THE CORRELATION MEASURES OF FINITE SEQUENCES: LIMITING DISTRIBUTIONS AND MINIMUM VALUES

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Abstract. Three measures of pseudorandomness of finite binary sequences were introduced by Mauduit and Sárközy in 1997 and have been studied extensively since then: the normality measure, the well-distribution measure, and the correlation measure of order r. Our main result is that the correlation measure of order r for random binary sequences converges strongly, and so has a limiting distribution. This solves a problem due to Alon, Kohayakawa, Mauduit, Moreira, and Rödl. We also show that the best known lower bounds for the minimum values of the correlation measures are simple consequences of a celebrated result due to Welch, concerning the maximum nontrivial scalar products over a set of vectors.

1. Introduction and main results

We consider finite binary sequences, namely elements A_n of $\{-1,1\}^n$. Mauduit and Sárközy [11] introduced three measures of pseudorandomness for finite binary sequences: the *well distribution measure* $W(A_n)$, the *normality measure* $\mathcal{N}(A_n)$, and the *r-th order correlation measure* $C_r(A_n)$. These measures have been studied extensively (see [11], [8], [4], [5], [1], [2], [3], for example). Finite binary sequences for which these measures are small are considered to possess a high 'level of randomness'.

In this paper, we are concerned with the correlation measures of finite binary sequences. Let $A_n = (a_1, a_2, \dots, a_n)$ be an element of $\{-1,1\}^n$. For $2 \le r \le n$, the *r-th order correlation measure* of A_n is defined as

$$C_r(A_n) = \max_{0\le u_1<u_2<\cdots<u_r<n} \max_{1\le m\le n-u_r} \left| \sum_{j=1}^{m} a_{j+u_1} a_{j+u_2} \cdots a_{j+u_r} \right|.$$

Following earlier work by Cassaigne, Mauduit, and Sárközy [8], Alon, Kohayakawa, Mauduit, Moreira, and Rödl [5] studied the behaviour of $W(A_n)$, $\mathcal{N}(A_n)$, and $C_r(A_n)$ when A_n is drawn at random from $\{-1,1\}^n$, equipped with the uniform probability measure. They posed the following problem.

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Problem A ([5, Problem 33])**.** Investigate the existence of the limiting distributions of

$$\left\{\frac{W(A_n)}{\sqrt{n}}\right\}_{n\ge 1} \quad \text{and} \quad \left\{\frac{\mathcal{N}(A_n)}{\sqrt{n}}\right\}_{n\ge 1}$$

and

$$\left\{\frac{C_r(A_n)}{\sqrt{n\log\binom{n}{r}}}\right\}_{n\ge r}. \tag{1}$$

Investigate these distributions.

The first two instances of Problem A have been solved recently: Aistleitner [2], [3] proved that the limiting distributions of $W(A_n)/\sqrt{n}$ and of $\mathcal{N}(A_n)/\sqrt{n}$ exist. Moreover, a tail characterisation of the limiting distribution of $W(A_n)/\sqrt{n}$ is provided in [2]. It is known that, if (1) has a limiting distribution, then it is a Dirac measure [5, Theorem 3]. We shall resolve the third instance of Problem A by proving strong convergence of (1). To do so, we consider the set Ω of infinite sequences of elements -1 or 1 and endow Ω in the standard way with the probability measure defined by

$$\Pr\big[(a_1,a_2,\dots)\in\Omega : a_1=c_1, a_2=c_2,\dots,a_n=c_n\big]=2^{-n} \tag{2}$$

for all $(c_1,c_2,\dots,c_n)\in\{-1,1\}^n$.

Theorem 1.1. *Let $(a_1,a_2,\dots)$ be drawn from Ω, equipped with the probability measure defined by (2), and write $A_n=(a_1,a_2,\dots,a_n)$. Let $r\ge 2$ be a fixed integer. Then, as $n\to\infty$,*

$$\frac{C_r(A_n)}{\sqrt{2n\log\binom{n}{r-1}}}\to 1 \quad \textit{almost surely.}$$

Alon, Kohayakawa, Mauduit, Moreira, and Rödl [5] also proved a result on the asymptotic order of $C_r(A_n)$ that holds uniformly for a large range of r.

Theorem B ([5, Theorem 2])**.** Let A_n be drawn uniformly at random from $\{-1,1\}^n$. Then the probability that

$$\frac{2}{5}\sqrt{n\log\binom{n}{r}}<C_r(A_n)<\sqrt{\left(2+\frac{\log\log n}{\log n}\right)n\log\left(n\binom{n}{r}\right)}$$

holds for all r satisfying $2\le r\le n/4$ tends to 1 as $n\to\infty$.

We improve the upper bound in Theorem B as follows.

Theorem 1.2. *Let A_n be drawn uniformly at random from $\{-1,1\}^n$ and let $\epsilon>0$ be real. Then, as $n\to\infty$,*

$$\Pr\left[C_r(A_n)\le(1+\epsilon)\sqrt{2n\log\tbinom{n}{r-1}} \quad \textit{for all } r \textit{ satisfying } 2\le r\le n\right]\to 1.$$

In view of Theorem 1.1, the bound in Theorem 1.2 is essentially best possible. We also note that Theorem 1.2 gives the currently strongest existence result. (The computation of the asymptotic behaviour of the correlation measures of individual binary sequences is a notoriously difficult problem and, in the light of Theorem 1.1, the currently known results tend to be unsatisfying, see for example [11, Theorem 1].)

We shall prove Theorem 1.2 in Section 2. In Section 3, we shall determine the limit of the expected value of (1) (Proposition 3.1). We shall then use this result in Section 4 to deduce Theorem 1.1.

We now turn to lower bounds for $C_r(A_n)$. It is known that

$$\min_{A_n \in \{-1,1\}^n} C_r(A_n) = 1 \quad \text{for odd } r,$$

which arises from the alternating sequence $(1, -1, 1, -1, \dots)$. Therefore, interesting results can only be expected for even r. Indeed the following result was established by Alon, Kohayakawa, Mauduit, Moreira, and Rödl [4].

Theorem C ([4, Theorem 1.1])**.** Let r and n be positive integers with $r \le n/2$. Then

$$C_{2r}(A_n) > \sqrt{\frac{1}{2}\left\lfloor \frac{n}{2r+1} \right\rfloor}$$

for all $A_n \in \{-1, 1\}^n$.

Theorem C gives an affirmative answer to a problem due to Cassaigne, Mauduit, and Sárközy [8, Problem 2], which was suspected to be 'really difficult' in [8, p. 109]. While the proof of Theorem C in [4] is quite involved, we shall show that Theorem C is a simple consequence of the so-called Welch bound [16]. This bound is an elementary result on the maximum nontrivial scalar products over a set of vectors.

We also establish, as another consequence of the Welch bound, the following result, which was proved in [4] without an explicit lower bound for c_k.

Theorem 1.3. *There exists a sequence of real numbers c_k, satisfying $c_k > 1/9$ for each $k \ge 3$ and $c_k \to 1/\sqrt{6e} = 0.2476\dots$ as $k \to \infty$, such that for all positive integers s and n with $s \le n/3$, we have*

$$\max\left\{C_2(A_n), C_4(A_n), \dots, C_{2s}(A_n)\right\} > c_n\sqrt{sn}$$

for all $A_n \in \{-1, 1\}^n$.

Theorems C and 1.3 will be proved in Section 5.

2. Typical upper bound

In this section, we shall prove Theorem 1.2. The key ingredient in the proof will be an estimate for the range of a random walk. Let $X_1, \dots, X_n$

be independent random variables, each taking the values -1 or 1, each with probability $1/2$. Define the random variable

$$R_n = \max_{1\le m_1\le m_2\le n}\left|\sum_{j=m_1}^{m_2} X_j\right|, \tag{3}$$

which is called the *range* of the random walk with steps $X_1, X_2, \dots$.

We begin with a minor generalisation of a lemma due to Aistleitner [2, Lemma 2.3].

Lemma 2.1. *Let p be a nonnegative integer and let n be an integer of the form*

$$j2^m, \text{ where } j,m\in\mathbb{Z},\ 2^p<j\le 2^{p+1}, \text{ and } m\ge 1.$$

Then, for $\lambda > 2\sqrt{n}$,

$$\Pr\Big[R_n > \lambda(1+12\cdot 2^{-p/2})\Big] \le 2^{2p+4}\exp\Big(-\frac{\lambda^2}{2n}\Big).$$

Aistleitner's lemma [2, Lemma 2.3] is obtained by setting $p = 10$ in Lemma 2.1. The general version can be proved by applying obvious modifications to the proof of [2, Lemma 2.3], which is proved using a dyadic decomposition technique. (Aistleitner's lemma has the additional assumption that n is sufficiently large, which however is not required in the proof.)

We now proceed similarly as in [2] and prove the following lemma, which holds for general n.

Lemma 2.2. *Let $\delta > 0$ be real. Then, there exists a constant $n_0 = n_0(\delta)$, such that for all $n \ge n_0$ and all $\lambda > 2\sqrt{n}$,*

$$\Pr\Big[R_n > \lambda(1+\delta)\Big] \le (\log n)\exp\Big(-\frac{\lambda^2}{2n}\Big).$$

Proof. Let p be a positive integer and let $\hat{n}$ be the smallest integer that satisfies $\hat{n}\ge n$ and is of the form

$$j2^m, \text{ where } j,m\in\mathbb{Z},\ 2^p<j\le 2^{p+1}, \text{ and } m\ge 1.$$

We readily verify that

$$\frac{\hat{n}}{n} \le 1+\frac{1}{2^p} \quad \text{for } n\ge 2^{p+1}. \tag{4}$$

Let $n\ge 2^{p+1}$ and $\lambda > 2\sqrt{n}$, so that $\lambda\sqrt{1+2^{-p}} > 2\sqrt{\hat{n}}$. Then

$$\begin{aligned}\Pr\Big[R_n > \lambda(1+12\cdot 2^{-p/2})\sqrt{1+2^{-p}}\Big] &\le \Pr\Big[R_{\hat{n}} > \lambda(1+12\cdot 2^{-p/2})\sqrt{1+2^{-p}}\Big]\\ &\le 2^{2p+4}\exp\Big(-\frac{\lambda^2(1+2^{-p})}{2\hat{n}}\Big)\\ &\le 2^{2p+4}\exp\Big(-\frac{\lambda^2}{2n}\Big),\end{aligned}$$

by Lemma 2.1 and (4). For $n > 2$, we take $p = p(n) = \lfloor \frac{1}{2} \log\log n \rfloor$, so that $n \geq 2^{p+1}$. Moreover

$$(1 + 12 \cdot 2^{-p/2})\sqrt{1 + 2^{-p}} \leq 1 + \delta$$

and $2^{2p+4} \leq \log n$ for all $n \geq n_0$, where n_0 depends only on δ. This completes the proof. □

Before proving Theorem 1.2, we record the following elementary, albeit very useful, fact.

Lemma 2.3. *Let $X_1, X_2, \ldots, X_n$ be mutually independent random variables, each taking each of the values -1 and 1 with probability $1/2$ and let $u_1, \ldots, u_r$ be integers satisfying*

$$0 \leq u_1 < u_2 < \cdots < u_r < n.$$

Then the $n - u_r$ products

$$X_{1+u_1} X_{1+u_2} \cdots X_{1+u_r}, \ldots, X_{n-u_r+u_1} X_{n-u_r+u_2} \cdots X_n$$

are mutually independent.

For $r = 2$, a formal proof of Lemma 2.3 is provided by Mercer [13, Proposition 1.1].

We now give a proof of Theorem 1.2. In this proof and in the remainder of this paper we make repeated use of the elementary bound

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k \quad \text{for } k, n \in \mathbb{Z} \text{ satisfying } 1 \leq k \leq n. \tag{5}$$

Proof of Theorem 1.2. Write $A_n = (a_1, a_2, \ldots, a_n)$ and notice that $C_r(A_n)$ can be rewritten as

$$C_r(A_n) = \max_{0<u_2<\cdots<u_r<n} \max_{1\leq m_1 \leq m_2 \leq n-u_r} \left| \sum_{j=m_1}^{m_2} a_j a_{j+u_2} \cdots a_{j+u_r} \right|. \tag{6}$$

Let r be an integer satisfying $2 \leq r \leq n$ and let $u_2, u_3, \ldots, u_r$ be integers satisfying

$$0 < u_2 < \cdots < u_r < n. \tag{7}$$

Write

$$\lambda = \sqrt{2n \log \tbinom{n}{r-1}}.$$

Then, in view of Lemma 2.3, the probability

$$\Pr\left[\max_{1\leq m_1 \leq m_2 \leq n-u_r} \left| \sum_{j=m_1}^{m_2} a_j a_{j+u_2} \cdots a_{j+u_r} \right| > \lambda(1+\epsilon) \right] \tag{8}$$

is at most $\Pr[R_n > \lambda(1+\epsilon)]$ with R_n defined as in (3). Write $1 + \epsilon = \sqrt{1+\gamma}(1+\delta)$ for some $\gamma, \delta > 0$. By Lemma 2.2, there is a constant n_0,

depending only on δ, such that for all $n \geq n_0$, the probability (8) is at most

$$(\log n) \exp\left(- \frac{\lambda^2(1+\gamma)}{2n} \right) = \frac{\log n}{\binom{n}{r-1}^{1+\gamma}}.$$

Summing over all possible tuples $(u_2, u_3, \dots, u_r)$ satisfying (7), we see from (6) that, for all $n \geq n_0$,

$$\begin{aligned} \Pr\left[C_r(A_n) > \lambda(1+\epsilon)\right] &\leq \frac{(\log n)\binom{n-1}{r-1}}{\binom{n}{r-1}^{1+\gamma}} \\ &< \frac{\log n}{\binom{n}{r-1}^{\gamma}}. \end{aligned} \tag{9}$$

To prove the theorem, it is enough to show that, as $n \to \infty$,

$$\sum_{r=2}^{n} \Pr\left[C_r(A_n) > \lambda(1+\epsilon)\right] \to 0.$$

From (9), for $n \geq n_0$, the left hand side is at most

$$\sum_{k=1}^{n-1} \frac{\log n}{\binom{n}{k}^{\gamma}}.$$

Let m be an integer such that $m\gamma > 1$. Then, for $n \geq m$, this last expression is at most

$$\begin{aligned} 2\sum_{k=1}^{m-1} \frac{\log n}{\binom{n}{k}^{\gamma}} + 2\sum_{k=m}^{\lfloor n/2 \rfloor} \frac{\log n}{\binom{n}{k}^{\gamma}} &\leq \frac{2m\log n}{n^{\gamma}} + \frac{n\log n}{\binom{n}{m}^{\gamma}} \\ &\leq \frac{2m\log n}{n^{\gamma}} + \frac{m^{m\gamma}\log n}{n^{m\gamma-1}}, \end{aligned}$$

using (5). Since $\gamma > 0$ and $m\gamma > 1$, the right hand side tends to zero as $n \to \infty$, as required. □

3. Asymptotic expected value

In this section, we prove the following result, which is a key step in the proof of Theorem 1.1.

Proposition 3.1. *Let A_n be drawn uniformly at random from $\{-1,1\}^n$. Then, as $n \to \infty$,*

$$\frac{\mathrm{E}\left[C_r(A_n)\right]}{\sqrt{2n\log\binom{n}{r-1}}} \to 1.$$

To prove this proposition, we make repeated use of the following lemma, which follows from well known results on concentration of probability measures (see McDiarmid [12], for example).

Lemma 3.2 ([5, Inequality (99)])**.** *Let A_n be drawn uniformly at random from $\{-1,1\}^n$. Then, for $\theta \geq 0$,*

$$\Pr\Big[\big|C_r(A_n) - \mathrm{E}[C_r(A_n)]\big| \geq \theta\Big] \leq 2\exp\bigg(-\frac{\theta^2}{2r^2n}\bigg).$$

By combining Lemma 3.2 and Theorem 1.2, it is readily verified that

$$\limsup_{n\to\infty} \frac{\mathrm{E}\left[C_r(A_n)\right]}{\sqrt{2n\log\binom{n}{r-1}}} \leq 1. \tag{10}$$

In studying a problem that is related to the second order correlation measure of finite binary sequences, the author proved in [14] that

$$\liminf_{n\to\infty} \frac{\mathrm{E}\left[C_2(A_n)\right]}{\sqrt{2n\log n}} \geq 1,$$

which proves Proposition 3.1 for $r = 2$. Our proof of the general case is also based on the approach of [14].

Let $A_n = (a_1, a_2 \ldots, a_n)$ be an element of $\{-1,1\}^n$ and, for integers $u_2, \ldots, u_r$ satisfying

$$0 < u_2 < u_3 < \cdots < u_r < n,$$

define

$$S_{u_2,\ldots,u_r}(A_n) = \sum_{j=1}^{n-u_r} a_j a_{j+u_2} \cdots a_{j+u_r}.$$

The key ingredients to the proof of Proposition 3.1 are the following two lemmas on $S_{u_2,\ldots,u_r}(A_n)$, which generalise [14, Proposition 2.1] and [14, Proposition 2.7], respectively, from $r = 2$ to general $r \geq 2$. These lemmas can be proved by modifying the arguments used in [14]. As the modifications are not always obvious, we include proofs at the end of this section.

Lemma 3.3. *Let A_n be drawn uniformly at random from $\{-1,1\}^n$ and let $r \geq 2$ be an integer. Then there exists a constant $n_0 = n_0(r)$, such that for all $n \geq n_0$ and all*

$$0 < u_2 < u_3 < \cdots < u_r \leq \frac{n}{\log n},$$

we have

$$\Pr\Big[|S_{u_2,\ldots,u_r}(A_n)| \geq \sqrt{2n\log\tbinom{n}{r-1}}\Big] \geq \frac{1}{5e^{r-2}\binom{n}{r-1}\sqrt{\log\binom{n}{r-1}}}. \tag{11}$$

Lemma 3.4. *Let A_n be drawn uniformly at random from $\{-1,1\}^n$, let $r \geq 2$ be an integer, and write*

$$\lambda = \sqrt{2n\log\tbinom{n}{r-1}}.$$

Let $u_2 < u_3 < \cdots < u_r$ and $v_2 < v_3 < \cdots < v_r$ be positive integers strictly less than n satisfying $(u_2,\dots,u_r) \neq (v_2,\dots,v_r)$. Then there exists a constant $n_0 = n_0(r)$, such that for all $n \geq n_0$, we have

$$\Pr\big[|S_{u_2,\dots,u_r}(A_n)| \geq \lambda \cap |S_{v_2,\dots,v_r}(A_n)| \geq \lambda\big] \leq \frac{23}{\binom{n}{r-1}^2}. \tag{12}$$

We now prove Proposition 3.1.

Proof of Proposition 3.1. Let $\delta > 0$ and define the set

$$N(\delta) = \left\{ n \geq r : \frac{\mathrm{E}\left[C_r(A_n)\right]}{\sqrt{2n \log \binom{n}{r-1}}} < 1 - \delta \right\}. \tag{13}$$

We shall show that $N(\delta)$ has finite size for all choices of $\delta > 0$, which together with (10) proves the proposition. To do so, we define the set

$$W = \left\{ (u_2, u_3, \dots, u_r) \in \mathbb{Z}^{r-1} : 0 < u_2 < u_3 < \cdots < u_r \leq \frac{n}{\log n} \right\}.$$

Since

$$C_r(A_n) \geq \max_{(u_2,\dots,u_r)\in W} \big|S_{u_2,\dots,u_r}(A_n)\big|,$$

we find by the inclusion-exclusion principle that, for all real λ,

$$\begin{aligned}\Pr\big[C_r(A_n) \geq \lambda\big] \geq & \sum_{(u_2,\dots,u_r)\in W} \Pr\big[|S_{u_2,\dots,u_r}(A_n)| \geq \lambda\big] \\ & - \frac{1}{2} \sum_{\substack{(u_2,\dots,u_r),(v_2,\dots,v_r)\in W \\ (u_2,\dots,u_r)\neq(v_2,\dots,v_r)}} \Pr\big[|S_{u_2,\dots,u_r}(A_n)| \geq \lambda \cap |S_{v_2,\dots,v_r}(A_n)| \geq \lambda\big].\end{aligned}$$

Now take

$$\lambda = \sqrt{2n \log \tbinom{n}{r-1}} \tag{14}$$

and apply Lemmas 3.3 and 3.4 to get, for all sufficiently large n,

$$\Pr\big[C_r(A_n) \geq \lambda\big] \geq |W| \cdot \frac{1}{5e^{r-2}\binom{n}{r-1}\sqrt{\log \binom{n}{r-1}}} - \frac{|W|^2}{2} \cdot \frac{23}{\binom{n}{r-1}^2}. \tag{15}$$

We have

$$|W| = \binom{\lfloor n/\log n \rfloor}{r-1}$$

and by the elementary bounds (5) for binomial coefficients we find that, for all sufficiently large n,

$$|W| \leq \left(\frac{en}{(r-1)\log n}\right)^{r-1} \leq \left(\frac{e}{\log n}\right)^{r-1} \binom{n}{r-1}$$

and

$$|W| \geq \left(\frac{n}{2(r-1)\log n}\right)^{r-1} \geq \left(\frac{1}{2e\log n}\right)^{r-1}\binom{n}{r-1}.$$

Hence, from (15) we obtain, for all sufficiently large n,

$$\Pr\left[C_r(A_n) \geq \lambda\right] \geq \frac{1}{5e^{r-2}}\left(\frac{1}{2e\log n}\right)^{r-1}\frac{1}{\sqrt{r\log n}} - 12\left(\frac{e}{\log n}\right)^{2r-2}.$$

Since $r \geq 2$, the first term on the right hand side dominates, and so a crude estimate gives

$$\Pr\left[C_r(A_n) \geq \lambda\right] \geq \frac{1}{e^{3r}\sqrt{r}}\left(\frac{1}{\log n}\right)^{r-1/2} \tag{16}$$

for all sufficiently large n. By the definition (13) of $N(\delta)$, we have $\lambda > \mathrm{E}[C_r(A_n)]$ for all $n \in N(\delta)$, and thus find from Lemma 3.2 with $\theta = \lambda - \mathrm{E}[C_r(A_n)]$ that, for all $n \in N(\delta)$,

$$\Pr\left[C_r(A_n) \geq \lambda\right] \leq 2\exp\left(-\frac{(\lambda - \mathrm{E}[C_r(A_n)])^2}{2r^2 n}\right).$$

Comparison with (16) then gives, for all sufficiently large $n \in N(\delta)$,

$$\frac{1}{e^{3r}\sqrt{r}}\left(\frac{1}{\log n}\right)^{r-1/2} \leq 2\exp\left(-\frac{(\lambda - \mathrm{E}[C_r(A_n)])^2}{2r^2 n}\right),$$

or equivalently, after substituting the value (14) for λ,

$$\frac{\mathrm{E}[C_r(A_n)]}{\sqrt{2n\log\binom{n}{r-1}}} \geq 1 - \sqrt{\frac{r^2(r-1/2)\log\log n + r^2\log(2e^{3r}\sqrt{r})}{\log\binom{n}{r-1}}}.$$

Hence, by the definition (13) of $N(\delta)$, we see that $N(\delta)$ has finite size for all choices of $\delta > 0$, as required. □

In the remainder of this section, we provide proofs of Lemmas 3.3 and 3.4.

Proof of Lemma 3.3. We adopt the standard notation $x_n \sim y_n$ to mean that $x_n = y_n(1+o(1))$ as $n \to \infty$. By Lemma 2.3, $S_{u_2,\ldots,u_r}(A_n)$ is a sum of $n-u_r$ mutually independent random variables, each taking each of the values -1 and $+1$ with probability $1/2$. We use a normal approximation to estimate the tail of the distribution of $|S_{u_2,\ldots,u_r}(A_n)|$ (see Feller [9, Chapter VII, (6.7)], for example): If $\xi_n \to \infty$ in such a way that $\xi_n^3/\sqrt{n} \to 0$ as $n \to \infty$, then

$$\Pr\left[\left|S_{u_2,\ldots,u_r}(A_n)\right| \geq \xi_n\sqrt{n-u_r}\right] \sim \sqrt{\frac{2}{\pi}}\cdot\frac{1}{\xi_n}\exp\left(-\frac{\xi_n^2}{2}\right).$$

Taking

$$\xi_n = \sqrt{\frac{2n}{n-u_r}\log\binom{n}{r-1}}$$

gives, since $\frac{n}{n-u_r} \sim 1$,

$$\Pr\Big[\big|S_{u_2,\dots,u_r}(A_n)\big| \ge \sqrt{2n\log\tbinom{n}{r-1}}\Big] \sim \frac{1}{\sqrt{\pi\log\tbinom{n}{r-1}}}\exp\bigg(-\frac{n}{n-u_r}\log\binom{n}{r-1}\bigg). \tag{17}$$

Using $u_r \le \frac{n}{\log n}$, we have

$$\exp\bigg(-\frac{n}{n-u_r}\log\binom{n}{r-1}\bigg) \ge \exp\bigg(-\frac{\log n}{\log n-1}\log\binom{n}{r-1}\bigg),$$

and then, since

$$\exp\bigg(-\frac{\log n}{\log n-1}\log\binom{n}{r-1}\bigg) \sim \frac{1}{e^{r-1}\binom{n}{r-1}}$$

and $e\sqrt{\pi} < 5$, we find from (17) that

$$\Pr\Big[\big|S_{u_2,\dots,u_r}(A_n)\big| \ge \sqrt{2n\log\tbinom{n}{r-1}}\Big] \ge \frac{1}{5e^{r-2}\binom{n}{r-1}\sqrt{\log\binom{n}{r-1}}}$$

for all sufficiently large n. □

To prove Lemma 3.4, it is convenient to use the following notation.

Definition 3.5. A tuple $(x_1,\dots,x_{2m})$ is *d-even* if there exists a permutation σ of $\{1,2,\dots,2m\}$ such that $x_{\sigma(2i-1)} = x_{\sigma(2i)}$ for each $i \in \{1,2,\dots,d\}$ and d is the largest integer with this property. An m-even tuple $(x_1,\dots,x_{2m})$ is just called *even*.

For example, $(1,3,1,4,3,4)$ is even, while $(2,1,1,2,1,3)$ is 2-even. In the next two lemmas we state two results about even tuples.

Recall that, for a positive integer k, the double factorial

$$(2k-1)!! = \frac{(2k)!}{k!\,2^k} = (2k-1)(2k-3)\cdots 3\cdot 1$$

is the number of ways to arrange $2k$ objects into k unordered pairs. The following lemma is immediate.

Lemma 3.6 ([14, Lemma 2.4])**.** *Let m and q be positive integers. Then the number of even tuples in $\{1,\dots,m\}^{2q}$ is at most $(2q-1)!!\,m^q$.*

The following lemma generalises [14, Lemma 2.5].

Lemma 3.7. *Let n, q, and t be positive integers satisfying $0 \le t < q$ and let $u_2 < u_3 < \cdots < u_r$ and $v_2 < v_3 < \cdots < v_r$ be positive integers strictly less than n satisfying $(u_2,\dots,u_r) \ne (v_2,\dots,v_r)$. Write $I = \{1,\dots,2q\}$ and let S be the subset of $\{1,\dots,n\}^{4rq}$ containing all even elements*

$$(x_i, x_i+u_2,\dots,x_i+u_r, y_i, y_i+v_2,\dots,y_i+v_r)_{i\in I}$$

such that $(x_i)_{i\in I}$ *is* d*-even for some* $d < q-t$*. Then*

$$|S| \le (4rq-1)!!\, n^{2q-(t+1)/3}.$$

Proof. We will construct a set of tuples that contains S as a subset. For convenience write $u_1 = v_1 = 0$. Arrange the $4rq$ variables

$$x_i + u_k,\ y_i + v_k \quad \text{for } i \in I \text{ and } k \in \{1,2,\dots,r\} \tag{18}$$

into $2rq$ unordered pairs $\{a_1,b_1\},\{a_2,b_2\},\dots,\{a_{2rq},b_{2rq}\}$ such that there are at most $q-t-1$ pairs $\{x_i,x_j\}$. This can be done in at most $(4rq-1)!!$ ways. We formally set $a_i = b_i$ for all $i \in \{1,2,\dots,2rq\}$. If this assignment does not yield a contradiction, then we call the arrangement of (18) into $2rq$ pairs *consistent*. For example, if there are pairs of the form $\{x_i,y_j\}$, $\{x_i+u_2,y_j+v_2\},\dots,\{x_i+u_r,y_j+v_r\}$, then the arrangement is not consistent since $(u_2,\dots,u_r) \ne (v_2,\dots,v_r)$ by assumption.

Notice that, if there is a pair of the form $\{x_i+u_k, x_j+u_\ell\}$ in a consistent arrangement, then $i \ne j$ and x_i determines x_j. Likewise, if there is a pair of the form $\{y_i+v_k, y_j+v_\ell\}$ in a consistent arrangement, then $i \ne j$ and y_i determines y_j. On the other hand, if a consistent arrangement contains a pair of the form $\{x_i+u_k, y_j+v_\ell\}$, then x_i determines y_j and at least one other variable in the list

$$x_1,\dots,x_{2q},y_1,\dots,y_{2q}. \tag{19}$$

To see this, it is enough to show that a consistent arrangement cannot contain r pairs involving only the $2r$ variables

$$x_i+u_1,\dots,x_i+u_r,y_j+v_1,\dots,y_j+v_r. \tag{20}$$

Indeed, since $u_1 < \cdots < u_r$ and $v_1 < \cdots < v_r$ and $u_1 = v_1 = 0$, the only possibility for such r pairs would be $\{x_i+u_1,y_j+v_1\},\dots,\{x_i+u_r,y_j+v_r\}$. However, as already mentioned above, this implies that the arrangement is not consistent. Hence at least one of the variables in the list (20) must be paired with a variable not in the list (20), and so x_i determines another variable in the list (19) different from y_j.

Now, by assumption, each consistent arrangement contains at most $q-t-1$ pairs of the form $\{x_i,x_j\}$ and at most q pairs of the form $\{y_i,y_j\}$, and so at most

$$q-t-1+q+\tfrac{1}{3}(2t+2) = 2q-\tfrac{1}{3}(t+1)$$

of the variables in (19) can be chosen independently. We assign to each of these a value of $\{1,\dots,n\}$. In this way, we construct a set of at most $(4rq-1)!!\, n^{2q-(t+1)/3}$ tuples that contains S as a subset. □

The next lemma, whose proof is modelled on that of [14, Lemma 2.6], provides the key step in the proof of Lemma 3.4.

Lemma 3.8. *Let* p *and* h *be integers satisfying* $0 \le h < p$ *and let* A_n *be drawn uniformly at random from* $\{-1,1\}^n$*. Let* $u_2 < u_3 < \cdots < u_r$

and $v_2 < v_3 < \cdots < v_r$ be positive integers strictly less than n satisfying $(u_2,\dots,u_r) \neq (v_2,\dots,v_r)$. Then

$$\mathrm{E}\Big[\big(S_{u_2,\dots,u_r}(A_n)S_{v_2,\dots,v_r}(A_n)\big)^{2p}\Big] \le n^{2p}\big[(2p-1)!!\big]^2\bigg(1+\frac{(4rp)^{4rh}}{n^{1/3}}+\frac{(4rp)^{2rp}}{n^{(h+1)/3}}\bigg). \tag{21}$$

Proof. Write $A_n = (a_1, a_2, \dots, a_n)$. Expand to see that the left hand side of (21) equals

$$\sum_{i_1,\dots,i_{2p}=1}^{n-u_r}\ \sum_{j_1,\dots,j_{2p}=1}^{n-v_r} \mathrm{E}\big[a_{i_1}a_{i_1+u_2}\cdots a_{i_1+u_r}\cdots a_{i_{2p}}a_{i_{2p}+u_2}\cdots a_{i_{2p}+u_r} a_{j_1}a_{j_1+v_2}\cdots a_{j_1+v_r}\cdots a_{j_{2p}}a_{j_{2p}+v_2}\cdots a_{j_{2p}+v_r}\big]. \tag{22}$$

Write $I = \{1, 2, \dots, 2p\}$ and let T be the set containing all even tuples in $\{1,\dots,n\}^{4rp}$ of the form

$$(x_i, x_i+u_2, \dots, x_i+u_r, y_i, y_i+v_2, \dots, y_i+v_r)_{i\in I}. \tag{23}$$

Since $a_1,\dots,a_n$ are mutually independent, $\mathrm{E}[a_j] = 0$, and $a_j^2 = 1$ for all $j \in \{1,\dots,n\}$, we find from (22) that the left hand side of (21) equals $|T|$. It remains to show that $|T|$ is at most the right hand side of (21).

We define the following subsets of T.

- T_1 contains all elements (23) of T such that $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ are even.
- T_2 contains all elements (23) of T such that $(x_i)_{i\in I}$ is d_1-even and $(y_i)_{i\in I}$ is d_2-even for some d_1 and d_2 satisfying $p-h \le d_1, d_2 \le p$, at least one of them strictly less than p.
- T_3 contains all elements (23) of T such that $(x_i)_{i\in I}$ or $(y_i)_{i\in I}$ is d-even for some $d < p-h$.

It is readily verified that T_1, T_2, and T_3 partition T. We now bound the cardinalities of T_1, T_2, and T_3.

The set T_1. Using Lemma 3.6 applied with $q = p$, we have

$$|T_1| \le \big[(2p-1)!!\big]^2 n^{2p}. \tag{24}$$

The set T_2. Consider an element (23) of T_2. Then there exist $(2p-2h)$-element subsets J and K of I such that $(x_i)_{i\in J}$ and $(y_i)_{i\in K}$ are even and

$$(x_i)_{i\in I\setminus J} \tag{25}$$

is not even (if $(x_i)_{i\in I\setminus J}$ were even, then $(y_i)_{i\in I\setminus K}$ would also be even, which contradicts the definition of the elements of T_2). Since $(x_i)_{i\in J}$ and $(y_i)_{i\in K}$ are even and the tuple (23) is even, we find that

$$(x_i, x_i+u_2, \dots, x_i+u_r, y_j, y_j+v_2, \dots, y_j+v_r)_{i\in I\setminus J,\, j\in I\setminus K} \tag{26}$$

is also even. There are $\binom{2p}{2h}$ subsets J and $\binom{2p}{2h}$ subsets K. By Lemma 3.6 applied with $q = p-h$, for each such J and K, there are at most $(2p-2h-1)!!\, n^{p-h}$ even tuples $(x_i)_{i\in J}$ satisfying $1 \le x_i \le n$ for each $i \in J$ and

at most $(2p-2h-1)!!\,n^{p-h}$ even tuples $(y_i)_{i\in K}$ satisfying $1 \le y_i \le n$ for each $i \in K$. By Lemma 3.7 applied with $q = h$ and $t = 0$, the number of even tuples in $\{1,\dots,n\}^{4rh}$ of the form (26) such that the tuple in (25) is not even is at most $(4rh-1)!!\,n^{2h-1/3}$. Therefore,

$$\begin{aligned} |T_2| &\le (4rh-1)!!\,n^{2h-1/3}\left[\tbinom{2p}{2h}(2p-2h-1)!!\,n^{p-h}\right]^2 \\ &\le n^{2p-1/3}\big[(2p-1)!!\big]^2(4rp)^{4rh}. \end{aligned} \tag{27}$$

The set T_3. By Lemma 3.7 applied with $q = p$ and $t = h$ and by symmetry, we have

$$|T_3| \le 2(4rp-1)!!\,n^{2p-(h+1)/3} \le n^{2p-(h+1)/3}(4rp)^{2rp}. \tag{28}$$

Now from (24), (27), and (28) we get an upper bound for $|T|$, from which we can deduce (21). □

We now prove Lemma 3.4.

Proof of Lemma 3.4. Let X_1 and X_2 be a random variables and let p be a positive integer. Then by Markov's inequality, for $\theta_1, \theta_2 > 0$,

$$\Pr\big[|X_1| \ge \theta_1 \cap |X_2| \ge \theta_2\big] \le \frac{\mathrm{E}\big[(X_1X_2)^{2p}\big]}{(\theta_1\theta_2)^{2p}}.$$

Let h be an integer satisfying $0 \le h < p$. Lemma 3.8 shows that the left hand side of (12) is at most

$$\frac{\big[(2p-1)!!\big]^2}{(2\log\binom{n}{r-1})^{2p}}\big[1 + K_1(n,p,h) + K_2(n,p,h)\big], \tag{29}$$

where

$$\begin{aligned} K_1(n,p,h) &= n^{-1/3}\,(4rp)^{4rh}, \\ K_2(n,p,h) &= n^{-(h+1)/3}\,(4rp)^{2rp}. \end{aligned}$$

We take $p = \lfloor \log\binom{n}{r-1} \rfloor$ and $h = \lfloor \alpha \log\log n \rfloor$ for some $\alpha > 0$, to be determined later, and show that (29) is at most $23/\binom{n}{r-1}^2$ for all sufficiently large n. Notice that $h < p$ for all sufficiently large n, as assumed. By Stirling's approximation

$$\sqrt{2\pi k}\;k^k e^{-k} \le k! \le \sqrt{3\pi k}\;k^k e^{-k},$$

we have

$$\frac{\big[(2p-1)!!\big]^2}{(2\log\binom{n}{r-1})^{2p}} \le \frac{3p^{2p}e^{-2p}}{(\log\binom{n}{r-1})^{2p}} \le \frac{3e^2}{\binom{n}{r-1}^2}.$$

Moreover

$$\begin{aligned} K_1(n,p,h) &\le K_1(n, r\log n, \alpha \log\log n) \\ &= n^{-\frac{1}{3}}\, n^{\frac{2\alpha \log\log n(\log r + \log\log n)}{\log n}} \\ &= O(n^{-\frac{1}{4}}) \end{aligned}$$

and

$$\begin{aligned} K_2(n,p,h) &\le K_2(n, r\log n, (\alpha-1) \log\log n) \\ &= n^{-\frac{1}{3} - (\frac{\alpha-1}{3} - 2r^2)\log\log n + 2r^2\log(4r^2)} \\ &= O(n^{-\log\log n}) \end{aligned}$$

by taking $\alpha = 10r^2$, say. The lemma follows since $3e^2 < 23$. □

4. Almost sure convergence

In this section we prove Theorem 1.1. We begin with the following standard result (see [6, Theorem A.1.1], for example).

Lemma 4.1. *Let $X_1, \dots, X_n$ be independent random variables, each taking the values -1 and 1, each with probability $1/2$. Then, for $\lambda \ge 0$,*

$$\Pr\left[\left|\sum_{j=1}^{n} X_j\right| > \lambda\right] \le 2\exp\Big(-\frac{\lambda^2}{2n}\Big).$$

Lemma 4.1 is used to deduce the following result.

Lemma 4.2. *Let $(a_1, a_2, \dots)$ be drawn from Ω, equipped with the probability measure defined by (2), and write $A_n = (a_1, a_2, \dots, a_n)$. Let $n_1, n_2, \dots$ be a strictly increasing sequence of integers greater than or equal to r. Then, almost surely,*

$$C_r(A_{n_{k+1}}) - C_r(A_{n_k}) \le \sqrt{10(n_{k+1}-n_k)\log\tbinom{n_{k+1}}{r-1}}$$

for all sufficiently large k.

Proof. Write

$$\lambda = \sqrt{10(n_{k+1}-n_k)\log\tbinom{n_{k+1}}{r-1}}. \tag{30}$$

If

$$C_r(A_{n_{k+1}}) - C_r(A_{n_k}) > \lambda, \tag{31}$$

then

$$\left|\sum_{j=\max(1,n_k-u_r+1)}^{m} a_{j+u_1}a_{j+u_2}\cdots a_{j+u_r}\right| > \lambda \tag{32}$$

for at least one tuple $(u_1, u_2, \dots, u_r)$ satisfying

$$0 \le u_1 < u_2 < \cdots < u_r < n_{k+1} \tag{33}$$

and at least one m satisfying

$$n_k - u_r + 1 \leq m \leq n_{k+1} - u_r. \tag{34}$$

Let $(u_1, u_2, \ldots, u_r)$ be a tuple of integers satisfying (33) and let m be an integer satisfying (34). By Lemma 2.3, the sum in (32) is a sum of at most $n_{k+1} - n_k$ independent random variables, each taking each of the values 1 and -1 with probability $1/2$. Thus, by Lemma 4.1, the probability of (32) is at most

$$2\exp\left(-\frac{\lambda^2}{2(n_{k+1}-n_k)}\right) = 2\binom{n_{k+1}}{r-1}^{-5},$$

after substituting (30). Summing over all possible tuples $(u_1, u_2, \ldots, u_r)$ and all possible m, the probability that (32) happens for some $(u_1, u_2, \ldots, u_r)$ satisfying (33) and some integer m satisfying (34) is at most

$$2(n_{k+1}-n_k)\binom{n_{k+1}}{r}\binom{n_{k+1}}{r-1}^{-5}.$$

This is also an upper bound for the probability of (31), and so

$$\Pr\left[C_r(A_{n_{k+1}}) - C_r(A_{n_k}) > \lambda\right] \leq 2(n_{k+1})^2\binom{n_{k+1}}{r-1}^{-4} \leq \frac{2}{(n_{k+1})^2}.$$

Thus,

$$\sum_{k=1}^{\infty}\Pr\left[C_r(A_{n_{k+1}}) - C_r(A_{n_k}) > \lambda\right] \leq \sum_{k=1}^{\infty}\frac{2}{(n_{k+1})^2} < \infty,$$

and the result follows from the Borel-Cantelli Lemma. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. Write

$$\vartheta_n = \sqrt{2n\log\tbinom{n}{r-1}}$$

and let n_k be the smallest integer that is at least $e^{k^{1/2}}$. We first show that the theorem holds for the subsequence n_k, namely that, as $k \to \infty$,

$$\frac{C_r(A_{n_k})}{\vartheta_{n_k}} \to 1 \quad \text{almost surely.} \tag{35}$$

To do so, choose an $\epsilon > 0$ and observe that by the triangle inequality, the probability

$$\Pr\left[\left|\frac{C_r(A_n)}{\vartheta_n} - 1\right| > \epsilon\right]$$

is bounded from above by

$$\Pr\left[\left|\frac{C_r(A_n)}{\vartheta_n} - \frac{\mathrm{E}[C_r(A_n)]}{\vartheta_n}\right| > \tfrac{1}{2}\epsilon\right] + \Pr\left[\left|\frac{\mathrm{E}[C_r(A_n)]}{\vartheta_n} - 1\right| > \tfrac{1}{2}\epsilon\right].$$

By Proposition 3.1, the second probability equals zero for all sufficiently large n. The first probability can be bounded using Lemma 3.2, showing that

$$\Pr\left[\left|\frac{C_r(A_n)}{\vartheta_n}-1\right|>\tfrac{1}{2}\epsilon\right]\le 2\exp\left(-\frac{\epsilon^2}{4r^2}\log\binom{n}{r-1}\right)$$

for all sufficiently large n. We can further bound this expression very crudely by $1/(\log n)^3$, say, for all sufficiently large n. Thus, since $n_k\ge e^{k^{1/2}}$, we have for sufficiently large k_0,

$$\sum_{k=k_0}^{\infty}\Pr\left[\left|\frac{C_r(A_{n_k})}{\vartheta_{n_k}}-1\right|>\tfrac{1}{2}\epsilon\right]\le\sum_{k=k_0}^{\infty}\frac{1}{(\log n_k)^3}\le\sum_{k=k_0}^{\infty}\frac{1}{k^{3/2}}<\infty$$

and (35) follows from the Borel-Cantelli Lemma.

We shall now complete the proof by showing that, as $k\to\infty$,

$$\max_{n_k\le n\le n_{k+1}}\left|\frac{C_r(A_n)}{\vartheta_n}-1\right|\to 0\quad\text{almost surely.}\tag{36}$$

We apply the triangle inequality to find that

$$\begin{aligned}\max_{n_k\le n\le n_{k+1}}\left|\frac{C_r(A_n)}{\vartheta_n}-1\right|&\le\left|1-\frac{C_r(A_{n_{k+1}})}{\vartheta_{n_{k+1}}}\right|\\&+\max_{n_k\le n\le n_{k+1}}\left|\frac{C_r(A_{n_{k+1}})}{\vartheta_{n_{k+1}}}-\frac{C_r(A_n)}{\vartheta_{n_{k+1}}}\right|+\max_{n_k\le n\le n_{k+1}}\left|\frac{C_r(A_n)}{\vartheta_{n_{k+1}}}-\frac{C_r(A_n)}{\vartheta_n}\right|.\end{aligned}\tag{37}$$

Since $C_r(A_n)$ is non-decreasing, we find from Lemma 4.2 that

$$\max_{n_k\le n\le n_{k+1}}\left|\frac{C_r(A_{n_{k+1}})}{\vartheta_{n_{k+1}}}-\frac{C_r(A_n)}{\vartheta_{n_{k+1}}}\right|\le\sqrt{\frac{5(n_{k+1}-n_k)}{n_{k+1}}}$$

almost surely for all sufficiently large k. From

$$\lim_{k\to\infty}\frac{n_{k+1}}{n_k}=\lim_{k\to\infty}e^{(k+1)^{1/2}-k^{1/2}}=1\tag{38}$$

we conclude that, as $k\to\infty$,

$$\max_{n_k\le n\le n_{k+1}}\left|\frac{C_r(A_{n_{k+1}})}{\vartheta_{n_{k+1}}}-\frac{C_r(A_n)}{\vartheta_{n_{k+1}}}\right|\to 0\quad\text{almost surely.}\tag{39}$$

The third term on the right hand side of (37) can be bounded as

$$\max_{n_k\le n\le n_{k+1}}\left|\frac{C_r(A_n)}{\vartheta_{n_{k+1}}}-\frac{C_r(A_n)}{\vartheta_n}\right|\le\frac{C_r(A_{n_{k+1}})}{\vartheta_{n_{k+1}}}\left|1-\frac{\vartheta_{n_{k+1}}}{\vartheta_{n_k}}\right|.$$

Using (38), it is readily verified that

$$\lim_{k\to\infty}\frac{\vartheta_{n_{k+1}}}{\vartheta_{n_k}}=1$$

and, after combination with (35), we conclude that, as $k \to \infty$,

$$\max_{n_k \le n \le n_{k+1}} \left| \frac{C_r(A_n)}{\vartheta_{n_{k+1}}} - \frac{C_r(A_n)}{\vartheta_n} \right| \to 0 \quad \text{almost surely.} \tag{40}$$

The required convergence (36) follows by combining (37), (35), (39), and (40). □

5. Minimum values

Recall that the *scalar product* between two vectors $x = (x_1, \ldots, x_\ell)$ and $y = (y_1, \ldots, y_\ell)$ in $\mathbb{C}^\ell$ is $\langle x, y \rangle = \sum_{j=1}^{\ell} x_j \overline{y_j}$, where bar means complex conjugation. We shall see that Theorems C and 1.3 follow from well known results on the maximum magnitude of the nontrivial scalar products over a set of vectors in $\mathbb{C}^\ell$; a good overview is given by Kumar and Liu [10]. The most famous such result is the following bound due to Welch [16].

Lemma 5.1 (Welch [16])**.** *For positive integers ℓ and $m \ge 2$, let $v_1, \ldots, v_m$ be elements of $\mathbb{C}^\ell$ satisfying $\|v_i\|_2^2 = \ell$ for each i. Then, for integral $k \ge 1$,*

$$\max_{i \ne i'} \left| \langle v_i, v_{i'} \rangle \right| \ge \left[\frac{\ell^{2k}}{m-1} \left(\frac{m}{\binom{\ell+k-1}{k}} - 1 \right) \right]^{1/2k}.$$

This lemma can be proved by observing

$$m\ell^{2k} + m(m-1) \max_{i \ne i'} \left| \langle v_i, v_{i'} \rangle \right|^{2k} \ge \sum_{i,i'} \left| \langle v_i, v_{i'} \rangle \right|^{2k}$$

and deriving a lower bound for the right hand side. We remark that, for $k > 1$ and when the vectors have entries in $\{-1, 1\}$, the bound in Lemma 5.1 can be slightly improved by a bound due to Sidelnikov [15]. Lemma 5.1 is now used to give a straightforward proof of Theorem C.

Proof of Theorem C. Let $A_n = (a_1, a_2, \ldots, a_n)$ be an element of $\{-1, 1\}^n$. Write $\ell = \lfloor n/(2r+1) \rfloor$. For $\ell = 0$, the theorem is trivial, so assume that $\ell \ge 1$. Let $S_1, S_2, \ldots, S_m$ be $m = \lfloor (n - \ell + 1)/r \rfloor$ pairwise disjoint r-element subsets of $\{0, \ldots, n - \ell\}$. For each such set S_i, define the vector $v_i = (v_{i,1}, \ldots, v_{i,\ell})$ by

$$v_{i,j} = \prod_{x \in S_i} a_{j+x} \quad \text{for each } j \in \{1, \ldots, \ell\}.$$

Since all of the sets $S_1, \ldots, S_m$ have size r and are pairwise disjoint, we have

$$C_{2r}(A_n) \ge \max_{i \ne i'} \left| \langle v_i, v_{i'} \rangle \right|. \tag{41}$$

Observe that

$$m = \left\lfloor \frac{n - \lfloor n/(2r+1) \rfloor + 1}{r} \right\rfloor \ge \left\lfloor \frac{2n}{2r+1} \right\rfloor \ge 2\ell.$$

Hence, $m \geq 2$ and we can apply Lemma 5.1 with $k = 1$ to (41) to conclude

$$\big[C_{2r}(A_n)\big]^2 \geq \frac{\ell^2}{m-1}\Big(\frac{m}{\ell} - 1\Big) > \ell\Big(1 - \frac{\ell}{m}\Big) \geq \frac{\ell}{2},$$

as required. □

Slight improvements of Theorem C are possible for particular values r, by choosing ℓ more carefully in the proof (see Anantharam [7] for $r = 2$).

We now prove Theorem 1.3.

Proof of Theorem 1.3. Let $A_n = (a_1, a_2, \ldots, a_n)$ be an element of $\{-1, 1\}^n$. We have $n \geq 3$. Let $\ell = \lfloor n/3 \rfloor$ and let $S_1, S_2, \ldots, S_m$ be all $m = \binom{n-\ell+1}{s}$ s-element subsets of $\{0, 1, \ldots, n-\ell\}$. For each such set S_i, define the vector $v_i = (v_{i,1}, \ldots, v_{i,\ell})$ by

$$v_{i,j} = \prod_{x \in S_i} a_{j+x} \quad \text{for each } j \in \{1, \ldots, \ell\}.$$

Then

$$\max\big\{C_2(A_n), C_4(A_n), \ldots, C_{2s}(A_n)\big\} \geq \max_{i \neq i'} \big|\langle v_i, v_{i'}\rangle\big|.$$

We apply Lemma 5.1 with $k = s$ to get

$$\Big[\max\big\{C_2(A_n), C_4(A_n), \ldots, C_{2s}(A_n)\big\}\Big]^{2s} \geq \frac{\ell^{2s}}{m-1}\left(\frac{m}{\binom{\ell+s-1}{s}} - 1\right).$$

Write $n = 3\ell + \delta$ for some $\delta \in \{0, 1, 2\}$. Then by (5) the leading term on the right hand side is

$$\frac{\ell^{2s}}{m-1} \geq \frac{\ell^{2s}}{m} = \frac{(\frac{n-\delta}{3})^{2s}}{\binom{(2n+\delta+3)/3}{s}} \geq \left(\frac{s(n-\delta)^2}{3e(2n+\delta+3)}\right)^s > \left(\frac{sn}{9^2}\right)^s,$$

using $n \geq 3$ and distinguishing the cases that $n \in \{3, 4, 5, 6\}$ and $n \geq 7$ to get the last inequality.

We complete the proof by showing that $m/\binom{\ell+s-1}{s} - 1$ is greater than 1. Define $f : \{1, 2, \ldots, \lfloor n/3 \rfloor\} \to \mathbb{Q}$ by

$$f(s) = \frac{\binom{n-\ell+1}{s}}{\binom{\ell+s-1}{s}}.$$

A standard calculation shows that f is monotonically increasing for $s \leq (n - 2\ell + 2)/2$ and is monotonically decreasing for $s \geq (n - 2\ell + 2)/2$. Therefore, the minimum value of $f(s)$ is either $f(1)$ or $f(\lfloor n/3 \rfloor) = f(\ell)$. Moreover, we readily verify that $f(1) > 2$ and

$$f(\ell) \geq \frac{\binom{2\ell+1}{\ell}}{\binom{2\ell-1}{\ell}} = \frac{2(2\ell+1)}{\ell+1} \geq 3.$$

Hence f satisfies $f(s) > 2$ on its entire domain, as required. □

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