mathfrak{S}_n$ let*

$$F(x_1, \ldots, x_n) := \sum_{\sigma \in H} (x_1^1 x_2^2 \cdots x_n^n)^\sigma.$$

Then $\mathrm{Stab}_{\mathfrak{S}_n}(F) = H$.

In practice it is not very efficient to use this polynomial. Our aim is to find an invariant of small degree. We use a very simple approach to compute H-invariant G-relative polynomials:

1 Set $d := 1$.
2 Compute all homogeneous invariants of H of degree d.
3 Check if these invariants are G-relative.
4 If there are H-invariant G-relative polynomials, return one with the smallest number of monomials.
5 Set $d := d + 1$ and go to 2.

For step 2 we use the algorithms implemented in Magma (Kemper and Steel, 1999). This part is the most expensive step of our algorithm. In the sequel we give some improvements which are useful in our situation.

Remark 3.3. *Let $H \leq U \leq G$ be permutation groups acting on $\{x_1, \dots, x_n\}$ and $F \in \mathbb{Z}[x_1, \dots, x_n]$ be an H-invariant U-relative polynomial. Define $\tilde{U} := \mathrm{Stab}_G(F)$. Then the following holds:*

1 $\tilde{U} \cap U = H$.
2 Let $\tilde{F} \in \mathbb{Z}[x_1, \dots, x_n]$ be an H-invariant $\tilde{U}$-relative polynomial. If F and $\tilde{F}$ are homogeneous of different degrees then $F + \tilde{F}$ is H-invariant G-relative.
3 If U is the only group with $H \lneqq U \lneqq G$, then F is H-invariant G-relative.

Proof. The first part follows directly from the definitions of U and $\tilde{U}$. In the second clearly $F+\tilde{F}$ is H-invariant. If $g \in G$ with $(F+\tilde{F})^g = F+\tilde{F}$ then $(F^g-F)+(\tilde{F}^g-\tilde{F}) = 0$. Since F and $\tilde{F}$ have different degrees this implies $F^g - F = 0$ and $\tilde{F}^g - \tilde{F} = 0$. This forces $g \in U \cap \tilde{U} = H$, proving that $F + \tilde{F}$ is H-invariant G-relative.
In the third part we get $\tilde{U} = H$, therefore F is H-invariant G-relative. □

Using 2 we hope to find a smaller H-invariant G-relative polynomial. Apart from this remark we only consider homogeneous polynomials F.
We remarked that the most expensive part of this algorithm is the computation of H-invariants. In our examples we consider groups H of size up to 10^{10} acting on up to 84 points. In these cases it is impossible to compute the invariants of H of degree 3 or higher. The following theorem gives a criterion which under suitable conditions allows us to reduce to the computation of the invariants of a smaller group.

Theorem 3.4. *Let $H \leq U \leq G$ be permutation groups acting on $\{x_1, \dots, x_n\}$. Assume that $U = \mathrm{Stab}_G(\Delta) := \{g \in G \mid \Delta^g = \Delta\}$ is the stabilizer of a block $\Delta \subseteq \{x_1, \dots, x_n\}$. Define $\bar{H} := H|_\Delta$ and $\bar{U} := U|_\Delta$. If $[\bar{U} : \bar{H}] = [U : H]$, then an $\bar{H}$-invariant $\bar{U}$-relative polynomial F is H-invariant U-relative.*

Proof. From $H \leq U$ we get $\Delta^H = \Delta$. Since F only involves indeterminates $x_i \in \Delta$ we see that $F^{\bar{\sigma}} = F^\sigma$ for any $\sigma \in U$ with image $\bar{\sigma} \in \bar{U}$, and in particular F is H-invariant. Now let $\sigma \in U \setminus H$, so σ lies in a non-trivial $\bar{H}$-coset. But then by assumption its image $\bar{\sigma} \in \bar{U}$ lies in a non-trivial H-coset, so $\bar{\sigma} \notin \bar{H}$. Since F is $\bar{H}$-invariant $\bar{U}$-relative this implies $F^\sigma = F^{\bar{\sigma}} \neq F$, and F is H-invariant U-relative. □

3.3. The computation of fixed fields

Let $f \in \mathbb{Z}[x]$ be a monic irreducible polynomial with zeros $\Omega := \{\alpha_1, \ldots, \alpha_n\}$ and G be the Galois group of f acting on Ω. Let $\tilde{G}$ be another permutation group with point stabilizer $\tilde{H}$. Assume given an epimorphism $\phi : G \to \tilde{G}$. Our aim is the computation of a polynomial g with Galois group $\tilde{G}$. We denote by H the preimage of $\tilde{H}$ under ϕ. Then G acts on the right cosets G/H as $\tilde{G}$. Now we can compute g as the minimal polynomial of a primitive element of the fixed field $\mathrm{Fix}(H)$.

In the following we give a procedure to compute a fixed field.

Suppose that we have computed H, F, and $R_{G,H,F}$, where F is an H-invariant G-relative polynomial. Since G is acting on Ω it is easy to see that $F(\alpha_1, \ldots, \alpha_n) \in \mathrm{Fix}(H)$ and $R_{G,H,F}$ is the corresponding characteristic polynomial. If $R_{G,H,F}$ is irreducible we have $\mathrm{Gal}(R_{G,H,F}) = \tilde{G}$. Otherwise $R_{G,H,F}$ is the power of an irreducible polynomial which is easy to check. In this case we have to apply a Tschirnhausen transformation to the roots of f and repeat the procedure. Girstmair (1983) has proven that one can compute a finite set of elements, which can be used for Tschirnhausen transformations. At least one of these transformations yields an irreducible $R_{G,H,F}$.

Algorithm 3.5. (ComputePolynomialFactorGroup)

Input: *A monic polynomial $f \in \mathbb{Z}[x]$, $G := \mathrm{Gal}(f)$, a group $\tilde{G}$, the roots $\{\alpha_1, \ldots, \alpha_n\}$ of f such that G acts on $\{\alpha_1, \ldots, \alpha_n\}$.*

Output: *A polynomial $g \in \mathbb{Z}[x]$ with Galois group $\tilde{G}$ if and only if there is an epimorphism from $G \to \tilde{G}$, otherwise an indication of failure.*

Step 1: *Try to find an epimorphism $\phi : G \to \tilde{G}$. If none exists then return "Failure".*

Step 2: *Compute the preimage H of a point stabilizer of $\tilde{G}$ under ϕ.*

Step 3: *Compute an H-invariant G-relative polynomial $F \in \mathbb{Z}[x_1, \ldots, x_n]$.*

Step 4: *Compute $R_{G,H,F}(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}[x]$.*

Step 5: *If $\gcd(R_{G,H,F}(\alpha_1, \ldots, \alpha_n), R'_{G,H,F}(\alpha_1, \ldots, \alpha_n)) = 1$ then return $R_{G,H,F}(\alpha_1, \ldots, \alpha_n)$.*

Step 6: *Apply a Tschirnhausen transformation to f, update the roots $\{\alpha_1, \ldots, \alpha_n\}$ and go to step 1.*

We remark that it is not necessary that the polynomial f is irreducible, i.e., that the group G acts transitively on the roots of f. The sorting of the roots can be computed with the Galois function of KASH. In the following sections we give special constructions which give the sorting of the roots.

3.4. The computation of polynomials for direct and subdirect products

Let $G = G_1 \times G_2$ be a permutation group which is a direct product. Suppose we know two irreducible polynomials $f_1, f_2 \in \mathbb{Z}[x]$ such that $G_1 = \mathrm{Gal}(f_1)$ and $G_2 = \mathrm{Gal}(f_2)$. We denote with N_1 and N_2 the splitting fields of f_1 and f_2, respectively. Supposing $N_1 \cap N_2 = \mathbb{Q}$ we know that $\mathrm{Gal}(N_1 N_2/\mathbb{Q}) \cong G$. In this case it remains to compute a polynomial f such that $\mathrm{Gal}(f) = G$.

Let $\alpha = \alpha_1, \ldots, \alpha_n$ and $\beta = \beta_1, \ldots, \beta_m$ be the roots of f_1 and f_2, respectively. We want to compute a minimal polynomial of a primitive element of the extension $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}$. There is a well known algorithm for this task.

ALGORITHM 3.6. (ComputePrimitiveElement)

Input: *Minimal polynomials* $f_1, f_2 \in \mathbb{Z}[x]$ *of* α, β.

Output: *A minimal polynomial of a primitive element of* $\mathbb{Q}(\alpha, \beta)/\mathbb{Q}$.

Step 1: *Set* $a := 1$.

Step 2: *Compute* $r(x) := \mathrm{resultant}_y(f_1(x-y), f_2(y/a)) \in \mathbb{Z}[x]$.

Step 3: *If* $\gcd(r, r') = 1$ *then return* r.

Step 4: *Set* $a := a + 1$ *and go to Step 2.*

PROOF. By the definition of the resultant the roots of r are $\alpha_i + a\beta_j$ $(1 \le i \le n, 1 \le j \le m)$. If r has no multiple roots it follows that r is irreducible. If there are multiple roots there exist $1 \le i, k \le n, 1 \le j, l \le m$ with $(i,j) \ne (k,l)$ such that $\alpha_i + a\beta_j = \alpha_k + a\beta_l$. This is equivalent to: $a = (\alpha_i - \alpha_k)/(\beta_l - \beta_j)$. This proves that there are only finitely many values for a which yield a reducible r. $\square$

Using algorithm 3.6 we can give a first algorithm to compute a polynomial $f \in \mathbb{Z}[x]$ such that $\mathrm{Gal}(f) = G$. After computing the polynomial $r \in \mathbb{Z}[x]$ with algorithm 3.6 we have to compute its Galois group including the action on the roots. After this we can apply the function ComputePolynomialFactorGroup to compute f. The above approach has two disadvantages. It may happen that it is impractical to compute the Galois group of r in its action on the roots. Even if it is possible to compute this Galois group, this approach tends to yield polynomials with large coefficients. The following algorithm avoids this.

ALGORITHM 3.7. (Computation of a polynomial for a direct product.)

Input: *Polynomials* $f_1, f_2 \in \mathbb{Z}[x]$ *of degree* n, m *with linearly disjoint splitting fields,* $G \cong \mathrm{Gal}(f_1) \times \mathrm{Gal}(f_2)$.

Output: *A polynomial* $f \in \mathbb{Z}[x]$ *with* $\mathrm{Gal}(f) = G$.

Step 1: *Compute the Galois groups* G_1 *and* G_2 *of* f_1 *and* f_2*, respectively, including their action on the roots.*

Step 2: *Set* $\tilde{f} := f_1 f_2$.

Step 3: *Set* $\tilde{G}$ *to the direct product of* G_1 *and* G_2 *acting on* $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\}$ *such that* $G|_{\{\alpha_1,\ldots,\alpha_n\}} = G_1$ *and* $G|_{\{\beta_1,\ldots,\beta_m\}} = G_2$.

Step 4: *Set* $f :=$ *ComputePolynomialFactorGroup*$(\tilde{f}, \tilde{G}, G, \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\})$.

Step 5: *Return* f.

The algorithm for computing polynomials for subdirect products is similar to algorithm 3.7. Recall the definition of a subdirect product.

Definition 3.8. *Let G_1, G_2, H be groups and μ_i epimorphisms from G_i to H $(i = 1, 2)$. The we define:*

$$G_1 \times_H G_2 := \{(g_1, g_2) \mid g_1 \in G_1,\ g_2 \in G_2,\ \mu_1(g_1) = \mu_2(g_2)\}.$$

Remark 3.9. *Note that the notation $G_1 \times_H G_2$ may be ambiguous if there exist several epimorphisms $\mu_i : G_i \to H$ for $i = 1$ or $i = 2$.*

1 *For example there exist two non-isomorphic subdirect products of the form $D_4 \times_{Z_2} S_3$, the reason being that D_4 has three different factor groups isomorphic to Z_2 (these are the groups denoted $12T12$ and $12T13$ in our tables).*
2 *Consider subdirect products of the Frobenius group F_{20} with itself of the form $F_{20} \times_{Z_4} F_{20}$. There is only one factor group of F_{20} isomorphic to Z_4. Since the automorphism group of Z_4 has two elements, there are two epimorphisms μ_1 and μ_2 from F_{20} to Z_4. Define $G_1 := \{(g_1, g_2 \mid g_1, g_2 \in F_{20},\ \mu_1(g_1) = \mu_1(g_2)\}$ and $G_2 := \{(g_1, g_2 \mid g_1, g_2 \in F_{20},\ \mu_1(g_1) = \mu_2(g_2)\}$. These two subdirect products are not isomorphic. (Only one of them, denoted $10T10$, has a faithfull permutation representation of degree at most 15.)*

Suppose given two polynomials $f_1, f_2 \in \mathbb{Z}[x]$ such that $\mathrm{Gal}(f_1) = G_1,\ \mathrm{Gal}(f_2) = G_2$, and the Galois group of the intersection of the corresponding splitting fields is isomorphic to H. Then we know that the splitting field of $f_1 f_2$ has Galois group isomorphic to $G_1 \times_H G_2$ for suitable epimorphisms $\mu_i : G_i \to H$. From a computational point of view we are given G_1 and G_2 only up to automorphisms. In the following algorithm we fix the groups G_1 and G_2 and the epimorphism μ_2. We want to find an epimorphism μ_1 such that μ_1 and μ_2 fit together.

Lemma 3.10. *Let G, H be finite groups and $\mu : G \to H$ an epimorphism with kernel N. Then all epimorphisms $\nu : G \to H$ with kernel N are given by:*

$$\{\tau \circ \mu \mid \tau \in \mathrm{Aut}(H)\}.$$

Proof. It is clear that $\tau \circ \mu$ is an epimorphism with kernel N. Furthermore we have $\tau_1 \circ \mu \neq \tau_2 \circ \mu$ for $\tau_1 \neq \tau_2 \in \mathrm{Aut}(H)$. Let μ_1, μ_2 be two epimorphisms with kernel N. Then $\mu_1 \circ \mu_2^{-1}$ is an automorphism of H. □

Algorithm 3.11. (Computation of a polynomial for a subdirect product.)

Input: *Monic polynomials $f_1, f_2 \in \mathbb{Z}[x]$ of degree n, m, $G \cong \mathrm{Gal}(f_1) \times_H \mathrm{Gal}(f_2)$.*

Output: *A polynomial $f \in \mathbb{Z}[x]$ with $\mathrm{Gal}(f) = G$.*

Step 1: *Compute $G_1 := \mathrm{Gal}(f_1)$ and $G_2 := \mathrm{Gal}(f_2)$ including the action on the roots $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_m\}$, respectively.*

Step 2: *Compute all normal subgroups $N_1 \trianglelefteq G_1$ such that $G_1/N_1 \cong H$. For each N_1 compute the fixed field using algorithm 3.5.*

Step 3: *Compute all normal subgroups $N_2 \trianglelefteq G_2$ such that $G_2/N_2 \cong H$. For each N_2 compute the fixed field using algorithm 3.5.*

Step 4: *Choose $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$ such that the corresponding fixed fields are equal.*

Step 5: *Compute epimorphisms $\mu_i : G_i \to H$ with kernel N_i $(i = 1, 2)$.*

Step 6: *Compute $A := \mathrm{Aut}(H)$.*

Step 7: *For all $\tau \in A$ do*

1. *$\tilde{\mu}_1 := \tau \circ \mu_1$.*
2. *Set $\tilde{G}$ to the subdirect product of G_1 and G_2 acting on $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\}$ corresponding to the epimorphisms $\tilde{\mu}_1, \mu_2$.*
3. *Set $f :=$ComputePolynomialFactorGroup($f_1 f_2, \tilde{G}, G, \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\}$).*
4. *If $f \in \mathbb{Z}[x]$ then return f.*

Proof. After having identified the correct kernels N_1 and N_2 our epimorphisms μ_1, μ_2 have to satisfy $\mu_1(N_1) = \mu_2(N_2) = 1$. Applying Lemma 3.10 we compute all epimorphisms from G_1 to H with kernel N_1. We know that one of these epimorphisms fits to our representations of the given Galois groups. □

We have to modify the function ComputePolynomialFactorGroup in such a way that this function can detect that the result is not in $\mathbb{Z}[x]$ which happens when the ordering of the roots was not correct. In Step 3 we identify the normal subgroups which are the kernels of the corresponding epimorphisms. In this step we can check which subdirect product we get (compare Remark 3.9).

3.5. Wreath products

Let G, H be finite groups such that we can realize G over $\mathbb{Q}$ and H regularly over $\mathbb{Q}(t)$. Recall from 2.2.3 that then the wreath product $H \wr G$ (with respect to any faithful transitive permutation representation of G) also occurs as Galois group over $\mathbb{Q}$. Without loss (see Section 3.3) we may assume that the realization of G is already given by a polynomial f such that the induced permutation representation on the roots of f is the desired one. Let $g(t, x) \in \mathbb{Z}(t)[x]$ be a generating polynomial for a regular H-extension.

For a number field $K/\mathbb{Q}$ we denote by Norm the norm function extended to the polynomial ring $K[x]$.

Algorithm 3.12. (WreathProduct)

Input: *A monic polynomial $g \in \mathbb{Z}[x]$ with $\mathrm{Gal}(g) = G$, $h(t, x) \in \mathbb{Z}[t][x]$ with $\mathrm{Gal}(h) = H$ regular over $\mathbb{Q}(t)$.*

Output: *A monic polynomial $f \in \mathbb{Z}[x]$ such that $\mathrm{Gal}(f) = H \wr G$.*

Step 1: *Let $K = \mathbb{Q}(\alpha)$, where α is a zero of g.*

Step 2: *Choose a random algebraic integer γ of K.*

Step 3: *Compute* $f := \mathrm{Norm}(h(\gamma, x)) \in \mathbb{Z}[x]$.

Step 4: *If* $\gcd(f, f') \neq 1$ *then go to Step 2.*

Step 5: *If* f *is reducible then go to Step 2.*

Step 6: *If* $\mathrm{Gal}(f) = H \wr G$ *then return* f, *otherwise go to Step 2.*

The time critical step of the above algorithm is Step 6. This can be improved by using the fact that the Galois group is a subgroup of $H \wr G$ which is very useful for the Galois group computation. We remark that the right γ in Step 2 abound (see Section 2.2.3).

3.6. Split extensions with abelian kernel

Suppose we know that G is a factor group of $A \wr H$, where A is an abelian group. When we are able to realize H over $\mathbb{Q}$ we can compute a polynomial $g \in \mathbb{Z}[x]$ for the wreath product. It may happen that we are not able to compute the Galois group including the ordering of the roots with general methods. We are looking for a special method in this case. In the following we denote with $\mathcal{O}_K$ the ring of algebraic integers of K. We only give an algorithm when A is a cyclic group, but it is not difficult to extend it to abelian groups.

Algorithm 3.13. (SplitExt)

Input: *A group* G *which is a factor group of* $A \wr H$ *(*A *cyclic), a monic polynomial* $g \in \mathbb{Z}[x]$ *with* $\mathrm{Gal}(g) = H$, $K = \mathbb{Q}(\alpha)$, *where* α *is a root of* g, *a polynomial* $h \in \mathcal{O}_K[x]$ *such that* $\mathrm{Gal}(\mathrm{Norm}(h)) \cong A \wr H$.

Output: *A polynomial* $f \in \mathbb{Z}[x]$ *with* $\mathrm{Gal}(f) = G$.

Step 1: *Define* $L := K(\beta)$, *where* β *is a root of* h.

Step 2: *Compute the automorphisms of the abelian extension* L/K.

Step 3: *Let* $\tilde{h} := \mathrm{Norm}(h)$ *and* $\alpha_1, \ldots, \alpha_n$ *be the roots of* g.

Step 4: *Let* $\sigma \in \mathrm{Aut}(L/K)$ *be a generator of order* m.

Step 5: *Compute* $\mathrm{Gal}(g)$ *including the action on* $\{\alpha_1, \ldots, \alpha_n\}$.

Step 6: *Compute the images* $h^{(i)}$ *of* h *under the mapping* $\alpha \mapsto \alpha_i$ $(1 \leq i \leq n)$.

Step 7: *Compute a root* $\gamma_{i,1}$ *of* $h^{(i)}$ $(1 \leq i \leq n)$.

Step 8: *Set* $\gamma_{i,j} := \gamma_{i,1}^{\sigma^{j-1}}$ $(1 \leq i \leq n, 2 \leq j \leq m)$.

Step 9: *Let* $H = \langle \tilde{\tau}_1, \ldots, \tilde{\tau}_r \rangle$. *Denote by* $\tau_1, \ldots, \tau_r$ *the permutations which operate simultaneously on* $\{\gamma_{1,j}, \ldots, \gamma_{n,j}\}$ $(1 \leq j \leq m)$ *in the same way as* $\tilde{\tau}_1, \ldots, \tilde{\tau}_r$ *on* $\{\alpha_1, \ldots, \alpha_n\}$. *Define* $\sigma_i = (\gamma_{i,1}, \ldots, \gamma_{i,m})$ $(1 \leq i \leq n)$.

Step 10: *Set* $\tilde{G} := \langle \sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_r \rangle$.

Step 11: *Set* $f :=$ *ComputeFactorGroup*$(\tilde{h}, \tilde{G}, G)$.

Step 12: *Return* f.

3.7. Finding "nicer" polynomials

Let $f \in \mathbb{Z}[x]$ be an irreducible monic polynomial of degree n and K be the stem field of f. We want to find an irreducible polynomial $g \in \mathbb{Z}[x]$ such that the stem fields of f and g are isomorphic and g is simpler or nicer in some sense. For instance that could mean that the coefficients or the discriminant of g are small. It is very difficult to solve this problem. A description of a possible approach to this problem can be found in Cohen (1993), section 4.4.2. The algorithm presented there works as follows. Let $\beta \in \mathcal{O}_K$ and m_β be the characteristic polynomial. We denote with $\beta_1, \ldots, \beta_n \in \mathbb{C}$ the zeros of m_β. The idea of the algorithm is to find a $\beta \in \mathcal{O}_K$ such that the T_2-norm

$$T_2(\beta) := \sum_{i=1}^{n} |\beta_i|^2$$

is small. If such an element is primitive one hopes that the corresponding polynomial m_β has small coefficients.

The first step of this algorithm is to compute the maximal order $\mathcal{O}_K$ of K. Then we can associate a lattice to $\mathcal{O}_K$. Elements with small norm in the lattice are associated to elements of $\mathcal{O}_K$ with small T_2-norm. By computing an LLL-basis (Lenstra *et al.*, 1982) of the lattice we get a basis of the lattice consisting of elements of small norms. This algorithm is implemented in PARI (polred) and in KASH (OrderShort).

There are three main problems of the polred and the OrderShort function. The first one is the computation of the maximal order. Here the main obstacle is the factorization of the polynomial discriminant. The second problem is the computation of the LLL reduced basis of the corresponding lattice. When the index of the equation order in the maximal order is very large we have to work with a high real precision in the lattice reduction. The third problem occurs when the given field has many subfields. In this case it is very probable that short vectors in the lattice correspond to elements which are not primitive. In the following section we give some ideas on how to overcome the second and third problem.

We remark that Fieker (1997) gives a relative version of this algorithm, where it is possible to find a nicer polynomial for an extension K/L. When a number field K has non-trivial subfields, very often the following procedure is useful.

1 Compute a subfield L of $K = \mathbb{Q}(\alpha)$.
2 Compute the minimal polynomial of α over L.
3 Try to find a nicer polynomial for the relative extension K/L.
4 Compute a primitive element from the new representation.

We remark that subfield computation can be done with algorithms described in Klüners and Pohst (1997) and Klüners (1999).

3.8. The computation of maximal orders

One important task in algorithmic number theory is the computation of the maximal order of a number field. Well known algorithms for this are described for instance in Pohst and Zassenhaus (1989) and Cohen (1993). All these algorithms have in common that the first step is the factorization of the polynomial discriminant or some related large number. In our present applications it is often impractical to factorize this number. But since we have constructed our polynomials in a special way we are in a better situation. For instance in many cases we know the field discriminant in advance from theory. The purpose of this section is to describe how to use special information to compute the maximal order of a number field. The main idea of all approaches consists in avoiding the factorization of large integers.

Let $K = \mathbb{Q}(\alpha)$ be a number field, where α is an algebraic integer. We first describe an algorithm which works if the field extension $K/\mathbb{Q}$ is not primitive, i.e., has proper intermediate fields. Using the subfield algorithm described in Klüners and Pohst (1997) and Klüners (1999) we can compute a subfield $\mathbb{Q}(\beta)$ of K including the embedding

$$\beta := \frac{1}{d}\sum_{i=0}^{n-1} b_i\alpha^i, \quad (d \in \mathbb{N}, b_i \in \mathbb{Z})$$

into K. We can now proceed as follows.

1 Compute all subfields of K.
2 Compute the maximal orders of all proper subfields.
3 Set $A := \{1, \alpha, \ldots, \alpha^{n-1}\}$.
4 For each subfield L do:
Move each basis element of $\mathcal{O}_L$ to a representation in K and add it to the set A.
5 Compute a $\mathbb{Z}$-basis of the multiplicative closure of A.
6 Set $\mathcal{O}$ to the order generated by this $\mathbb{Z}$-basis.
7 Compute the maximal order $\mathcal{O}_K$ starting from $\mathcal{O}$.

The idea of the above algorithm is to find integral elements which are not contained in $\mathbb{Z}[\alpha]$. The computation of a $\mathbb{Z}$-basis can be done using the Hermite normal form algorithm.

In practice the discriminant of the order $\mathcal{O}$ is much smaller than the discriminant of $\mathbb{Z}[\alpha]$. For primitive extensions $K/\mathbb{Q}$ we have two other possibilities to find integral elements. When the extension is normal we can compute the automorphisms which in general produces elements not contained in $\mathbb{Z}[\alpha]$. The other possibility is to factorize the minimal polynomial of α over K. When the factorization is not trivial (a linear factor and a factor of degree $n-1$) the coefficients of the factors are in general not contained in $\mathbb{Z}[\alpha]$. This approach is very useful for dihedral groups, but it cannot work when the group is at least 2-transitive.

Another approach is based on a theorem of Buchmann and Lenstra (1994).

THEOREM 3.14. *There are polynomial time algorithms that given an algebraic number field K and one of (a), (b), determine the other:*

(a) the ring of algebraic integers of K,
(b) the largest square free divisor of the discriminant of K.

Since our fields are constructed in a special way, we are very often in the situation that we know (b) of the above theorem. This result is not only theoretical. Buchmann and Lenstra (1994) also provide an algorithm which works well in practice. This has been implemented in KASH (Daberkow *et al.*, 1997) by the first author.

4. Determination of polynomials with groups of degree ≤ 15

In the preceding sections we have presented some general methods applicable to all groups with a certain structure. When trying to apply these to a particular group G, one has to investigate the structure of G to see whether it falls into one of the above categories. This was done for the transitive groups of degree at most 15 by using the computer algebra system GAP (Schönert *et al.*, 1997). The results are collected in Section 5. It turned out that for the overwhelming majority of these groups at least one of the general methods can be used to construct a polynomial. Here, we do not consider the method of a subdirect product as a general method, since it depends on first finding fields with the desired properties.

In this section we consider the few remaining groups of degree between 12 and 15 and show how they can be realized explicitly as Galois groups. Finally, we give the proof of Theorem 1.1.

4.1. The groups 12T57, 12T91, 12T93, 12T104

The permutation groups 12T91 and 12T93 are isomorphic as abstract groups. Moreover, all four groups contain a normal subgroup N isomorphic to 12T57. In the groups 12T91, 12T93, and 12T104 there exists an elementary abelian normal subgroup such that N is a supplement. Similar to the concept of semi-abelian groups we can prove that such a group is a factor group of a suitable wreath product. We computed the following:

1 12T91 and 12T93 are a factor group of $Z_2 \wr G$, where G operates on 24 points and is isomorphic to 12T57. We have $G = \langle (1,3)(2,4)(7,11)(8,12), (1,13,5)(2,14,6)$ $(3,17,9)(4,18,10)(7,23,15)(8,24,16)(11,22,19)(12,21,20), (1,11,3,7)(2,12,4,8)$ $(5,10)(6,9)(13,21)(14,22)(15,16)(17,24)(18,23)(19,20)\rangle$.
2 12T104 is a factor group of $Z_2 \wr 12T57$.

It thus remains to construct a polynomial for the group $12T57$. We know that $12T57 \cong SL_2(3) \times_{\mathfrak{A}_4} [4^2]3 \cong 8T12 \times_{\mathfrak{A}_4} 12T31$. One can prove that all subdirect products of this type are isomorphic. Therefore it is sufficient to find an $\mathfrak{A}_4$-extension which is embeddable into both an $SL_2(3)$- and a $[4^2]3$-extension. Since these are Frattini embedding problems the solutions are regular provided the $\mathfrak{A}_4$ extension is regular.

Let L/k be a normal $\mathfrak{A}_4$ extension and K/k be the unique subfield of degree 3. Then K/k is cyclic and L/K has Galois group $Z_2 \times Z_2$. The theorem of Kochendörffer (see Malle and Matzat (1999), Th. IV.8.2) states in our case that

1 L/k is embeddable into an $SL_2(3)$ extension, if and only if L/K is embeddable into a Q_8 extension.
2 L/k is embeddable into a $[4^2]3$ extension, if and only if L/K is embeddable into a $Z_4 \times Z_4$ extension.

Ledet (1996), Theorem 5.3, proved the following:

Theorem 4.1. *Let k be a Hilbertian field of characteristic $\neq 2$, and let K/k be a cyclic extension of degree 3 with Galois group $\langle\sigma\rangle$. Let $a \in K \backslash k$ be an element with the following properties:*

1 $a\sigma(a) \notin K^2$,
2 $a = 1 + r^2 + r^2\sigma(r)^2$ *for some* $r \in K \setminus k$.

Then $K(\sqrt{ab}, \sqrt{bc})/k$ *is a normal* $\mathfrak{A}_4$ *extension, where* $b := \sigma(a)$ *and* $c := \sigma(b)$. *Furthermore* $K(\sqrt{ab}, \sqrt{bc})/k$ *is embeddable into a* $SL_2(3)$ *extension.*

This theorem gives us an approach how to construct an $SL_2(3)$ extension. It is well known that a cyclic extension of degree 2 is embeddable into a Z_4 extension if and only if the generator of the quadratic extension is the sum of two squares. Suppose that a, b are sums of two squares, then ab is the sum of two squares, too. We want to choose r in the above theorem in such a way that $1 + r^2$ is a square. In this case a and therefore b, c are sums of two squares. When we can prove that ab is not a square, we know that $K(\sqrt{ab}, \sqrt{bc})/k$ is embeddable into an $SL_2(3)$ and into a $[4^2]3$ extension and we are done. We have to solve two problems:

1 Find $r \in K \setminus k$ such that $1 + r^2$ is a square.
2 Check, that ab is not a square in K.

A parametrization of all k-points on the genus 0 curve

$$s^2 = 1 + r^2$$

is given by

$$r = \frac{2\lambda}{1-\lambda^2} \quad (\lambda \neq \pm 1).$$

It remains to find an example such that ab is not a square in K. Let $f(t, x) := x^3 + 3tx^2 - 9x - 3t \in \mathbb{Q}(t)[x]$. Since f is irreducible and $\mathrm{disc}(f) = 2^2 \cdot 3^4 \cdot (t^2 + 3)^2$ is a square, we get that $\mathrm{Gal}(f) = Z_3$. Define $K := \mathbb{Q}(t)(\alpha)$, where α is a root of f. One can check that $\beta := \frac{1}{2}(-3t - 6 + (3t - 1)\alpha + \alpha^2)$ is another root of f, hence $\beta = \sigma(\alpha)$ for a suitable generator σ of $\mathrm{Gal}(f)$. Now we compute:

$$r := \frac{2\alpha}{1-\alpha^2} = \frac{1}{4}(-3t - \alpha),\ \sigma(r) = \frac{2\beta}{1-\beta^2} = \frac{1}{8}(-3t + 6 + (-3t + 1)\alpha - \alpha^2).$$

Therefore we can compute $a := 1 + r^2 + r^2\sigma(r)^2 =$

$$\frac{1}{512}(81t^4 - 135t^3 + 423t^2 - 9t + 512 + (81t^4 + 18t^2 + 216t - 27)\alpha + (27t^3 - 9t^2 - 3t + 41)\alpha^2).$$

Now we have to check that $a\sigma(a)$ is not a square in K or equivalently that $K(\sqrt{a})/\mathbb{Q}(t)$ is not cyclic. The latter one can be easily checked by specializations and we get that $\mathrm{Gal}(K(\sqrt{a})/\mathbb{Q}(t)) = Z_2 \wr Z_3$. Therefore we have proved that $K(\sqrt{a\sigma(a)})/\mathbb{Q}(t)$ is an $\mathfrak{A}_4(6)$ extension which can be embedded into 12T31 and 8T12.

4.2. The group 12T124

This group is the (unique) subdirect product of Z_4 and $\mathfrak{S}_5$ with kernel Z_2. In particular 12T124 is not solvable. In order to construct a 12T124-extension we need $\mathfrak{S}_5$ and Z_4-extensions with a common subfield of degree 2. The following polynomials $f_i \in \mathbb{Q}(t_i)[x]$,

$i \in \{1,2\}$, generate regular Galois extensions of $\mathbb{Q}(t_i)$ with Galois group $\mathfrak{S}_5$ and Z_4, respectively.

$$f_1(x,t) := x^5 - 5x^4 + 256\,t_1, \quad f_2(x,t) := x^4 + t_2\,x^3 - 6x^2 - t_2\,x + 1\,.$$

Their quadratic subfields are generated by the square roots of the polynomial discriminants and equal $\mathbb{Q}(\sqrt{5t_1(t_1-1)})$, $\mathbb{Q}(\sqrt{t_2^2+16})$ respectively. If we specialize

$$t_1 := 4(5t^2 - 6t + 5)/(4t^2 - 5), \quad t_2 := -3(4t^2 - 15t + 5)/(4t^2 - 5),$$

then the Galois groups are conserved, but the polynomial discriminants of f_1, f_2 coincide up to squares. This means that the splitting fields of f_1 and f_2 contain the same quadratic subfield. It follows that the splitting field of $f_1 f_2$ has Galois group isomorphic to 12T124 over $\mathbb{Q}(t)$. Furthermore this extension is regular since $\mathfrak{S}_5' = \mathfrak{A}_5$ is simple.

4.3. The groups 12T143, 12T192, 12T242

The group 12T143 is solvable but not semi-abelian. The normal subgroup Z_2^4 has a supplement which is isomorphic to 12T93.

The group 12T192 is semi-abelian, but not a split extension with abelian kernel. The normal subgroup Z_2^5 has a supplement isomorphic to 12T98.

The group 12T242 is a split extension with abelian kernel: the normal subgroup Z_3 has complement 8T22. It follows that 12T242 is a factor group of the regular wreath product $Z_3 \wr_r 8T22$. We have not succeeded in finding a smaller wreath product.

4.4. The groups 12T256, 12T287, 12T297, 12T298, 14T59, 14T60, 15T94, 15T95, 15T97, 15T98, 15T99, 15T100

The groups G mentioned in the title all occur as subgroups of index 2 in suitable wreath products $\tilde{G}$ or in groups with a rationally rigid class triple. We can realize them as Galois groups by first constructing a regular extension with group $\tilde{G}$ and then using the descent trick explained in Section 2.1.2. This works whenever only two divisors of degree 1 ramify in the fixed field of G. We explain this by means of the following table.

$\tilde{G}$		$\tilde{\mathbf{C}}$	G	$\mathbf{C}$
$12T270$	$= 2 \wr \mathfrak{S}_5$	$2^6 - 8 - 10$	$12T256$	$4^2 - 10 - 10$
$12T293$	$= 2 \wr \mathfrak{S}_6$	$2^2 - 10 - 12$	$12T287$	$5^2 - 12 - 12$
$12T299$	$= \mathfrak{S}_6 \wr 2$	$2 - 10.2 - 12$	$12T297$	$10.2 - 10.2 - 6^2$
$12T299$	$= \mathfrak{S}_6 \wr 2$	$2 - 10.2 - 12$	$12T298$	$5^2 - 12 - 12$
$14T61$	$= \mathfrak{S}_7 \wr 2$	$2 - 12.2 - 14$	$14T59$	$12.2 - 12.2 - 7^2$
$14T61$	$= \mathfrak{S}_7 \wr 2$	$2 - 12.2 - 14$	$14T60$	$6^2 - 14 - 14$
$15T97$		$2^6 - 10 - 12.3$	$15T94$	$2^6 - 2^6 - 5^2 - 6^2.3$
$15T97$		$2^6 - 10 - 12.3$	$15T95$	$5^2 - 12.3 - 12.3$
$15T101$	$= \mathfrak{S}_5 \wr 3$	$2 - 2 - 4^2 - 15 - 15$	$15T98$	$4^2 - 4^2 - 15 - 15 - 15 - 15$
$15T102$	$= \mathfrak{S}_5 \wr \mathfrak{S}_3$	$2 - 2^5 - 8.2 - 15$	$15T99$	$8.2 - 8.2 - 15 - 15$
$15T102$	$= \mathfrak{S}_5 \wr \mathfrak{S}_3$	$2 - 2^5 - 8.2 - 15$	$15T100$	$2^5 - 2^5 - 4^2 - 15 - 15$

It contains a list of groups $\tilde{G}$ with class vector $\tilde{\mathbf{C}}$. A generating polynomial for a stem field of a $\tilde{G}$-Galois extension $N/\mathbb{Q}(t)$ with ramification type $\tilde{\mathbf{C}}$ can be obtained by the wreath product construction described above. Any of the missing groups G is a subgroup of index at most 2 in one of the groups $\tilde{G}$ and we indicate these subgroups in the third column. The fixed field of G is a rational function field, and in the fourth column we give the corresponding class vector $\mathbf{C}$ for the Galois extension N/N^G.

4.5. The groups 14T33, 14T42

The group 14T33 is the non-split extension of the elementary abelian group of order 8 with the simple group $L_3(2)$ (the split extension already has a faithful permutation representation of degree 8), and 14T42 is the direct product of 14T33 with the group of order 2. Let us denote by $3A$, $4A$ respectively the conjugacy classes of $L_3(2)$ containing elements of permutation types 3^2, 4.2 in the degree 7 permutation representation. There exists a regular Galois extension of $\mathbb{Q}(t)$ with group $L_3(2)$ and class vector $\mathbf{C} = (3A, 4A, 4A)$, with generating polynomial given by

$$f := (y^3 - 41\,y^2 + 5\,y - 1)\,(y^3 - 20\,y^2 + 19\,y - 162)\,(y - 2) - (y - 1)^3\,y\,t\,,$$

ramified in ∞ and $\pm 1372\sqrt{7}$ (see Malle and Matzat (1985)). The behaviour at the finite ramification points can be seen by calculating the resultant of f and $t^2 - 13176688$ as

$$(y^2 - 4\,y - 3)^4\,(y^2 - 32\,y + 4)^2\,(y^2 - 46\,y + 81)\,.$$

It turns out that 14T42 has a generating system with three generators of permutation types 3^4, 8.2^2, 8.2^2 in the degree 14 representation. In the degree 7 permutation representation of the factor group $L_3(2)$, these map to elements in the class vector $\mathbf{C}$. The above equation for the ramification thus shows that we obtain a Galois extension of $\mathbb{Q}(t)$ with group 14T42 by extracting a square root from $y^2 - 4\,y - 3$ in the stem field $\mathbb{Q}(y)/\mathbb{Q}(t)$ of the $L_3(2)$ extension. A generating polynomial is given by

$$x^{14} - 1764\,x^{12} - 27\,(4\,t - 14063)\,x^{10} - 378\,(2\,t - 7203)\,x^8 - (x^2 + 3)\,(x^2 + 6)^3\,(t^2 - 13176688).$$

It can be checked that the corresponding Galois extension of $\mathbb{Q}(t)$ with group 14T42 is regular. The group 14T33 is now obtained either via its fixed field of genus zero, or alternatively as a factor group in the way described in the previous section. We get an explicit polynomial over $\mathbb{Q}(u)$ using the substitution $t := 1372(6u^2 - 4u + 3)/(2u^2 - 1)$.

4.6. Primitive non-solvable groups

Polynomials for the primitive non-solvable groups of permutation degree at most 15 were known previously, their determination was completed in Malle (1987). References for these polynomials can either be found in loc. cit., or in the Appendix of the forthcoming book of Malle and Matzat (1999).

More precisely, this concerns the alternating and symmetric groups and the almost simple groups given in the following table:

G		$\mathbf{C}$
$12T218$	$\cong PGL_2(11)$	$2^5 - 4^3 - 11$
$12T295$	$\cong M_{12}$	$4^2 - 4^2 - 10.2$
$13T7$	$\cong L_3(3)$	$2^4 - 8.4 - 8.4$
$14T30$	$\cong L_2(13)$	$2^6 - 6^2 - 6^2$
$14T39$	$\cong PGL_2(13)$	$2^7 - 4^3 - 6^2$

All of these groups were realized by the rigidity method described in Section 2.1.

4.7. Proof of Theorem 1.1

Let G be a transitive subgroup of $\mathfrak{S}_d$ for $d \leq 15$. If $d \leq 11$ then the assertion of Theorem 1.1 follows for G by the tables in Malle and Matzat (1999), Appendix, except for the 7 groups for which no polynomial is given there. But all seven groups are semidirect products with abelian kernel and complement already known to occur regularly. Thus G itself occurs regularly by Section 2.2.4. So now assume that $d \geq 12$. The arguments given in Section 2 show that G occurs regularly over $\mathbb{Q}(t)$ unless G is one of the groups treated in this section. But for all of these we have given explicit constructions of regular extensions with group G. This completes the proof of Theorem 1.1.

5. The groups of degree $12 \leq d \leq 15$

In the following tables we give the information needed to realize the transitive groups of degree $12 \leq d \leq 15$ as Galois groups. We denote by dTm the m-th transitive group of degree d. This is the same group one gets with TransitiveGroup(d, m) in GAP or Magma. For each group G we give a line with five columns. The first column gives the number of G, the second one the size of G. A * in the second column indicates that G acts primitively. In the third column we give the information if G is a wreath product, a direct product, or a subdirect product. In the fourth column we give the information whether G is a factor group of a non-trivial wreath product. This information was computed by checking whether G is a split extension with abelian kernel or an abelian normal subgroup of G has a proper supplement. In the case that there is no entry this means that no non trivial normal abelian subgroup has a supplement. In the last column we give groups such that G is a factor group. Here we give groups acting on less than d points and/or a list of the numbers of transitive groups acting on d points. In the degree 12 case we do not list all groups with that property. If a group is a factor group of a group G then all groups which are bigger than G having G as a factor group are not listed.

For the group 14T5 the following information is given in the table:

No	Size	Group	Factor	Factor
5	42	7T3 x 2T1	$Z_7 \wr 6T1$	[14, 18, 44]

This means that this group has 42 elements and is a direct product of 7T3 and 2T1 $(7.3 \times Z_2)$. Furthermore this group can be realized as a factor group of $Z_7 \wr 6T1$ $(Z_7 \wr Z_6)$, 14T14, 14T18, and 14T44.

5.1. Transitive Groups of degree 12

In the following table the group G is defined in section 4.1.

No	Size	Group	Factor	Factor
1	12	4T1 x 3T1	$Z_4 \wr 3T1$	[19, 29, 73]
2	12	4T2 x 3T1	$Z_3 \wr 4T2$	[14, 18, 25]
3	12	3T2 x 2T1	$Z_3 \wr 4T2$	6T3 [10, 11, 12, 13, 16, 18, 21]
4	12		$Z_2 \wr 3T1$	4T4 [6, 20, 31, 32, 122, 132]
5	12	$3T2 \times_{2T1} 4T1$	$Z_3 \wr 4T1$	[19, 27, 72]

6	24	4T4 x 2T1	$Z_2 \wr 3T1$	6T6 [7, 25, 29, 43, 55, 56, 58, 60, 91, 176]
7	24	4T4 x 2T1	$Z_2 \wr 3T1$	6T6 [6, 25, 29, 43, 55, 56, 58, 60, 91, 176]
8	24		$Z_2 \wr 3T2$	4T5 [9, 21, 27, 44, 45, 62, 66, 157, 175, 177]
9	24		$Z_2 \wr 3T2$	4T5 [8, 21, 27, 44, 45, 62, 66, 157, 175, 177]
10	24	3T2 x 4T2	$Z_3 \wr 8T3$	[28, 37, 48]
11	24	3T2 x 4T1	$Z_3 \wr 8T2$	[39, 53, 119]
12	24	$4T3 \times_{2T1} 3T2$	$Z_4 \wr 6T2$	[38, 54, 118]
13	24	$4T3 \times_{2T1} 3T2$	$Z_2 \wr 6T2$	[15, 38, 42, 49, 116]
14	24	4T3 x 3T1	$Z_2 \wr 6T1$	[42, 51, 121]
15	24	$4T3 \times_{2T1} 3T2$	$Z_2 \wr 6T2$	[13, 38, 42, 49, 116]
16	36	3T2 x 3T2	$Z_3 \wr 4T2$	6T9 [37, 38, 39, 70, 71, 83, 161, 268, 283]
17	36		$Z_3 \wr 4T1$	6T10 [40, 72, 73, 160]
18	36	6T1 x 3T2	$Z_3 \wr 4T2$	[42, 70, 158]
19	36	12T5 x 3T1	$Z_3 \wr 4T1$	[131, 159]
20	36	4T4 x 3T1	$Z_2 \wr 9T2$	[85, 194]
21	48	4T5 x 2T1	$Z_2 \wr 3T2$	6T11 [22, 23, 24, 48, 49, 53, 54, 83, 95, 100, 108, 112, 213]
22	48	4T5 x 2T1	$Z_2 \wr 3T2$	6T11 [21, 23, 24, 48, 49, 53, 54, 83, 95, 100, 108, 112, 213]
23	48	4T5 x 2T1	$Z_2 \wr 3T2$	6T11 [21, 22, 24, 48, 49, 53, 54, 83, 95, 100, 108, 112, 213]
24	48	4T5 x 2T1	$Z_2 \wr 3T2$	6T11 [21, 22, 23, 48, 49, 53, 54, 83, 95, 100, 108, 112, 213]
25	48	4T4 x 4T2	$Z_2 \wr 6T1$	[26, 51, 87, 89, 90, 143]
26	48	4T4 x 4T2	$Z_2 \wr 6T1$	[25, 51, 87, 89, 90, 143]
27	48	$4T5 \times_{2T1} 4T1$	$Z_2 \wr 12T5$	[30, 98, 102]
28	48	4T3 x 3T2	$Z_2 \wr 6T3$	[81, 86, 156]
29	48	4T4 x 4T1	$Z_4 \wr 4T4$	[94, 99]
30	48	$4T5 \times_{2T1} 4T1$	$Z_2 \wr 12T5$	[27, 98, 102]
31	48		$Z_4 \wr 3T1$	[55, 57]
32	48	$4T4 \times_{3T1} 4T4$	$Z_2 \wr 6T4$	8T32 [56, 144, 271]
33	60			5T4 [75, 230]
34	72		$Z_3 \wr 4T3$	6T13 [35, 36, 77, 79, 116, 118, 121, 172, 200]
35	72	$6T2 \wr 2T1$	$Z_3 \wr 4T3$	6T13 [34, 36, 77, 79, 116, 118, 121, 172, 200]
36	72		$Z_3 \wr 4T3$	6T13 [34, 35, 77, 79, 116, 118, 121, 172, 200]
37	72	6T3 x 3T2	$Z_3 \wr 8T3$	[81, 117, 195]
38	72	$6T9 \times_{4T2} 4T3$	$Z_3 \wr 8T4$	[169, 196]
39	72	$6T9 \times_{2T1} 4T1$	$Z_3 \wr 8T2$	[170, 197]
40	72	6T10 x 2T1	$Z_3 \wr 8T2$	[41, 82, 119, 171, 198]
41	72	6T10 x 2T1	$Z_3 \wr 8T2$	[40, 82, 119, 171, 198]
42	72	$6T1 \wr 2T1$	$Z_2 \wr 6T5$	[167, 208]
43	72	4T4 x 3T2	$Z_2 \wr 9T4$	[206, 234]
44	72	$4T5 \times_{2T1} 3T2$	$Z_2 \wr 9T5$	[127, 233]
45	72	4T5 x 3T1	$Z_2 \wr 9T4$	[205, 231]
46	72		$Z_3 \wr 8T1$	9T15 [173]
47	72		$Z_3 \wr 8T5$	9T14 [174]
48	96	4T5 x 4T2	$Z_2 \wr 6T3$	[86, 136, 138, 139]
49	96	$4T5 \times_{2T1} 4T3$	$Z_2 \wr 6T2$	[50, 52, 135, 145, 147]
50	96	$4T5 \times_{2T1} 4T3$	$Z_2 \wr 6T2$	[49, 52, 135, 145, 147]
51	96	4T4 x 4T3	$Z_2 \wr 6T1$	[134, 141]
52	96	$4T5 \times_{2T1} 4T3$	$Z_2 \wr 6T2$	[49, 50, 135, 145, 147]
53	96	4T5 x 4T1	$Z_4 \wr 4T5$	[150, 153]

54	96	$4T5 \times_{2T1} 4T3$	$Z_2 \wr 12T12$	[151, 152]
55	96	12T31 x 2T1	$Z_4 \wr 6T1$	[94]
56	96	12T32 x 2T1	$Z_2 \wr 6T4$	[90, 187, 189]
57	96	$12T31 \times_{4T4} 8T12$		
58	96		$Z_2 \wr 6T1$	8T33 [59, 87, 99, 280]
59	96		$Z_2 \wr 6T1$	8T33 [58, 87, 99, 280]
60	96		$Z_4 \wr 6T1$	[61, 89, 104]
61	96		$Z_4 \wr 6T1$	[60, 89, 104]
62	96		$Z_4 \wr 3T2$	[63, 64, 65, 95, 98]
63	96		$Z_4 \wr 3T2$	[62, 64, 65, 95, 98]
64	96		$Z_4 \wr 3T2$	[62, 63, 65, 95, 98]
65	96		$Z_4 \wr 3T2$	[62, 63, 64, 95, 98]
66	96	$4T5 \times_{3T2} 4T5$	$Z_2 \wr 6T2$	8T34 [67, 68, 69, 100, 102, 184, 192, 281, 282]
67	96	$4T5 \times_{3T2} 4T5$	$Z_2 \wr 6T2$	8T34 [66, 68, 69, 100, 102, 184, 192, 281, 282]
68	96	$4T5 \times_{3T2} 4T5$	$Z_2 \wr 6T2$	8T34 [66, 67, 69, 100, 102, 184, 192, 281, 282]
69	96	$4T5 \times_{3T2} 4T5$	$Z_2 \wr 6T2$	8T34 [66, 67, 68, 100, 102, 184, 192, 281, 282]
70	108	6T5 x 3T2	$Z_3 \wr 4T2$	[130]
71	108	$6T9 \times_{2T1} 3T2$	$Z_3 \wr 4T2$	[130]
72	108	$6T10 \times_{2T1} 3T2$	$Z_3 \wr 4T1$	[131]
73	108	6T10 x 3T1	$Z_3 \wr 4T1$	[131]
74	120			5T5 [123, 124, 256, 257]
75	120	5T4 x 2T1	$Z_2 \wr 5T4$	10T11 [76, 255]
76	120	5T4 x 2T1	$Z_2 \wr 5T4$	10T11 [75, 255]
77	144	6T13 x 2T1	$Z_3 \wr 8T9$	[78, 125, 156, 210, 235]
78	144	6T13 x 2T1	$Z_3 \wr 8T9$	[77, 125, 156, 210, 235]
79	144	$6T13 \times_{2T1} 4T1$	$Z_3 \wr 8T10$	[80, 209, 211, 237]
80	144	$6T13 \times_{2T1} 4T1$	$Z_3 \wr 8T10$	[79, 209, 211, 237]
81	144	$6T9 \times_{2T1} 4T3$	$Z_2 \wr 6T9$	[217, 240]
82	144	$6T10 \times_{2T1} 4T3$	$Z_3 \wr 8T10$	[209, 241]
83	144	4T5 x 3T2	$Z_2 \wr 9T8$	[239, 258]
84	144		$Z_3 \wr 8T8$	9T19 [212]
85	144	4T4 x 4T4	$Z_2 \wr 9T2$	[164]
86	192	4T5 x 4T3	$Z_2 \wr 6T3$	[185, 186]
87	192	8T33 x 2T1	$Z_2 \wr 6T1$	[88, 134]
88	192	8T33 x 2T1	$Z_2 \wr 6T1$	[87, 134]
89	192	12T60 x 2T1	$Z_4 \wr 12T2$	[92, 141]
90	192	$4T2 \wr 3T1$	$Z_2 \wr 6T6$	[222]
91	192		$Z_2 \wr G$	[93]
92	192	12T60 x 2T1	$Z_2 \wr 12T60$	[89, 141]
93	192		$Z_2 \wr G$	[91]
94	192	$4T1 \wr 3T1$	$Z_4 \wr 3T1$	
95	192	12T65 x 2T1	$Z_2 \wr 12T65$	[96, 97, 150, 151]
96	192	12T64 x 2T1	$Z_2 \wr 12T64$	[95, 97, 150, 151]
97	192	12T65 x 2T1	$Z_2 \wr 12T65$	[95, 96, 150, 151]
98	192	$12T65 \times_{2T1} 4T1$	$Z_4 \wr 12T5$	
99	192	$8T33 \times_{2T1} 4T1$	$Z_2 \wr 12T1$	[105]
100	192	8T34 x 2T1	$Z_2 \wr 6T2$	[101, 103, 106, 135, 139, 221, 223, 225, 226]
101	192	8T34 x 2T1	$Z_2 \wr 6T2$	[100, 103, 106, 135, 139, 221, 223, 225, 226]

102	192	$8T34 \times_{2T1} 4T1$	$Z_2 \wr 12T5$	[107]
103	192	8T34 x 2T1	$Z_2 \wr 6T2$	[100, 101, 106, 135, 139, 221, 223, 225, 226]
104	192		$Z_2 \wr 12T57$	
105	192	$8T33 \times_{2T1} 4T1$	$Z_2 \wr 12T1$	[99]
106	192	8T34 x 2T1	$Z_2 \wr 6T2$	[100, 101, 103, 135, 139, 221, 223, 225, 226]
107	192	$8T34 \times_{2T1} 4T1$	$Z_2 \wr 12T5$	[102]
108	192		$Z_2 \wr 6T3$	8T41 [109, 110, 111, 136, 145, 152, 153, 289]
109	192		$Z_2 \wr 6T3$	8T41 [108, 110, 111, 136, 145, 152, 153, 289]
110	192		$Z_2 \wr 6T3$	8T41 [108, 109, 111, 136, 145, 152, 153, 289]
111	192		$Z_2 \wr 6T3$	8T41 [108, 109, 110, 136, 145, 152, 153, 289]
112	192		$Z_4 \wr 6T3$	[113, 114, 115, 138, 147]
113	192		$Z_4 \wr 6T3$	[112, 114, 115, 138, 147]
114	192		$Z_4 \wr 6T3$	[112, 113, 115, 138, 147]
115	192		$Z_4 \wr 6T3$	[112, 113, 114, 138, 147]
116	216	$6T13 \times_{2T1} 3T2$	$Z_3 \wr 4T3$	[120, 167, 169]
117	216	6T9 x 3T2	$Z_3 \wr 8T3$	[168]
118	216	$6T13 \times_{2T1} 3T2$	$Z_3 \wr 8T4$	[169]
119	216	6T10 x 3T2	$Z_3 \wr 8T2$	[170]
120	216	$6T13 \times_{2T1} 3T2$	$Z_3 \wr 4T3$	[116, 167, 169]
121	216	6T13 x 3T1	$Z_3 \wr 4T3$	[167]
122	216		$Z_3 \wr 8T12$	9T23 [232]
123	240	5T5 x 2T1	$Z_2 \wr 5T5$	10T22 [270]
124	240	$5T5 \times_{2T1} 4T1$		
125	288	$6T3 \wr 2T1$	$Z_3 \wr 8T18$	[248, 260]
126	288	$6T4 \wr 2T1$	$Z_2 \wr 6T5$	8T42 [128, 129, 158, 159, 205, 206]
127	288	$4T5 \times_{2T1} 4T5$	$Z_2 \wr 9T5$	[204]
128	288		$Z_2 \wr 6T5$	8T42 [126, 129, 158, 159, 205, 206]
129	288		$Z_2 \wr 6T5$	8T42 [126, 128, 158, 159, 205, 206]
130	324	$3T1 \wr 4T2$	$Z_3 \wr 4T2$	
131	324	$3T1 \wr 4T1$	$Z_3 \wr 4T1$	
132	324		$Z_3 \wr 4T4$	9T25 [133, 194, 284]
133	324		$Z_3 \wr 4T4$	9T25 [132, 194, 284]
134	384	$2T1 \wr 6T1$	$Z_2 \wr 6T1$	[142]
135	384	$2T1 \wr 6T2$	$Z_2 \wr 6T2$	[148]
136	384	8T41 x 2T1	$Z_2 \wr 6T3$	[137, 186]
137	384	8T41 x 2T1	$Z_2 \wr 6T3$	[136, 186]
138	384	12T112 x 2T1	$Z_2 \wr 12T112$	[140, 185]
139	384	$4T2 \wr 3T2$	$Z_2 \wr 6T11$	[250]
140	384	12T112 x 2T1	$Z_2 \wr 12T112$	[138, 185]
141	384	$12T51 \times_{6T6} 12T61$	$Z_4 \wr 6T1$	
142	384		$Z_2 \wr 6T1$	[134]
143	384		$Z_2^4 \wr_r 12T93$	
144	384		$Z_2 \wr 12T4$	
145	384	$8T41 \times_{4T2} 4T3$	$Z_2 \wr 12T13$	[146]
146	384	$8T41 \times_{4T2} 4T3$	$Z_2 \wr 12T13$	[145]
147	384	$12T112 \times_{4T2} 4T3$	$Z_4 \wr 12T13$	[149]
148	384		$Z_2 \wr 6T2$	[135]
149	384	$12T112 \times_{4T2} 4T3$	$Z_4 \wr 12T13$	[147]

150	384	$4T1 \wr 3T2$	$Z_4 \wr 3T2$	
151	384	$12T65 \times_{2T1} 4T3$	$Z_4 \wr 6T2$	
152	384	$8T41 \times_{4T2} 4T3$	$Z_2 \wr 12T12$	[154]
153	384	$8T41 \times_{2T1} 4T1$	$Z_2 \wr 12T11$	[155]
154	384	$8T41 \times_{4T2} 4T3$	$Z_2 \wr 12T12$	[152]
155	384	$8T41 \times_{2T1} 4T1$	$Z_2 \wr 12T11$	[153]
156	432	6T13 x 3T2	$Z_3 \wr 8T9$	[217]
157	432		$Z_3 \wr 8T23$	9T26 [259]
158	576	8T42 x 2T1	$Z_2 \wr 6T5$	[208]
159	576	$8T42 \times_{2T1} 4T1$	$Z_2 \wr 12T19$	
160	576		$Z_2 \wr 6T10$	8T46 [162, 198]
161	576		$Z_2 \wr 6T9$	8T45 [163, 165, 195, 196, 197, 239]
162	576		$Z_2 \wr 6T10$	8T46 [160, 198]
163	576		$Z_2 \wr 6T9$	8T45 [161, 165, 195, 196, 197, 239]
164	576	$12T85 \times_{3T1} 4T4$	$Z_2 \wr 9T2$	
165	576		$Z_2 \wr 6T9$	8T45 [161, 163, 195, 196, 197, 239]
166	576		$Z_2 \wr 9T1$	
167	648	$6T5 \wr 2T1$	$Z_3 \wr 4T3$	
168	648	$6T9 \times_{2T1} 6T9$	$Z_3 \wr 8T3$	
169	648	$6T13 \times_{4T2} 6T9$	$Z_3 \wr 8T4$	
170	648	$6T9 \times_{2T1} 6T10$	$Z_3 \wr 8T2$	
171	648	$6T10 \times_{2T1} 6T10$	$Z_3 \wr 8T2$	
172	648	$6T13 \times_{4T3} 6T13$	$Z_3 \wr 8T4$	
173	648	$9T15 \times_{8T1} 9T15$	$Z_3 \wr 8T1$	
174	648	$9T14 \times_{8T5} 9T14$	$Z_3 \wr 8T5$	
175	648		$Z_3 \wr 4T5$	9T29 [231, 290]
176	648		$Z_3 \wr 6T6$	9T28 [234, 292]
177	648		$Z_3 \wr 6T8$	9T30 [178, 233, 291]
178	648		$Z_3 \wr 6T8$	9T30 [177, 233, 291]
179	660*			11T5
180	720	6T15 x 2T1	$Z_2 \wr 6T15$	[286]
181	720			10T31
182	720			10T30
183	720			6T16 [219, 285, 287]
184	768		$Z_2 \wr 12T8$	[190, 191]
185	768	$12T86 \times_{6T11} 12T114$	$Z_4 \wr 6T3$	
186	768		$Z_2 \wr 6T3$	[193]
187	768		$Z_2 \wr 6T4$	[188]
188	768	$2T1 \wr 6T4$	$Z_2 \wr 6T4$	[187]
189	768		$Z_4 \wr 6T4$	
190	768		$Z_2 \wr 12T8$	[184, 191]
191	768		$Z_2 \wr 12T8$	[184, 190]
192	768		$Z_2^5 \wr_r 12T98$	
193	768	$2T1 \wr 6T3$	$Z_2 \wr 6T3$	[186]
194	972	$3T1 \wr 4T4$	$Z_3 \wr 4T4$	
195	1152	8T45 x 2T1	$Z_2 \wr 6T9$	[240]
196	1152	$8T45 \times_{4T2} 4T3$	$Z_2 \wr 12T38$	
197	1152	$8T45 \times_{2T1} 4T1$	$Z_2 \wr 12T39$	

198	1152	8T46 x 2T1	$Z_2 \wr 6T10$	[199, 241]
199	1152	8T46 x 2T1	$Z_2 \wr 6T10$	[198, 241]
200	1152	$6T8 \wr 2T1$	$Z_2 \wr 6T13$	8T47 [201, 202, 203, 235, 237]
201	1152		$Z_2 \wr 6T13$	8T47 [200, 202, 203, 235, 237]
202	1152		$Z_2 \wr 6T13$	8T47 [200, 201, 203, 235, 237]
203	1152	$6T7 \wr 2T1$	$Z_2 \wr 6T13$	8T47 [200, 201, 202, 235, 237]
204	1152	$12T127 \times_{3T2} 4T5$	$Z_2 \wr 9T5$	
205	1152	$8T42 \times_{3T2} 4T5$	$Z_2 \wr 9T4$	
206	1152	$8T42 \times_{3T1} 4T4$	$Z_2 \wr 9T4$	
207	1152		$Z_2 \wr 9T3$	
208	1152	$6T6 \wr 2T1$	$Z_2 \wr 6T5$	
209	1296	$6T13 \times_{2T1} 6T10$	$Z_3 \wr 8T10$	
210	1296	$6T13 \times_{4T2} 6T13$	$Z_3 \wr 8T9$	
211	1296	$6T13 \times_{4T2} 6T13$	$Z_3 \wr 8T10$	
212	1296	$9T19 \times_{8T8} 9T19$	$Z_3 \wr 8T8$	
213	1296		$Z_3 \wr 6T11$	9T31 [258, 294]
214	1296		$Z_3 \wr 8T11$	
215	1296		$Z_3 \wr 8T7$	
216	1296		$Z_3 \wr 8T6$	
217	1296	$6T13 \times_{2T1} 6T9$	$Z_3 \wr 8T9$	
218	1320*			
219	1440	6T16 x 2T1	$Z_2 \wr 6T16$	[293]
220	1440			10T35
221	1536		$Z_4 \wr 6T7$	
222	1536	$4T3 \wr 3T1$	$Z_2 \wr 6T6$	
223	1536		$Z_2 \wr 6T8$	[224]
224	1536	$2T1 \wr 6T8$	$Z_2 \wr 6T8$	[223]
225	1536		$Z_4 \wr 6T8$	
226	1536		$Z_2 \wr 6T7$	[227]
227	1536	$2T1 \wr 6T7$	$Z_2 \wr 6T7$	[226]
228	1728		$Z_2 \wr 9T6$	
229	1728		$Z_2 \wr 9T7$	
230	1920		$Z_2 \wr 15T5$	
231	1944	$3T1 \wr 4T5$	$Z_3 \wr 4T5$	
232	1944		$Z_3 \wr 8T12$	
233	1944	$9T30 \times_{2T1} 3T2$	$Z_3 \wr 8T14$	
234	1944	$9T28 \times_{2T1} 3T2$	$Z_3 \wr 8T13$	
235	2304	8T47 x 2T1	$Z_2 \wr 6T13$	[236, 260]
236	2304	8T47 x 2T1	$Z_2 \wr 6T13$	[235, 260]
237	2304	$8T47 \times_{2T1} 4T1$	$Z_2 \wr 12T79$	[238]
238	2304	$8T47 \times_{2T1} 4T1$	$Z_2 \wr 12T79$	[237]
239	2304	$8T45 \times_{3T2} 4T5$	$Z_2 \wr 9T8$	
240	2304	$2T1 \wr 6T9$	$Z_2 \wr 6T9$	
241	2304	$2T1 \wr 6T10$	$Z_2 \wr 6T10$	
242	2592		$Z_3 \wr_r 8T22$	
243	2592		$Z_3 \wr 8T15$	
244	2592		$Z_3 \wr 8T16$	
245	2592		$Z_3 \wr 8T19$	[246, 247]

246	2592		$Z_3 \wr 8T19$	[245, 247]
247	2592		$Z_3 \wr 8T19$	[245, 246]
248	2592	$6T9 \wr 2T1$	$Z_3 \wr 8T18$	
249	2592	$6T10 \wr 2T1$	$Z_3 \wr 8T17$	
250	3072	$4T3 \wr 3T2$	$Z_2 \wr 6T11$	
251	3456		$Z_2 \wr 9T11$	[253]
252	3456		$Z_2 \wr 9T12$	
253	3456		$Z_2 \wr 9T11$	[251]
254	3456		$Z_2 \wr 9T10$	
255	3840	$2T1 \wr 6T12$	$Z_2 \wr 6T12$	
256	3840		$Z_2^5 \wr_r 12T124$	
257	3840		$Z_2 \wr 15T10$	
258	3888	$9T31 \times_{2T1} 3T2$	$Z_3 \wr 8T24$	
259	3888		$Z_3 \wr 8T23$	
260	4608	$6T11 \wr 2T1$	$Z_2 \wr 6T13$	
261	5184	$3T2 \wr 4T2$	$Z_3 \wr 8T29$	[267]
262	5184		$Z_3 \wr 8T27$	[264]
263	5184		$Z_3 \wr 8T30$	
264	5184	$3T2 \wr 4T1$	$Z_3 \wr 8T27$	[262]
265	5184	$4T4 \wr 3T1$	$Z_2 \wr 9T17$	
266	5184		$Z_3 \wr 8T26$	
267	5184		$Z_3 \wr 8T29$	[261]
268	6912		$Z_2 \wr 9T18$	
269	7200	$6T12 \wr 2T1$		10T40
270	7680	$2T1 \wr 6T14$	$Z_2 \wr 6T14$	
271	7776		$Z_3 \wr 8T32$	
272	7920*			11T6
273	10368		$Z_2 \wr 9T22$	
274	10368	$6T13 \wr 2T1$	$Z_2 \wr 8T35$	
275	10368	$4T4 \wr 3T2$	$Z_2 \wr 9T20$	
276	10368		$Z_2 \wr 9T21$	
277	11520		$Z_2 \wr 15T20$	
278	14400			10T42
279	14400			10T41
280	15552	$3T2 \wr 4T4$	$Z_3 \wr 8T38$	
281	15552		$Z_3 \wr 8T39$	
282	15552		$Z_3 \wr 8T40$	
283	20736		$Z_2 \wr 9T24$	
284	20736		$Z_2 \wr 9T25$	
285	23040		$Z_2 \wr 15T28$	
286	23040	$2T1 \wr 6T15$	$Z_2 \wr 6T15$	
287	23040			
288	28800	$6T14 \wr 2T1$		10T43
289	31104	$3T2 \wr 4T5$	$Z_3 \wr 8T44$	
290	41472		$Z_2 \wr 9T29$	
291	41472		$Z_2 \wr 9T30$	
292	41472	$4T5 \wr 3T1$	$Z_2 \wr 9T28$	
293	46080	$2T1 \wr 6T16$	$Z_2 \wr 6T16$	

294	82944	$4T5 \wr 3T2$	$Z_2 \wr 9T31$
295	95040*		
296	259200	$6T15 \wr 2T1$	
297	518400		
298	518400		
299	1036800	$6T16 \wr 2T1$	
300	12!/2*		
301	12!*		

5.2. Transitive Groups of degree 14

No	Size	Group	Factor	Factor
1	14	7T1 x 2T1	$Z_7 \wr 2T1$	[8, 9, 29]
2	14		$Z_7 \wr 2T1$	7T2 [3, 8, 13, 27, 28, 38]
3	28	7T2 x 2T1	$Z_7 \wr 4T2$	[13, 38]
4	42		$Z_7 \wr 6T1$	7T4 [7, 14, 24, 40, 41, 48]
5	42	7T3 x 2T1	$Z_7 \wr 6T1$	[14, 18, 44]
6	56		$Z_2 \wr 7T1$	8T25 [9, 21, 29]
7	84	7T4 x 2T1	$Z_7 \wr 12T2$	[24, 48]
8	98	$7T1 \wr 2T1$	$Z_7 \wr 2T1$	
9	112	8T25 x 2T1	$Z_2 \wr 7T1$	[29]
10	168			7T5 [17, 19, 33, 34, 42, 43, 50, 51]
11	168		$Z_2 \wr 7T3$	8T36 [18, 35, 44]
12	196		$Z_7 \wr 4T1$	
13	196	7T2 x 7T2	$Z_7 \wr 4T2$	
14	294	$7T4 \times_{3T1} 7T3$	$Z_7 \wr 6T1$	
15	294		$Z_7 \wr 3T2$	
16	336			8T43
17	336	7T5 x 2T1	$Z_2 \wr 7T5$	[19, 42, 43, 51]
18	336	8T36 x 2T1	$Z_2 \wr 7T3$	[44]
19	336	7T5 x 2T1	$Z_2 \wr 7T5$	[17, 42, 43, 51]
20	392	$7T2 \wr 2T1$	$Z_7 \wr 4T3$	
21	448	$8T25 \times_{7T1} 8T25$	$Z_2 \wr 7T1$	[29]
22	588		$Z_7 \wr 12T5$	
23	588		$Z_7 \wr 12T1$	
24	588	$7T4 \times_{3T1} 7T4$	$Z_7 \wr 12T2$	
25	588		$Z_7 \wr 6T3$	
26	882	$7T3 \wr 2T1$	$Z_7 \wr 6T5$	
27	896		$Z_2 \wr 7T2$	[28, 38]
28	896		$Z_2 \wr 7T2$	[27, 38]
29	896	$2T1 \wr 7T1$	$Z_2 \wr 7T1$	
30	1092*			
31	1176		$Z_7 \wr 12T13$	
32	1176		$Z_7 \wr 12T14$	
33	1344			[42]
34	1344		$Z_2 \wr 7T5$	8T48 [43, 50, 51]
35	1344	$8T36 \times_{7T3} 8T36$	$Z_2 \wr 7T3$	[44]

36	1764		$Z_7 \wr 12T19$	
37	1764		$Z_7 \wr 12T18$	
38	1792	$2T1 \wr 7T2$	$Z_2 \wr 7T2$	
39	2184*			
40	2688		$Z_2 \wr 7T4$	[41, 48]
41	2688		$Z_2 \wr 7T4$	[40, 48]
42	2688	14T33 x 2T1	$Z_2 \wr 14T33$	
43	2688	8T48 x 2T1	$Z_2 \wr 7T5$	[51]
44	2688	$2T1 \wr 7T3$	$Z_2 \wr 7T3$	
45	3528	$7T4 \wr 2T1$	$Z_7 \wr 12T42$	
46	5040			7T7 [49, 54, 55, 57]
47	5040	7T6 x 2T1	$Z_2 \wr 7T6$	[56]
48	5376	$2T1 \wr 7T4$	$Z_2 \wr 7T4$	
49	10080	7T7 x 2T1	$Z_2 \wr 7T7$	[57]
50	10752		$Z_2 \wr 7T5$	[51]
51	21504	$2T1 \wr 7T5$	$Z_2 \wr 7T5$	
52	56448	$7T5 \wr 2T1$		
53	161280		$Z_2 \wr 7T6$	[56]
54	322560		$Z_2 \wr 7T7$	[55, 57]
55	322560		$Z_2 \wr 7T7$	[54, 57]
56	322560	$2T1 \wr 7T6$	$Z_2 \wr 7T6$	
57	645120	$2T1 \wr 7T7$	$Z_2 \wr 7T7$	
58	12700800	$7T6 \wr 2T1$		
59	25401600			
60	25401600			
61	50803200	$7T7 \wr 2T1$		
62	14!/2*			
63	14!*			

5.3. Transitive Groups of degree 15

No	Size	Group	Factor	Factor
1	15	5T1 x 3T1	$Z_5 \wr 3T1$	[25, 36]
2	30	$5T2 \times_{2T1} 3T2$	$Z_5 \wr 6T2$	[31, 45]
3	30	5T2 x 3T1	$Z_5 \wr 6T1$	[30, 46]
4	30	3T2 x 5T1	$Z_5 \wr 3T2$	[32, 44]
5	60			5T4 [15, 16, 23, 53, 61, 69, 76, 88, 90]
6	60	$5T3 \times_{2T1} 3T2$	$Z_5 \wr 12T5$	[37, 54]
7	60	5T2 x 3T2	$Z_5 \wr 6T3$	[40, 55]
8	60	5T3 x 3T1	$Z_5 \wr 12T1$	[38, 56]
9	75		$Z_5 \wr 3T1$	[25]
10	120			5T5 [21, 22, 24, 29, 62, 63, 70, 77, 78, 83, 89, 91, 93]
11	120	5T3 x 3T2	$Z_5 \wr 12T11$	[49, 64]
12	150		$Z_5 \wr 6T1$	[30]
13	150		$Z_5 \wr 3T2$	[14, 31, 32]
14	150		$Z_5 \wr 3T2$	[13, 31, 32]
15	180	5T4 x 3T1	$Z_3 \wr 5T4$	[16, 69]

16	180	5T4 x 3T1	$Z_3 \wr 5T4$	[15, 69]
17	300		$Z_5 \wr 12T5$	[37]
18	300		$Z_5 \wr 6T3$	[40]
19	300		$Z_5 \wr 12T1$	[38]
20	360*			6T15
21	360	$5T5 \times_{2T1} 3T2$	$Z_3 \wr 10T12$	[22, 77]
22	360	$5T5 \times_{2T1} 3T2$	$Z_3 \wr 10T12$	[21, 77]
23	360	5T4 x 3T2	$Z_3 \wr 10T11$	[76]
24	360	5T5 x 3T1	$Z_3 \wr 5T5$	[78]
25	375	$5T1 \wr 3T1$	$Z_5 \wr 3T1$	
26	405		$Z_3 \wr 5T1$	[36]
27	600		$Z_5 \wr 12T11$	[49]
28	720*			6T16
29	720	5T5 x 3T2	$Z_3 \wr 10T22$	[83]
30	750	$15T12 \times_{2T1} 5T2$	$Z_5 \wr 6T1$	
31	750	$15T14 \times_{2T1} 5T2$	$Z_5 \wr 6T2$	
32	750	$5T1 \wr 3T2$	$Z_5 \wr 3T2$	
33	810		$Z_3 \wr 10T1$	[44]
34	810		$Z_3 \wr 5T2$	[35, 45, 46]
35	810		$Z_3 \wr 5T2$	[34, 45, 46]
36	1215	$3T1 \wr 5T1$	$Z_3 \wr 5T1$	
37	1500	$15T17 \times_{4T1} 5T3$	$Z_5 \wr 12T5$	
38	1500	$15T19 \times_{4T1} 5T3$	$Z_5 \wr 12T1$	
39	1500		$Z_5 \wr 4T4$	
40	1500	$15T18 \times_{2T1} 5T2$	$Z_5 \wr 6T3$	
41	1620		$Z_3 \wr 5T3$	[42, 54, 56]
42	1620		$Z_3 \wr 5T3$	[41, 54, 56]
43	1620		$Z_3 \wr 10T3$	[55]
44	2430	$15T33 \times_{2T1} 3T2$	$Z_3 \wr 10T1$	
45	2430	$15T35 \times_{2T1} 3T2$	$Z_3 \wr 10T2$	
46	2430	$3T1 \wr 5T2$	$Z_3 \wr 5T2$	
47	2520*			7T6
48	3000		$Z_5 \wr 6T8$	
49	3000	$15T27 \times_{4T1} 5T3$	$Z_5 \wr 12T11$	
50	3000	$5T2 \wr 3T1$	$Z_5 \wr 6T6$	
51	3000		$Z_5 \wr 4T5$	
52	3240		$Z_3 \wr 10T5$	[64]
53	4860		$Z_3 \wr 5T4$	[69]
54	4860	$15T42 \times_{2T1} 3T2$	$Z_3 \wr 10T4$	
55	4860	$15T43 \times_{2T1} 3T2$	$Z_3 \wr 10T3$	
56	4860	$3T1 \wr 5T3$	$Z_3 \wr 5T3$	
57	6000		$Z_5 \wr 12T31$	
58	6000		$Z_5 \wr 12T27$	
59	6000		$Z_5 \wr 12T29$	
60	6000	$5T2 \wr 3T2$	$Z_5 \wr 6T11$	
61	9720		$Z_3 \wr 10T11$	[76]
62	9720		$Z_3 \wr 10T12$	[77]
63	9720		$Z_3 \wr 5T5$	[78]

64	9720	$15T52 \times_{2T1} 3T2$	$Z_3 \wr 10T5$	
65	12000		$Z_5 \wr 12T63$	
66	12000		$Z_5 \wr 12T62$	
67	12000		$Z_5 \wr 12T55$	
68	12000		$Z_5 \wr 12T53$	
69	14580	$3T1 \wr 5T4$	$Z_3 \wr 5T4$	
70	19440		$Z_3 \wr 10T22$	[83]
71	19440		$Z_3 \wr 10T8$	
72	20160*			8T49
73	24000		$Z_5 \wr 12T98$	
74	24000		$Z_5 \wr 12T95$	
75	24000	$5T3 \wr 3T1$	$Z_5 \wr 12T94$	
76	29160	$15T61 \times_{2T1} 3T2$	$Z_3 \wr 10T11$	
77	29160	$15T62 \times_{2T1} 3T2$	$Z_3 \wr 10T12$	
78	29160	$3T1 \wr 5T5$	$Z_3 \wr 5T5$	
79	38880		$Z_3 \wr 10T16$	
80	38880		$Z_3 \wr 10T15$	
81	38880	$3T2 \wr 5T1$	$Z_3 \wr 10T14$	
82	48000	$5T3 \wr 3T2$	$Z_5 \wr 12T150$	
83	58320	$15T70 \times_{2T1} 3T2$	$Z_3 \wr 10T22$	
84	77760		$Z_3 \wr 10T25$	
85	77760		$Z_3 \wr 10T24$	
86	77760	$3T2 \wr 5T2$	$Z_3 \wr 10T23$	
87	155520	$3T2 \wr 5T3$	$Z_3 \wr 10T29$	
88	233280		$Z_3 \wr 10T34$	
89	466560		$Z_3 \wr 10T38$	
90	466560	$3T2 \wr 5T4$	$Z_3 \wr 10T36$	
91	466560		$Z_3 \wr 10T37$	
92	648000	$5T4 \wr 3T1$		
93	933120	$3T2 \wr 5T5$	$Z_3 \wr 10T39$	
94	1296000			
95	1296000			
96	1296000	$5T4 \wr 3T2$		
97	2592000			
98	2592000			
99	5184000			
100	5184000			
101	5184000	$5T5 \wr 3T1$		
102	10368000	$5T5 \wr 3T2$		
103	15!/2*			
104	15!*			

6. The polynomials

Degree 12

S_{12}	$x^{12}-x+1$
A_{12}	$x^{12}-866052x+2381643$
T_{299}	$x^{12}-x^9-x^7-x^6-x^4+x^3+x^2+x+1$
T_{298}	$x^{12}-72x^2-120x-50$
T_{297}	$x^{12}-24x^7+144x^2+200$
T_{296}	$x^{12}-12x^{11}+36x^{10}+6251x^6-37506x^5+9768751$
M_{12}	$x^{12}-375x^8-3750x^6-75000x^3+228750x^2-750000x+1265625$
T_{294}	$x^{12}-x^{11}+x^{10}+x^9+x^7+x^6+3x^4-x^3+x^2+1$
T_{293}	$x^{12}-x^6-x^2-1$
T_{292}	$x^{12}-3x^{11}+5x^9-3x^8+3x^7+2x^6-6x^5-3x^4+1$
T_{291}	$x^{12}-12x^9-9x^8-64x^3-144x^2-108x-27$
T_{290}	$x^{12}-2x^{10}-6x^9+x^8+8x^7-10x^5-7x^4+20x^3+2x^2-2x-4$
T_{289}	$x^{12}-x^{11}-x^{10}-x^7-x^5+3x^4-x^3+2x^2-x+1$
T_{288}	$x^{12}-4x^{11}+4x^{10}-50x^4+120x^3-112x^2+48x-8$
T_{287}	$x^{12}-3x^{10}-3x^8+2x^4+2x^2+2$
T_{286}	$x^{12}-x^6-3x^4-1$
T_{285}	$x^{12}-x^{10}-2x^4+1$
T_{284}	$x^{12}-12x^9-9x^8+64x^3+144x^2+108x+27$
T_{283}	$x^{12}-8x^9+24x^6+144x^5+96x^3+144x^2+48$
T_{282}	$x^{12}-7x^{10}-8x^9+7x^8+16x^7-x^6-16x^5-16x^4-16x^3+24x^2+40x-8$
T_{281}	$x^{12}-12x^{10}+54x^8-129x^6+207x^4-70x^3-189x^2+210x-153$
T_{280}	$x^{12}-4x^{11}+6x^{10}-2x^9-5x^8+6x^7-4x^5+2x^4+2$
T_{279}	$x^{12}-4x^{11}+4x^{10}+4x^7-6x^6-4x^5+36x^2+36x+9$
T_{278}	$x^{12}+20x^8-80x^6+50x^4-320x^3-912x^2+1280x+800$
T_{277}	$x^{12}+3x^6+3x^2+4$
T_{276}	$x^{12}+192x^6-288x^5+108x^4+256x^3-576x^2+432x-108$
T_{275}	$x^{12}-9x^8+288x^6-756x^5+324x^4+1728x^3-1296x^2-1944x+2916$
T_{274}	$x^{12}+x^{11}+x^{10}-x^9+x^6+3x^5+x^3-x+1$
T_{273}	$x^{12}-12x^9+9x^8+192x^3-432x^2+324x-81$
M_{11}	$x^{12}-x^{11}-16x^{10}+15x^9+145x^8-8x^7-392x^6+88x^5+415x^4-255x^3$ $-64x^2+89x-41$
T_{271}	$x^{12}-135x^8-180x^7+399x^6+918x^5+693x^4+352x^3+216x^2+96x+16$
T_{270}	$x^{12}-2x^8-4x^2+8$
T_{269}	$x^{12}-5x^{10}-11x^8-5x^7-3x^6-60x^5-21x^4-20x^3+4x^2+70x-29$
T_{268}	$x^{12}+4x^9-3x^8-64x^3+144x^2-108x+27$
T_{267}	$x^{12}+12x^{10}-8x^9+54x^8-48x^7+132x^6-72x^5-33x^4-32x^3+8$
T_{266}	$x^{12}-8x^9-9x^8+24x^6+36x^5-81x^4-32x^3-36x^2+16$
T_{265}	$x^{12}+36x^{10}-24x^9+333x^8-288x^7+810x^6-72x^5-486x^4+80x^3$ $+162x^2-72x+9$
T_{264}	$x^{12}-4x^{11}+6x^{10}-3x^9-2x^8+3x^7-2x^5+x^4+x^3-x^2+1$
T_{263}	$x^{12}-162x^4-432x^3-432x^2-192x-32$
T_{262}	$x^{12}-72x^8-96x^7+184x^6+432x^5+369x^4+280x^3+216x^2+96x+16$
T_{261}	$x^{12}-4x^{11}+6x^{10}-4x^9+x^8+1$
T_{260}	$x^{12}-3x^2+3$
T_{259}	$x^{12}-12x^{10}+54x^8-110x^6+93x^4-4x^3-18x^2+12x-8$
T_{258}	$x^{12}-2x^9+2x^3+3$
T_{257}	$x^{12}+3x^8-2x^6+6x^4+1$

T_{256}	$x^{12}-3x^8+6x^4-8x^2+2$
T_{255}	$x^{12}-2x^8-6x^6+9x^4-1$
T_{254}	$x^{12}-12x^{11}-2x^{10}+316x^9+381x^8-2760x^7-11742x^6-26260x^5$ $-42490x^4-52184x^3-48664x^2-32640x-11943$
T_{253}	$x^{12}-4x^9-3x^8-32x^6-48x^5-18x^4+64x^3+144x^2+108x+27$
T_{252}	$x^{12}-12x^9+27x^8+12x^6-36x^5+27x^4-16x^3+36x^2+9$
T_{251}	$x^{12}+48x^6-72x^5+27x^4+64x^3-144x^2+108x-27$
T_{250}	$x^{12}+x^{10}+x^8-x^6+2$
T_{249}	$x^{12}-12x^{10}+54x^8-108x^6+81x^4-8x^3+24x+8$
T_{248}	$x^{12}+324x^6-648x^5+675x^4-744x^3+648x^2-288x+48$
T_{247}	$x^{12}-8x^9+24x^6+162x^4-32x^3+16$
T_{246}	$x^{12}+81x^4-216x^3+216x^2-96x+16$
T_{245}	$x^{12}-12x^{10}-54x^8-72x^7+96x^6+9x^4+200x^3+108x^2-4$
T_{244}	$x^{12}-10x^9+60x^7+195x^6+180x^5+105x^4-200x^3+600x+400$
T_{243}	$x^{12}-9x^8-12x^7-4x^6-81x^4-216x^3-216x^2-96x-16$
T_{242}	$x^{12}-4x^9+18x^8-4x^6-36x^5+81x^4+16x^3+108x^2+16$
T_{241}	$x^{12}+x^{10}-3x^8-x^6+6x^4-3$
T_{240}	$x^{12}+4x^8-2x^6+4x^4-x^2+7$
T_{239}	$x^{12}-12x^9+9x^8-32x^6+48x^5-18x^4-64x^3+144x^2-108x+27$
T_{238}	$x^{12}-2x^{10}-3x^8+4x^6+2x^4+4x^2-2$
T_{237}	$x^{12}-2x^{10}-x^8-4x^6+2$
T_{236}	$x^{12}+x^{10}+x^8-x^6+4x^4+1$
T_{235}	$x^{12}-2x^{10}-x^8+4x^4+4x^2+2$
T_{234}	$x^{12}-x^9-3x^3+4$
T_{233}	$x^{12}-4x^3-6$
T_{232}	$x^{12}-13x^8-26x^7-11x^6+6x^5+25x^4+78x^3+114x^2+76x+19$
T_{231}	$x^{12}+36x^8-48x^7-65x^6+162x^5+459x^4-1488x^3+1512x^2-672x+112$
T_{230}	$x^{12}+x^{10}-3x^8+4x^4+1$
T_{229}	$x^{12}-24x^9+108x^8-720x^6+324x^5+2349x^4-1728x^3-1296x^2$ $+5832x+5103$
T_{228}	$x^{12}-6x^{10}-24x^9-15x^8+96x^7+786x^6-912x^5+1974x^4-6992x^3$ $+16896x^2-19728x+12609$
T_{227}	$x^{12}-3x^{10}-3x^8-x^6+2$
T_{226}	$x^{12}+x^{10}-x^8-2x^6+x^2+1$
T_{225}	$x^{12}-3x^{10}+2x^6+2x^4-3$
T_{224}	$x^{12}-2x^{10}+x^8+3x^4-2x^2+3$
T_{223}	$x^{12}-3x^{10}-5x^6+6x^4+3x^2-3$
T_{222}	$x^{12}-4x^6+3x^2-1$
T_{221}	$x^{12}-2x^{10}-x^8+6x^6-x^4-4x^2-1$
T_{220}	$x^{12}-4x^9-12x^8+34x^6-12x^5+45x^4+42x^2+10$
T_{219}	$x^{12}-x^6+2x^4+x^2+1$
T_{218}	$x^{12}-2x^{11}+22x^9-88x^7+176x^5-176x^3+64x+4$
T_{217}	$x^{12}-6x^6-8x^3-4$
T_{216}	$x^{12}-12x^{10}-8x^9+162x^4+432x^3+432x^2+192x+32$
T_{215}	$x^{12}-12x^{10}-12x^9+54x^8+108x^7-84x^6-324x^5-63x^4+436x^3$ $+216x^2-336x-304$
T_{214}	$x^{12}-12x^{10}+8x^9+216x^6-432x^5+207x^4+152x^3-216x^2+96x-16$
T_{213}	$x^{12}-2x^9-1$
T_{212}	$x^{12}-16x^9-72x^8-192x^7-800x^6-1824x^4+4608x^2-2048$
T_{211}	$x^{12}-135x^8-180x^7+210x^6+540x^5+765x^4+1160x^3+1080x^2$ $+480x+80$

T_{210}	$x^{12}-4x^9+8x^6-36x^5+105x^4-120x^3+90x^2-36x+9$
T_{209}	$x^{12}-8x^9+18x^8-24x^7+24x^6-33x^4-16x^3-48x-8$
T_{208}	$x^{12}-3x^{10}+3x^6+3x^4+3$
T_{207}	$x^{12}-6x^{10}-8x^9+9x^8-60x^6+207x^4-256x^3-1494x^2-1848x-793$
T_{206}	$x^{12}-12x^9+15x^8-12x^5+18x^4-64x^3+96x^2-36x+9$
T_{205}	$x^{12}-208x^6-312x^5-117x^4-832x^3-1872x^2-1404x-351$
T_{204}	$x^{12}+260x^9+63x^8-648x^7-780x^6+108x^5+2133x^4-32x^3$ $-900x^2+125$
T_{203}	$x^{12}-x^{10}-x^4+x^2+1$
T_{202}	$x^{12}-2x^{10}-4x^6+6x^4+4x^2+4$
T_{201}	$x^{12}-3x^{10}+3x^6+6x^4+3x^2+3$
T_{200}	$x^{12}+6x^{10}+9x^8-8x^6-24x^4+52$
T_{199}	$x^{12}-2x^{10}-4x^8-x^6+x^4+4$
T_{198}	$x^{12}-2x^{10}-x^8+6x^4-4x^2+2$
T_{197}	$x^{12}-14x^{10}+70x^8-152x^6+144x^4-50x^2+5$
T_{196}	$x^{12}-2x^8-4x^6+6x^4+4x^2-1$
T_{195}	$x^{12}-x^8+4x^6-5x^4+4x^2+1$
T_{194}	$x^{12}-4x^{11}-32x^9+198x^8+216x^7+1032x^6-384x^5+801x^4$ $-452x^3+32x^2+4$
T_{193}	$x^{12}+6x^6+6x^4+3$
T_{192}	$x^{12}-6x^{10}+x^8+36x^6-30x^4-28x^2+18$
T_{191}	$x^{12}+x^{10}+2x^8-x^6+2x^4-3x^2+1$
T_{190}	$x^{12}+2x^{10}-13x^8+36x^6+15x^4-38x^2-19$
T_{189}	$x^{12}-6x^{10}+7x^8+12x^6-16x^4-8x^2+5$
T_{188}	$x^{12}-2x^{10}+5x^6+5x^2-1$
T_{187}	$x^{12}-6x^8-2x^6-3x^2+1$
T_{186}	$x^{12}-2x^8-3x^2-1$
T_{185}	$x^{12}-x^8-x^4+2$
T_{184}	$x^{12}-2x^{10}-2x^8+4x^6-x^4+6x^2+1$
T_{183}	$x^{12}-6x^{10}+49x^8-2464x^6+388x^4+80x^2+4$
T_{182}	$x^{12}-8x^9+6x^8+20x^6-24x^5+18x^4-16x^3+24x^2+8$
T_{181}	$x^{12}-18x^8-36x^6-72x^5+54x^4-144x^3-216x^2-72$
T_{180}	$x^{12}-2x^{10}+5x^8-8x^6+6x^4-4x^2+1$
$\mathrm{L}_2(11)$	$x^{12}-4x^{11}+396x^8-1056x^7+2112x^6+52272x^4-69696x^3+278784x^2$ $-211968x+2336832$
T_{178}	$x^{12}-x^9+4x^3-1$
T_{177}	$x^{12}-4x^9+4x^3+2$
T_{176}	$x^{12}+4x^6-8x^3+8$
T_{175}	$x^{12}-12x^{10}-8x^9+36x^8+48x^7-65x^6-162x^5+135x^4+624x^3$ $+648x^2+288x+48$
T_{174}	$x^{12}-12x^{10}-8x^9-120x^6+432x^5-828x^4-32x^3-864x^2-1920x-128$
T_{173}	$x^{12}-36x^8-48x^7-32x^6+162x^4-288x^2+128$
T_{172}	$x^{12}-2x^{11}+16x^{10}-68x^9-530x^8-300x^7+5380x^6+19304x^5+27280x^4$ $+19880x^3+10476x^2+2704x+676$
T_{171}	$x^{12}-8x^9-36x^8-72x^5+81x^4+64x^3-144x^2+64$
T_{170}	$x^{12}-x^9+2x^6+4x^3+3$
T_{169}	$x^{12}-8x^3+18$
T_{168}	$x^{12}-10x^6-12x^3-2$
T_{167}	$x^{12}-3x^3+3$
T_{166}	$x^{12}+18x^{10}+135x^8+348x^6+63x^4-512x^3-270x^2+729$
T_{165}	$x^{12}-16x^9+12x^8+256x^3-576x^2+432x-108$

T_{164}	$x^{12}+3x^{11}-6x^{10}-33x^9-30x^8+54x^7+155x^6+180x^5+192x^4+272x^3$ $+300x^2+192x+64$
T_{163}	$x^{12}-x^8-2x^6+x^4-2x^2+1$
T_{162}	$x^{12}-2x^8-8x^6+14x^4-16x^2+4$
T_{161}	$x^{12}-x^8+2x^6+x^4+2x^2+1$
T_{160}	$x^{12}+x^{10}+x^8+x^6-4x^4+5$
T_{159}	$x^{12}+4x^{10}-4x^8-24x^6-x^4+32x^2+8$
T_{158}	$x^{12}-x^8-2x^6+2x^2+1$
T_{157}	$x^{12}-8x^9+24x^7+44x^6-51x^4+48x^3-72x^2+16$
T_{156}	$x^{12}-6x^6-8x^3-1$
T_{155}	$x^{12}-2x^{10}-3x^8+2$
T_{154}	$x^{12}+10x^8-4x^6+49x^4+52x^2+104$
T_{153}	$x^{12}-3x^{10}+5x^8-8x^6+10x^4-12x^2+8$
T_{152}	$x^{12}-4x^8-2x^6+4x^4-1$
T_{151}	$x^{12}-3x^8-2$
T_{150}	$x^{12}-x^6-3x^4+2x^2+2$
T_{149}	$x^{12}-9x^4-6$
T_{148}	$x^{12}-2x^{10}+2x^8-2x^6-2x^4-2x^2-1$
T_{147}	$x^{12}-9x^4-12$
T_{146}	$x^{12}-2x^{10}-x^8-2x^6-2x^4-8x^2+8$
T_{145}	$x^{12}+6x^8+4x^6-18x^4-24x^2-8$
T_{144}	$x^{12}+6x^{10}+4x^8-24x^6-21x^4+22x^2+4$
T_{143}	$x^{12}-6x^{10}+24x^8-56x^6+93x^4-90x^2+51$
T_{142}	$x^{12}+3x^8+4x^6+6x^4+3$
T_{141}	$x^{12}+3x^8-3$
T_{140}	$x^{12}+4x^4-4$
T_{139}	$x^{12}+x^8-3x^6+x^4+1$
T_{138}	$x^{12}-x^4+1$
T_{137}	$x^{12}+x^{10}+x^8-x^4+x^2-1$
T_{136}	$x^{12}-2x^{10}-5x^4-2x^2+4$
T_{135}	$x^{12}-18x^8-24x^6+27x^4+36x^2-6$
T_{134}	$x^{12}-7x^{10}+14x^8-21x^4+7x^2+7$
T_{133}	$x^{12}+54x^8-72x^7+204x^6+216x^5+585x^4+8x^3+972x^2-216x+216$
T_{132}	$x^{12}-4x^{10}-36x^7-12x^6+144x^5+228x^4+208x^3+360x^2+464x+216$
T_{131}	$x^{12}-3x^{10}-2x^9+54x^8+72x^7+402x^6+756x^5+5445x^4+13288x^3$ $+13176x^2+5856x+976$
T_{130}	$x^{12}-x^9+5x^6-8x^3+4$
T_{129}	$x^{12}-24x^9+72x^7+2x^6-60x^5+9x^4+64x^3+30x^2-24x-17$
T_{128}	$x^{12}-32x^9+50x^8+248x^6-768x^5+579x^4+64x^3-160x^2+40x+76$
T_{127}	$x^{12}-54x^9-315x^8+4372x^6+3996x^5-22005x^4-13176x^3+17484x^2$ $-22518x+1775$
T_{126}	$x^{12}+x^8+x^6-2x^4-x^2+1$
T_{125}	$x^{12}-2x^8-2x^6+x^4+2x^2-1$
T_{124}	$x^{12}-2x^{10}-5x^8+35x^4-30x^2+5$
T_{123}	$x^{12}-2x^{10}+10x^6-8x^2+1$
T_{122}	$x^{12}-2x^{11}-3x^{10}-6x^9+21x^8-32x^7+37x^6-16x^5+11x^4+32x^3$ $-x^2+20x+1$
T_{121}	$x^{12}-x^9+2x^3+1$
T_{120}	$x^{12}-2x^9-6x^3+9$
T_{119}	$x^{12}-8x^6-8x^3-2$
T_{118}	$x^{12}+8x^6-8x^3+2$

T_{117}	$x^{12}-2x^9+x^6+5$
T_{116}	$x^{12}-2x^9+4x^3+4$
T_{115}	$x^{12}-2x^8+3x^4-4$
T_{114}	$x^{12}-x^4-1$
T_{113}	$x^{12}-x^8+4$
T_{112}	$x^{12}-3x^8+9x^4+9$
T_{111}	$x^{12}-6x^8+68x^6+105x^4+36x^2+12$
T_{110}	$x^{12}+x^8-x^6-x^4-1$
T_{109}	$x^{12}+x^{10}-4x^2+1$
T_{108}	$x^{12}-3x^8-4x^6+6x^4+4$
T_{107}	$x^{12}+6x^{10}+3x^8-28x^6-21x^4+30x^2+5$
T_{106}	$x^{12}+3x^{10}-2x^8-9x^6+5x^2+1$
T_{105}	$x^{12}-7x^{10}+7x^8+14x^6-16x^4-5x^2+5$
T_{104}	$x^{12}+6x^{10}+12x^8+8x^6-3x^4-6x^2-1$
T_{103}	$x^{12}-2x^8+5x^6-2x^4+1$
T_{102}	$x^{12}-5x^{10}+20x^8-70x^6+145x^4-280x^2+208$
T_{101}	$x^{12}-x^8+x^6-x^4+1$
T_{100}	$x^{12}-x^{10}+x^8+4x^6-x^4-x^2-1$
T_{99}	$x^{12}+12x^{10}+60x^8+160x^6+228x^4+144x^2+8$
T_{98}	$x^{12}+214x^{10}-1046693x^8+18491564x^6-29606993x^4+937950x^2+8560357$
T_{97}	$x^{12}+x^8+9x^4+1$
T_{96}	$x^{12}-3x^4-4$
T_{95}	$x^{12}-x^{10}+3x^6-2x^4-3x^2+1$
T_{94}	$x^{12}-57x^8-38x^6+318x^4-204x^2+17$
T_{93}	$x^{12}+10x^{10}+28x^8+6x^6-43x^4+6x^2+3$
T_{92}	$x^{12}-9x^4-9$
T_{91}	$x^{12}+5x^{10}+9x^8+8x^6+2x^4-12x^2+16$
T_{90}	$x^{12}+2x^{10}-x^6+2x^2+1$
T_{89}	$x^{12}-3x^4+1$
T_{88}	$x^{12}-6x^8-4x^6-3x^4-18x^2+3$
T_{87}	$x^{12}+6x^{10}+9x^8-4x^6-12x^4+1$
T_{86}	$x^{12}+2x^8-2$
T_{85}	$x^{12}-3x^{11}-3x^{10}+15x^9-15x^8-33x^7+29x^6+15x^5-30x^4-128x^3$ $-30x^2+198x+48$
T_{84}	$x^{12}-4x^{11}+2x^{10}+12x^9-20x^8+16x^6-6x^4-8x^3+4x^2+8x+4$
T_{83}	$x^{12}+3x^6-x^3+3$
T_{82}	$x^{12}-12x^{10}+54x^8-116x^6+129x^4-72x^2-16$
T_{81}	$x^{12}+x^6+2$
T_{80}	$x^{12}-90x^8+160x^6-135x^4+7200x^2-80$
T_{79}	$x^{12}-4x^8+4x^6+5x^4-4x^2+2$
T_{78}	$x^{12}-x^9+x^3+1$
T_{77}	$x^{12}-2x^{10}+x^8+6x^6-6x^4+1$
T_{76}	$x^{12}+2x^8-2x^6+5x^4-6x^2+1$
T_{75}	$x^{12}-2x^{10}-2x^8+6x^6+x^4-6x^2+1$
T_{74}	$x^{12}-x^{10}+2x^8+4x^6-3x^4-3x^2+1$
T_{73}	$x^{12}+12x^{10}-3x^9+54x^8-27x^7+122x^6-81x^5+165x^4-93x^3+126x^2$ $-36x+31$
T_{72}	$x^{12}+12x^{10}-40x^9+414x^8-1416x^7+3388x^6-6552x^5+8001x^4-7448x^3$ $+7056x^2+4704x+2744$
T_{71}	$x^{12}-4x^9+4x^6+3$
T_{70}	$x^{12}+9x^6-18x^3+9$

T_{69}	$x^{12}-3x^{10}-2x^8+9x^6-5x^2+1$
T_{68}	$x^{12}+x^{10}+6x^8+3x^6+6x^4+x^2+1$
T_{67}	$x^{12}-x^8-x^6-x^4+1$
T_{66}	$x^{12}+6x^{10}+12x^8+8x^6-3$
T_{65}	$x^{12}-3x^4+4$
T_{64}	$x^{12}-x^8+9x^4-1$
T_{63}	$x^{12}-6x^{10}+104x^6+93x^4+18x^2+4$
T_{62}	$x^{12}+6x^{10}-104x^6+93x^4-18x^2+4$
T_{61}	$x^{12}-3x^4-1$
T_{60}	$x^{12}-14x^8-7x^4+4$
T_{59}	$x^{12}-6x^{10}+6x^8-4x^6-3x^4+3$
T_{58}	$x^{12}-12x^8-14x^6+9x^4+12x^2+1$
T_{57}	$x^{12}-69x^{10}-2091x^8-7571x^6+134691x^4+960267x^2+1545049$
T_{56}	$x^{12}-2x^{10}+x^6-2x^2+1$
T_{55}	$x^{12}-30x^{10}+348x^8-1960x^6+5505x^4-7050x^2+3025$
T_{54}	$x^{12}-6x^8+9x^4+12$
T_{53}	$x^12+2x^8-16x^6+4x^4+8$
T_{52}	$x^{12}+12x^4-12$
T_{51}	$x^{12}+6x^8+9x^4+3$
T_{50}	$x^{12}-3x^4+6$
T_{49}	$x^{12}+3x^8-4x^6-3x^4-1$
T_{48}	$x^{12}-x^8+3x^4+1$
T_{47}	$x^{12}-6x^{10}+20x^9-72x^7+128x^6-96x^5+45x^4-8x^3-18x^2+12x-2$
T_{46}	$x^{12}+12x^{10}-19x^9+54x^8-171x^7+169x^6-513x^5+447x^4-573x^3$ $+549x^2-180x+16$
T_{45}	$x^{12}-3x^9-18x^8-24x^6-9x^5+69x^4-x^3+3x-1$
T_{44}	$x^{12}-6x^6-10x^3-6$
T_{43}	$x^{12}-6x^9+10x^6+4x^3+2$
T_{42}	$x^{12}-x^6+7$
T_{41}	$x^{12}-x^9-9x^6-x^3+1$
T_{40}	$x^{12}-7x^{10}+24x^8-36x^6+24x^4+13x^2+1$
T_{39}	$x^{12}-5x^3+5$
T_{38}	$x^{12}+x^6-3$
T_{37}	$x^{12}+x^6+9$
T_{36}	$x^{12}-x^9-x^6-x^3+1$
T_{35}	$x^{12}-x^9-x^6+x^3+1$
T_{34}	$x^{12}+12x^{10}+54x^8+108x^6+81x^4+16$
T_{33}	$x^{12}+2x^8+58x^6+301x^4+174x^2+25$
T_{32}	$x^{12}+7x^{10}-x^8-23x^6-x^4+7x^2+1$
T_{31}	$x^{12}-30x^{10}+343x^8-1860x^6+4760x^4-4600x^2+225$
T_{30}	$x^{12}-7x^{10}-14x^8+115x^6-70x^4-175x^2+125$
T_{29}	$x^{12}-36x^8-40x^6+36x^4+48x^2+8$
T_{28}	$x^{12}+2$
T_{27}	$x^{12}+12x^{10}+68x^8+220x^6+392x^4+360x^2+148$
T_{26}	$x^{12}-9x^8-8x^6-9x^4+1$
T_{25}	$x^{12}+5x^8+6x^4+1$
T_{24}	$x^{12}+4x^{10}+7x^8+4x^6-x^4-2x^2+1$
T_{23}	$x^{12}-4x^4+4$
T_{22}	$x^{12}-5x^{10}+7x^8-6x^7-17x^6-6x^5+7x^4-5x^2+1$
T_{21}	$x^{12}+3x^8-4x^6+3x^4+1$
T_{20}	$x^{12}-4x^9+72x^8-84x^7+236x^6-144x^5+324x^4-192x^3+72x^2+8$

T_{19}	$x^{12}-2x^{11}+4x^{10}+x^9+5x^8+8x^7+53x^6+44x^5+59x^4+19x^3+13x^2+5x+1$
T_{18}	$x^{12}-4x^6+16$
T_{17}	$x^{12}+4x^8+4x^6+5x^4+12x^2+2$
T_{16}	$x^{12}-x^6+4$
T_{15}	$x^{12}+3$
T_{14}	$x^{12}-9x^6+27$
T_{13}	$x^{12}-12$
T_{12}	$x^{12}+x^6-27$
T_{11}	$x^{12}+10x^6+5$
T_{10}	$x^{12}+16$
T_9	$x^{12}+3x^8+4x^6+3x^4+1$
T_8	$x^{12}-6x^{10}-8x^9+9x^8+12x^7-20x^6+9x^4-24x^3-4$
T_7	$x^{12}+4x^{10}-x^8-x^4+4x^2+1$
T_6	$x^{12}+2x^{10}-6x^8+2x^6-6x^4+2x^2+1$
T_5	$x^{12}-80x^{10}+1820x^8-13680x^6+29860x^4-2720x^2+32$
T_4	$x^{12}+6x^8+26x^6-63x^4+162x^2+81$
T_3	$x^{12}+36$
T_2	$x^{12}-x^6+1$
T_1	$x^{12}-x^{11}+x^{10}-x^9+x^8-x^7+x^6-x^5+x^4-x^3+x^2-x+1$

Degree 13

S_{13}	$x^{13}-x^{12}+x^4-x+1$
A_{13}	$x^{13}+156x-144$
T_7	$x^{13}-3x^{12}+3x^{11}+13x^{10}-24x^9+36x^8-99x^7+63x^6-18x^5+66x^4$ $+9x^3-9x^2-21x-9$
T_6	$x^{13}-2$
T_5	$x^{13}-x^{12}+20x^{10}-50x^9+141x^8-225x^7+420x^6-456x^5+470x^4$ $-101x^3+90x^2-8x+11$
T_4	$x^{13}+13x^{10}-26x^8+13x^7+52x^6-39x^4+26x^2+13x+2$
T_3	$x^{13}-39x^{11}+507x^9-156x^8-2925x^7+1872x^6+7605x^5-7488x^4-6435x^3$ $+10062x^2-2691x-306$
T_2	$x^{13}-3x^{12}-278x^{11}+205x^{10}+24414x^9+55909x^8-638959x^7-3888668x^6$ $-8795208x^5-9304040x^4-4272496x^3-446336x^2+199680x+41600$
T_1	$x^{13}-x^{12}-24x^{11}+19x^{10}+190x^9-116x^8-601x^7+246x^6+738x^5-215x^4$ $-291x^3+68x^2+10x-1$

Degree 14

S_{14}	$x^{14}-x+2$
A_{14}	$x^{14}-154x^{11}-643076643$
T_{61}	$x^{14}+49x^2-84x+36$
T_{60}	$x^{14}+147x^2-252x+108$
T_{59}	$x^{14}-14x^{13}+49x^{12}-4353564672$
T_{58}	$x^{14}+10976x^2-18816x+8064$
T_{57}	$x^{14}+2x^4+x^2+1$
T_{56}	$x^{14}+2x^{12}+2x^2-2$
T_{55}	$x^{14}-2x^4+x^2-1$
T_{54}	$x^{14}-2x^{13}-10x^{12}+73x^{11}-133x^{10}-108x^9+418x^8-300x^7-605x^6+772x^5$ $+717x^4-499x^3+444x^2+42x-37$
T_{53}	$x^{14}-x^8-x^6+3x^4-1$
T_{52}	$x^{14}-6x^{13}+11x^{12}-17x^{10}+6x^9+x^8+24x^7-40x^5-8x^4+32x^3+8x^2$ $-16x+8$
T_{51}	$x^{14}+2x^8-5x^6-3x^2+4$
T_{50}	$x^{14}-2x^8-5x^6-3x^2-4$
T_{49}	$x^{14}-3x^8+2x^4+1$
T_{48}	$x^{14}-7x^6+14x^4-7x^2+8$
T_{47}	$x^{14}+2x^{12}-2x^{10}+x^6-8x^4+5x^2+2$
T_{46}	$x^{14}+189x^8-234x^7+28224x^2+24192x+41472$
T_{45}	$x^{14}-x^{13}+3x^{12}+x^{11}-8x^{10}+2x^9+12x^8-35x^7+52x^6+26x^5-59x^4$ $-39x^3+99x^2-64x+19$
T_{44}	$x^{14}-14x^{10}+56x^6-56x^2+20$
T_{43}	$x^{14}-5x^8-5x^6+9x^4+2x^2+1$
T_{42}	$x^{14}-42\,x^{12}-2268\,x^{10}+110376\,x^8+650916\,x^6-62338248\,x^4$ $+446777856\,x^2-859963392$
T_{41}	$x^{14}-7x^{10}+14x^6-7x^2-4$
T_{40}	$x^{14}-x^{12}-3x^{10}-3x^8+12x^6-9x^2+9$
T_{39}	$x^{14}-x^{13}+26x^{10}+65x^6+13x^5+52x^2+12x+1$
T_{38}	$x^{14}+4x^{10}+2x^8+2x^6-3x^2-2$
T_{37}	$x^{14}-28x^{11}-14x^{10}-28x^9+280x^8+180x^7+833x^6-2548x^5+1400x^4$ $-728x^3+1316x^2-672x+136$
T_{36}	$x^{14}-7x^{12}-147x^{11}+399x^{10}-2282x^9+5068x^8+5510x^7+120393x^6$ $-706293x^5+1918966x^4-2714152x^3+2804872x^2-1751568x+551872$
T_{35}	$x^{14}+6x^{12}-12x^{10}-64x^8+16x^6+128x^4-64$
T_{34}	$x^{14}+5x^8-5x^6-9x^4+2x^2-1$
T_{33}	$x^{14}+84x^{12}+1848x^{10}+11872x^8+359730x^6+2314872x^4+43964424x^2$ -746496
T_{32}	$x^{14}-42x^{10}-16x^7+392x^6-560x^5-448x^3-392x^2-1120x-736$
T_{31}	$x^{14}+14x^{12}-7x^{11}+259x^{10}-21x^9+1680x^8-1885x^7+9114x^6-13391x^5$ $+27853x^4-31885x^3+21322x^2-20370x-13275$
T_{30}	$x^{14}-6x^{13}+13x^{12}-338x^9+845x^8+17576x^4+70304x+35152$
T_{29}	$x^{14}+7x^{12}-49x^8+98x^4-49x^2-7$
T_{28}	$x^{14}+7x^6+7x^4+7x^2-1$
T_{27}	$x^{14}+9x^{12}+37x^{10}+85x^8+163x^6+267x^4+439x^2+151$
T_{26}	$x^{14}+714x^{12}+16100x^{11}+238623x^{10}+5068476x^9+85951551x^8+40905306x^7$ $+3304719936x^6+129890941166x^5-423192245805x^4-3704820336210x^3$ $+42231389433079x^2-241689758907660x+670952296880775$
T_{25}	$x^{14}+42x^{12}-56x^{11}+371x^{10}-840x^9+1834x^8-5008x^7+7987x^6-6384x^5$ $+22358x^4-3416x^3+18753x^2-5544x+7702$

T_{24}	$x^{14}-3x^7+6$
T_{23}	$x^{14}-56x^{12}+245x^{11}+2534x^{10}-1372x^9-24528x^8-257782x^7$ $-1470049x^6+1229802x^5+12092913x^4+48481531x^3+478963261x^2$ $+704643912x-1494810659$
T_{22}	$x^{14}-63x^{12}-9555x^{11}+118671x^{10}-708246x^9-17922660x^8+859373823x^7$ $+2085856500x^6-117366985106x^5-335941176396x^4+4638317668005x^3$ $+17926524826973x^2+7429846568445x+91264986397629$
T_{21}	$x^{14}-x^{12}-12x^{10}+7x^8+28x^6-14x^4-9x^2-1$
T_{20}	$x^{14}+210x^{12}-3164x^{11}+63455x^{10}-534016x^9+7977046x^8-27661364x^7$ $+1002627612x^6+6022284016x^5-28570776528x^4+138886748224x^3$ $-3146649429952x^2+4701085568256x-59618052726016$
T_{19}	$x^{14}+6x^{12}+16x^{10}+28x^8+32x^6+26x^4+12x^2+2$
T_{18}	$x^{14}-5x^{12}-39x^{10}-7x^8+41x^6+7x^4-9x^2+1$
T_{17}	$x^{14}-18x^{12}-964x^{10}+32592x^8-353912x^6+1736792x^4-3987152x^2+3495368$
T_{16}	$x^{14}-7x^{12}+42x^{10}-98x^8+441x^6-196x^5-343x^4+392x^3+1372x^2$ $+588x+112$
T_{15}	$x^{14}-105x^{12}-147x^{11}+5271x^{10}+19838x^9-94150x^8-607634x^7+570164x^6$ $+12260920x^5+42847770x^4+95169270x^3+197804880x^2+348280352x$ $+336238208$
T_{14}	$x^{14}-31958x^7+656356768$
T_{13}	$x^{14}-2x^{13}-29x^{12}-222x^{11}-352x^{10}+3498x^9+18163x^8+46467x^7$ $+92188x^6+128405x^5+96637x^4+31142x^3+7064x^2+6304x+2432$
T_{12}	$x^{14}+28x^{12}-189x^{11}+756x^{10}-4004x^9+15953x^8-48856x^7+129262x^6$ $-251559x^5+330764x^4-272986x^3-123305x^2+739662x-577916$
T_{11}	$x^{14}+7x^{12}-21x^{10}-147x^8+91x^6+693x^4+385x^2-9$
T_{10}	$x^{14}+14x^8-84x^6+84x^4+21x^2-9$
T_9	$x^{14}+7x^{12}-49x^{10}-245x^8+588x^6+294x^4-7$
T_8	$x^{14}+28x^{11}+28x^{10}-28x^9+140x^8+360x^7+147x^6+196x^5+336x^4$ $-546x^3-532x^2+896x+823$
T_7	$x^{14}+2$
T_6	$x^{14}+28x^{12}+56x^{10}-245x^8-322x^6+406x^4-56x^2-1$
T_5	$x^{14}+28x^{12}+308x^{10}+1680x^8+4704x^6+6272x^4+3136x^2+484$
T_4	$x^{14}+7$
T_3	$x^{14}-8x^{12}+22x^{10}-8x^8-55x^6+48x^4+64x^2-71$
T_2	$x^{14}+9x^{12}+53x^{10}+333x^8+1251x^6+731x^4+5415x^2+8591$
T_1	$x^{14}+25x^{12}+214x^{10}+767x^8+1194x^6+686x^4+53x^2+1$

Degree 15

S_{15}	$x^{15}-x-1$
A_{15}	$x^{15}-240x-224$
T_{102}	$x^{15}+375x^3-900x^2+720x-192$
T_{101}	$x^{15}-15x^{11}-12x^{10}+375x^3+900x^2+720x+192$
T_{100}	$x^{15}+500x^3-1200x^2+960x-256$
T_{99}	$x^{15}-15x^{11}+75x^7-125x^3-32$
T_{98}	$x^{15}-5x^{11}+4x^{10}-1100x^7+1760x^6-704x^5+8000x^3-19200x^2$ $+15360x-4096$
T_{97}	$x^{15}-500x^3-1200x^2-960x-256$
T_{96}	$x^{15}+15x^{11}-12x^{10}+125x^3-300x^2+240x-64$
T_{95}	$x^{15}+15x^{11}-12x^{10}-125x^3+300x^2-240x+64$
T_{94}	$x^{15}-15x^{14}+75x^{13}-125x^{12}-180x^5+900x^4+15360$
T_{93}	$x^{15}+5x^{12}+8x^9+6x^6-x^5+3x^3+1$
T_{92}	$x^{15}+20x^{11}-8x^{10}+75x^7-60x^6-16x^5-125x^3-200x^2+80x+64$
T_{91}	$x^{15}-1620x^7+4320x^6-432x^5-11040x^4+16960x^3-11520x^2+3840x-512$
T_{90}	$x^{15}+5x^{12}+x^{11}+13x^9+3x^8+16x^6+8x^3+x^2+1$
T_{89}	$x^{15}-405x^{11}-4824x^{10}-25600x^9-76800x^8-137040x^7-138240x^6$ $-62208x^5+400x^3+1920x^2+3840x+2048$
T_{88}	$x^{15}+5x^{14}+5x^{13}-5x^{12}+7x^{11}+31x^{10}-5x^9-23x^8+13x^7+9x^6+45x^5$ $+19x^4-53x^3-5x^2-13x+1$
T_{87}	$x^{15}+2x^5-10x^4+20x^3-20x^2+10x-2$
T_{86}	$x^{15}+5x^{12}+5x^{11}+5x^9+15x^8+5x^7+18x^5+5x^4+5x^2+1$
T_{85}	$x^{15}-15x^{13}+90x^{11}-270x^9+405x^7-243x^5-16$
T_{84}	$x^{15}-765x^{11}-510x^{10}+2075x^9+2160x^8+15705x^7+20060x^6-27432x^5$ $-60150x^4-41395x^3-13860x^2-2310x-154$
T_{83}	$x^{15}-2x^{12}+2x^6-2$
T_{82}	$x^{15}-3x^{13}-4x^{12}+27x^{11}+6x^{10}-79x^9-342x^8+858x^7-632x^6-504x^5$ $+420x^4+716x^3-1728x^2+1296x-432$
T_{81}	$x^{15}+5x^{13}+10x^{11}-25x^7-5x^6-29x^5-10x^4-5x^2+10x-1$
T_{80}	$x^{15}-15x^{13}+90x^{11}-280x^9+495x^7-5x^6-513x^5+30x^4+330x^3$ $-45x^2-180x-16$
T_{79}	$x^{15}-5x^{13}+7x^{12}+10x^{11}-28x^{10}+16x^9+42x^8-73x^7+31x^6+77x^5$ $-111x^4+43x^3+59x^2-69x+29$
T_{78}	$x^{15}-44x^{13}+782x^{11}-x^{10}-7005x^9+5x^8+31605x^7-10x^6-57406x^5$ $+10x^4-692x^3-5x^2-9x+1$
T_{77}	$x^{15}-952x^{12}-512732x^9-8487340x^6+3418039096x^3+369608289704$
T_{76}	$x^{15}+2x^9-2x^6-x^3+2$
T_{75}	$x^{15}-x^{14}-16x^{13}+36x^{12}+159x^{11}-981x^{10}-487x^9+8972x^8-1724x^7$ $-41608x^6+49632x^5+9600x^4-41728x^3+27648x-13824$
T_{74}	$x^{15}-4x^{14}+29x^{13}-70x^{12}+249x^{11}-336x^{10}+267x^9-154x^8+698x^7$ $-400x^6-1620x^5+1548x^4+1628x^3-3744x^2+2016x-432$
T_{73}	$x^{15}+30x^{13}+30x^{12}+255x^{11}+674x^{10}+150x^9+3580x^8-2295x^7-4070x^6$ $+17912x^5-14800x^4-2705x^3+25520x^2-24180x+10942$
T_{72}	$x^{15}+x^{14}-37x^{13}-81x^{12}+394x^{11}+906x^{10}-1650x^9-3878x^8+1548x^7$ $+3724x^6+28484x^5-42844x^4-18504x^3+59864x^2-36216x+7288$
T_{71}	$x^{15}+3x^{14}-33x^{13}+18x^{12}+414x^{11}-1011x^{10}-1662x^9+5289x^8+6717x^7$ $-16690x^6-14886x^5+29238x^4+13966x^3-24117x^2-4551x+7303$
T_{70}	$x^{15}-x^3-1$
T_{69}	$x^{15}-47x^{13}-2x^{12}+871x^{11}+44x^{10}-7925x^9-286x^8+35415x^7+616x^6$ $-62625x^5-542x^4+1569x^3+172x^2-27x-2$

T_{68}	$x^{15} - 25x^{13} - 30x^{12} + 370x^{11} + 148x^{10} - 2560x^9 + 1120x^8 + 9000x^7 - 12880x^6 - 8272x^5 + 32800x^4 - 21760x^3 - 23040x^2 + 34560x - 13824$
T_{67}	$x^{15} + 27x^{13} - 76x^{12} + 2640x^{11} - 11358x^{10} - 21029x^9 + 130602x^8 - 22668x^7 - 465624x^6 + 645456x^5 - 210144x^4 - 151552x^3 + 36864x^2 + 34560x - 13824$
T_{66}	$x^{15} - 114x^{14} + 282185319x^{13} + 1247857228852x^{12} - 35114805704965233x^{11} - 141524337796433387826x^{10} + 2604584980442264028744009x^9 + 14153948932132918272984150384x^8 - 178273077248353369941327628479552x^7 - 1142953506821390914419260564494304768x^6 + 15975069142211276963134599495014990639616x^5 + 335166844383030880182173082532512773761597 44x^4 - 617589777108203716232396372453619309554471256064x^3 - 3975614124450669195454617623548846315018061748633 60x^2 + 22666571829085475706482454642151923578020471016281866 24x - 1302222456532760256406916223259306960561657428777814196224$
T_{65}	$x^{15} - 9x^{14} - 777124356x^{13} - 1133827391276x^{12} + 248667491445971640x^{11} + 198247003710891320076 0x^{10} - 2025167492138674323308966 4x^9 - 47758209690766095052311701 4816x^8 - 33088861073352644868101397 08481072x^7 + 78219641307131776967508554 9386001392x^6 + 13677138465476312597562863594491 3787162688x^5 + 4953057527153646148002049060206164940 62534208x^4 - 2654636123517266162930575109745213496431971332096x^3 - 26690849479086638949750226189970304815661445420472832x^2 - 847628875527545519279368656465832592426970344922081116 16x - 1024871671448323058965788498735385810803135249473163217674 24$
T_{64}	$x^{15} + 5x^9 + 5x^3 - 2$
T_{63}	$x^{15} - 15x^{13} + 90x^{11} - 247x^9 + 198x^7 - 20x^6 + 378x^5 + 120x^4 - 600x^3 - 180x^2 - 63x + 146$
T_{62}	$x^{15} + 23x^{12} + 130x^9 + 18x^6 - 27x^3 + 27$
T_{61}	$x^{15} + 2x^9 - 5x^6 + 2x^3 - 1$
T_{60}	$x^{15} - 5x^{12} + 10x^{11} - 12x^{10} + 10x^9 - 20x^8 + 40x^7 - 60x^6 + 73x^5 - 65x^4 + 35x^3 - 35x^2 + 30x - 9$
T_{59}	$x^{15} + 20x^{13} + 125x^{11} - 53x^{10} + 250x^9 - 275x^8 - 75x^7 - 245x^6 - 1009x^5 - 1025x^4 + 1825x^3 - 1150x^2 + 225x + 2059$
T_{58}	$x^{15} + 30x^{13} + 45x^{12} + 255x^{11} + 986x^{10} + 525x^9 + 4870x^8 + 2355x^7 - 6605x^6 + 21707x^5 - 7825x^4 - 12005x^3 + 38780x^2 - 30480x + 14578$
T_{57}	$x^{15} - 6x^{14} - 185929x^{13} - 7490711x^{12} + 10447793997x^{11} + 850215157784x^{10} - 183592676667989x^9 - 22599650147469301x^8 + 2688713422003070 43x^7 + 111386427537228333570x^6 + 2235951723532365786373x^5 - 15131063851167668518601 7x^4 - 508653781363084067735134 5x^3 - 5767418822154378301922100x^2 + 338165560497864890767793817x + 6129688867708593869647630 21$
T_{56}	$x^{15} - 45x^{13} + 810x^{11} - 2x^{10} - 7290x^9 + 10x^8 + 32805x^7 - 20x^6 - 59049x^5 + 20x^4 - 10x^2 + 2$
T_{55}	$x^{15} + 3x^9 - x^6 + 3x^3 + 3$
T_{54}	$x^{15} + 6x^{14} - 60x^{13} - 1074x^{12} - 4950x^{11} + 5454x^{10} + 90400x^9 + 136296x^8 - 428418x^7 - 1326320x^6 - 295140x^5 + 2170494x^4 + 2415630x^3 - 135012x^2 - 6410634x - 5989054$
T_{53}	$x^{15} + 7x^{14} + 22x^{13} + 44x^{12} + 67x^{11} + 90x^{10} + 133x^9 + 198x^8 + 198x^7 + 102x^6 + 36x^5 + 21x^4 - 15x^3 - 24x^2 + 3$
T_{52}	$x^{15} + 5x^9 + 5x^3 - 1$
T_{51}	$x^{15} - 50x^{13} - 5x^{12} + 330x^{11} + 266x^{10} + 9150x^9 - 12230x^8 - 55090x^7 + 142500x^6 - 156072x^5 + 136600x^4 - 150040x^3 + 126420x^2 - 48320x + 1024$

T_{50}	$x^{15}-5x^{13}-10x^{12}+139x^{10}+50x^9-1670x^8-1950x^7+6980x^6+16836x^5$ $+5450x^4-28890x^3-58420x^2-39745x-7279$
T_{49}	$x^{15}-x^5-2$
T_{48}	$x^{15}-x^{14}-42x^{13}+576x^{12}+6248x^{11}+1220x^{10}-123132x^9-275290x^8$ $+902478x^7+3670908x^6-147896x^5-15604028x^4-10561837x^3$ $+25131303x^2+22337416x-19837924$
T_{47}	$x^{15}-2x^{14}-6x^{13}-18x^{12}+823x^{11}-4627x^{10}+12741x^9-15872x^8-6962x^7$ $+59365x^6-91720x^5+43297x^4+53515x^3-108581x^2+80509x-28454$
T_{46}	$x^{15}-50x^{13}+965x^{11}-4x^{10}-8910x^9+20x^8+39435x^7-40x^6-68134x^5$ $+40x^4+4015x^3-20x^2-90x+4$
T_{45}	$x^{15}-6x^{14}+28x^{13}+133x^{12}-17x^{11}+508x^{10}+14373x^9+24160x^8-119521x^7$ $+95861x^6+681292x^5+97666x^4-2852755x^3+4387594x^2-2726259x$ $+777069$
T_{44}	$x^{15}-5x^{12}+15x^6-5x^3-7$
T_{43}	$x^{15}-2x^{12}+2x^9-x^6+1$
T_{42}	$x^{15}-6x^{13}+31x^{12}-270x^{10}-674x^9-54x^8+5670x^7+8818x^6+11790x^5$ $-372x^4+4043x^3-774x^2-2670x+1$
T_{41}	$x^{15}-3x^{14}+24x^{13}-4x^{12}-315x^{11}+453x^{10}+818x^9-2964x^8+2370x^7$ $+5990x^6-13470x^5+2028x^4+13804x^3-13104x^2+7320x-2872$
T_{40}	$x^{15}-5x^{14}+15x^{13}+45x^{12}-935x^{11}+2619x^{10}+17025x^9-19445x^8-171320x^7$ $+344390x^6+3336719x^5+2746335x^4-5890980x^3-1407410x^2$ $+2396000x+798080$
T_{39}	$x^{15}+5x^{14}-675x^{13}-5500x^{12}+83550x^{11}+1101150x^{10}+4157575x^9$ $+3236100x^8-11798025x^7-21372125x^6-63158850x^5-281431250x^4$ $-135052500x^3+1343230000x^2+2535100000x+1313680000$
T_{38}	$x^{15}-5x^{10}+4x^5+5$
T_{37}	$x^{15}-6x^{10}+2$
T_{36}	$x^{15}-55x^{13}-5x^{12}+1110x^{11}+104x^{10}-10400x^9-685x^8+45645x^7+1480x^6$ $-76639x^5-1295x^4+7660x^3+400x^2-90x+1$
T_{35}	$x^{15}-5x^{14}-8x^{13}+73x^{12}-55x^{11}-382x^{10}+1017x^9+482x^8-3994x^7$ $+1543x^6+2690x^5-4868x^4+3487x^3-1350x^2+279x-27$
T_{34}	$x^{15}-5x^{14}+7x^{13}+6x^{11}-44x^{10}+67x^9-53x^8+52x^7-48x^6+10x^5+12x^4$ $-11x^3+16x^2-12x+1$
T_{33}	$x^{15}-10x^9-5x^6+10x^3-1$
T_{32}	$x^{15}-3x^{14}-9x^{13}+26x^{12}-3x^{11}-60x^{10}+14x^9+72x^8+63x^7-190x^6$ $+150x^5-117x^4+79x^3-42x^2+15x-1$
T_{31}	$x^{15}-615x^{13}+8650x^{12}+196335x^{11}-1724326x^{10}-10046015x^9+511745770x^8$ $+2281327270x^7-30001270850x^6+7799981467x^5+2779038445300x^4$ $+8058450933185x^3-52551636276610x^2-201477388078380x$ $+674662192576472$
T_{30}	$x^{15}+5x^{14}-55x^{13}-220x^{12}+700x^{11}+2791x^{10}-1980x^9-14270x^8-8030x^7$ $+23050x^6+35952x^5+13270x^4-8695x^3-9555x^2-3200x-377$
T_{29}	$x^{15}-4x^{12}+6x^9-6x^6+4x^3-2$
T_{28}	$x^{15}+x^{12}-x^{11}+x^{10}-2x^9+x^8-2x^6+2x^5+x^3+1$
T_{27}	$x^{15}-x^5-1$
T_{26}	$x^{15}-150x^{13}-520x^{12}+2400x^{11}+12366x^{10}-1700x^9-73410x^8-60675x^7$ $+161150x^6+214578x^5-119280x^4-247825x^3-3750x^2+93525x+23255$
T_{25}	$x^{15}-110x^{13}+130x^{12}+4425x^{11}-10351x^{10}-71925x^9+269785x^8+212240x^7$ $-2211090x^6+3656145x^5-2402600x^4+722420x^3-99015x^2+5125x-41$
T_{24}	$x^{15}-x^{14}-2x^{13}+x^{12}+16x^{10}+22x^9-13x^8-57x^5+26x^4-1$
T_{23}	$x^{15}-4x^{12}+10x^9-12x^6+10x^3-4$

T_{22}	$x^{15}+6x^{14}+27x^{13}-65x^{12}-1818x^{11}-4800x^{10}+15172x^9+83334x^8+100197x^7$ $-95461x^6-358221x^5-215697x^4+185503x^3+195837x^2-101025x-138979$
T_{21}	$x^{15}-96x^{13}+59x^{12}+5031x^{11}-13182x^{10}-109799x^9+487485x^8+821655x^7$ $-6975764x^6+2066256x^5+46961715x^4-80656366x^3-8511147x^2$ $+57236934x+26273539$
T_{20}	$x^{15}+3x^{13}-9x^{11}-6x^{10}-32x^9-9x^8+5x^6+18x^5+6x^4-3x-1$
T_{19}	$x^{15}+x^{10}-2x^5-1$
T_{18}	$x^{15}-15x^{13}-15x^{12}+55x^{11}+149x^{10}+150x^9-165x^8-740x^7-1105x^6$ $-1153x^5-875x^4-460x^3-160x^2-25x+1$
T_{17}	$x^{15}+45x^{13}+630x^{11}+162x^{10}+3625x^9+1350x^8+9000x^7+1800x^6$ $+7305x^5-1375x^3-2250x-450$
T_{16}	$x^{15}-x^{14}+3x^{13}+8x^{12}+36x^{11}-17x^{10}-40x^9-3x^8+140x^7-90x^6-32x^5$ $+46x^4-8x^3-13x^2-2x+1$
T_{15}	$x^{15}-30x^{13}-12x^{12}+369x^{11}+171x^{10}-2769x^9-234x^8+16218x^7+2328x^6$ $-58374x^5-50094x^4+18009x^3+57132x^2-42849x+4761$
T_{14}	$x^{15}+5x^{14}+50x^{13}+220x^{12}+970x^{11}+1142x^{10}+7935x^9-2815x^8-156480x^7$ $+1127980x^6-3389737x^5+7033715x^4-9360980x^3+9369840x^2$ $-5090420x+1432484$
T_{13}	$x^{15}-60x^{13}-90x^{12}+1095x^{11}+1224x^{10}-14450x^9-26775x^8+59640x^7$ $+177525x^6+12810x^5-335025x^4-386555x^3-246465x^2-192375x-92889$
T_{12}	$x^{15}-235x^{13}+70x^{12}+15930x^{11}-14493x^{10}-325950x^9+112750x^8$ $+2876560x^7+708890x^6-11702794x^5-8278925x^4+19171805x^3$ $+20087950x^2-6362585x-9179941$
T_{11}	$x^{15}-2$
T_{10}	$x^{15}-5x^{14}-5x^{13}+65x^{12}-55x^{11}-301x^{10}+550x^9+505x^8-1925x^7$ $+400x^6+3450x^5-3670x^4-1690x^3+6335x^2-5195x+1547$
T_9	$x^{15}-470x^{13}-305x^{12}+71840x^{11}+85357x^{10}-4292700x^9-3714805x^8$ $+119761820x^7+25284495x^6-1542190154x^5+717324725x^4+7178878600x^3$ $-5452953875x^2-7998223215x+4461221029$
T_8	$x^{15}+32x^{10}-228x^5-8$
T_7	$x^{15}-2x^{12}+36x^9+304x^3-32$
T_6	$x^{15}-30x^{10}-3708x^5-2$
T_5	$x^{15}-4x^{14}-x^{13}+9x^{12}+2x^{11}-2x^{10}-32x^9+36x^8+39x^7-88x^6+57x^5$ $+7x^4-33x^3+30x^2-15x+5$
T_4	$x^{15}-8x^{12}-124x^9+480x^6+672x^3-32$
T_3	$x^{15}+2x^{14}+2x^{13}-2x^{11}-12x^{10}+37x^9-2x^8+37x^7+37x^6-39x^5$ $-47x^4-22x^3-6x^2+1$
T_2	$x^{15}-10x^{12}+15x^{11}-24x^{10}+95x^9-90x^8+90x^7-125x^6+27x^5+90x^4$ $+120x^3+195x^2+90x+33$
T_1	$x^{15}-27x^{13}-4x^{12}+252x^{11}+60x^{10}-976x^9-288x^8+1473x^7+384x^6$ $-765x^5-168x^4+150x^3+27x^2-9x-1$

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