

Simple Maximum-Likelihood Decoding of Generalized First-order Reed–Muller Codes

Kai-Uwe Schmidt and Adolf Finger

Abstract

An efficient decoder for the generalized first-order Reed–Muller code $\text{RM}_q(1, m)$ is essential for the decoding of various block-coding schemes for orthogonal frequency-division multiplexing with reduced peak-to-mean power ratio. We present an efficient and simple maximum-likelihood decoding algorithm for $\text{RM}_q(1, m)$. It is shown that this algorithm has lower complexity than other previously known maximum-likelihood decoders for $\text{RM}_q(1, m)$.

Index Terms

maximum-likelihood decoding, OFDM, peak power, Reed–Muller codes

I. INTRODUCTION

IN ORDER to ensure tight control of the peak-to-mean power ratio (PMPR) in orthogonal frequency-division multiplexing (OFDM) systems, various block-coding schemes have been proposed [1], [2]. These codes are obtained from unions of, say M , cosets of a q -ary generalization of the first-order Reed–Muller code $\text{RM}_q(1, m)$ (see Definition 1). Such codes have the potential to perform error correction and ensure a substantially reduced PMPR compared to uncoded transmission.

Most of the existing decoding algorithms for the above mentioned codes are based on the supercode decoding method, described in [3] for the binary case. Such a decoding scheme involves M decodings of $\text{RM}_q(1, m)$. Hence an efficient decoder for $\text{RM}_q(1, m)$ is required. Moreover, if a maximum-likelihood (ML) decoder is used to decode $\text{RM}_q(1, m)$, then the supercode decoder performs ML decoding too [3], [1].

There exist various decoding techniques for the code $\text{RM}_2(1, m)$. The fast Hadamard transform (FHT) can be used to obtain an ML decoder [4]. Suboptimal techniques include majority-logic decoding [4] and decoding based on the interpretation of the Reed–Muller codes as general concatenated codes [5], [6]. In [7] the latter approach was extended to a list-decoding scheme, and it was shown that ML decoding of $\text{RM}_2(1, m)$ is possible if the list length is equal to 2.

In [1] an iterative decoder for $\text{RM}_{2^h}(1, m)$ was proposed that relegates the decoding of $\text{RM}_{2^h}(1, m)$ to h decodings of $\text{RM}_2(1, m)$, for which efficient methods exist. A quite similar approach has been reported in [8]. Another suboptimal decoder was proposed in [9], where the majority-logic decoding method was extended to the nonbinary case. Implicitly, a decoder for $\text{RM}_q(1, m)$ was obtained in [10], by applying the method of ordered statistics to nonbinary codes. In [11] an ML decoder was given for $\text{RM}_4(1, m)$, by treating $\text{RM}_4(1, m)$ itself as a union of 2^m cosets of $\text{RM}_2(1, m + 1)$. A q -ary equivalent of the binary FHT has been reported in [8] and [12]. Consequently an ML decoder for $\text{RM}_q(1, m)$ for arbitrary q was obtained. In this letter we present a new ML decoder for $\text{RM}_q(1, m)$, which appears to have lower complexity than the above mentioned ML decoding schemes.

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II. DEFINITIONS AND PRELIMINARIES

Definition 1: The generalized first-order Reed–Muller code $\text{RM}_q(1, m)$ ($m \geq 1$) consists of the \mathbb{Z}_q -valued codewords of length 2^m $\{\mathbf{u} \cdot \mathbf{G}_m \pmod{q} \mid \mathbf{u} \in \mathbb{Z}_q^{m+1}\}$, where we define a generator matrix \mathbf{G}_m recursively by

$$\mathbf{G}_m = \begin{pmatrix} \mathbf{G}_{m-1} & \mathbf{G}_{m-1} \\ \mathbf{0}_{2^{m-1}} & \mathbf{1}_{2^{m-1}} \end{pmatrix} \quad \text{with} \quad \mathbf{G}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here $\mathbf{0}_n$ and $\mathbf{1}_n$ are row vectors of length n containing only zeros and ones, respectively.

The above definition implies the following recursive construction of the codewords of $\text{RM}_q(1, m)$

$$\mathbf{u} \cdot \mathbf{G}_m \pmod{q} = (\mathbf{u}' \cdot \mathbf{G}_{m-1} \mid \mathbf{u}' \cdot \mathbf{G}_{m-1} + u_m \mathbf{1}_{2^{m-1}}) \pmod{q}.$$

Here $(\cdot \mid \cdot)$ means concatenation and $\mathbf{u} = (u_0 \ u_1 \ \cdots \ u_m)$ and $\mathbf{u}' = (u_0 \ u_1 \ \cdots \ u_{m-1})$. This property will be used in the next section to derive an efficient ML decoding scheme for $\text{RM}_q(1, m)$.

After encoding, the \mathbb{Z}_q -valued codewords are mapped onto a q -ary phase-shift-keying (PSK) constellation. This means that for each \mathbb{Z}_q -valued codeword there exists a corresponding polyphase codeword, which is given by

$$\xi^{\mathbf{u} \cdot \mathbf{G}_m} = \xi^{\mathbf{c}} = (\xi^{c_0} \ \xi^{c_1} \ \cdots \ \xi^{c_{2^m-1}}).$$

Here $\xi = \exp(j2\pi/q)$ is a primitive q th root of unity, where $j^2 = -1$. Since there is a one-to-one correspondence between \mathbb{Z}_q -valued and polyphase codewords, in the remainder of this letter, we shall drop the distinction between them. The context should make clear to which one we refer.

III. DECODING OF $\text{RM}_q(1, m)$

We assume that the codeword $\xi^{\mathbf{u} \cdot \mathbf{G}_m}$ is transmitted over an additive white Gaussian noise (AWGN) channel. Then we receive

$$\mathbf{y} = \xi^{\mathbf{u} \cdot \mathbf{G}_m} + \mathbf{n},$$

where $\mathbf{n} \in \mathbb{C}^{2^m}$ is the complex white Gaussian noise vector. An ML decoder now finds the codeword $\hat{\mathbf{x}}$ that is closest to \mathbf{y} in the Euclidean sense. We shall call $\hat{\mathbf{x}}$ the ML codeword of \mathbf{y} . That is to say

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \in \text{RM}_q(1, m)} \|\mathbf{y} - \mathbf{x}\|^2,$$

where

$$\|\mathbf{y} - \mathbf{x}\|^2 = \sum_{k=0}^{2^m-1} |y_k - x_k|^2.$$

It is straightforward to verify

$$\|\mathbf{y} - \mathbf{x}\|^2 = \|\mathbf{y}\|^2 + \|\mathbf{x}\|^2 - 2 \operatorname{Re}\{\mathbf{y} \cdot \mathbf{x}^H\},$$

where $()^H$ denotes Hermitian (conjugate transpose), such that $\mathbf{y} \cdot \mathbf{x}^H$ is the dot product of \mathbf{y} and \mathbf{x} . Notice that $\|\mathbf{y}\|^2$ and $\|\mathbf{x}\|^2$ are independent of the decoding result, since $\|\mathbf{x}\|^2 = n$ for each $\mathbf{x} \in \text{RM}_q(1, m)$. Hence finding the codeword that is closest to \mathbf{y} is equivalent to finding the codeword for which the real part of the dot product with the received vector is maximized. To be precise

$$\hat{\mathbf{x}} = \arg \max_{\mathbf{x} \in \text{RM}_q(1, m)} \operatorname{Re}\{\mathbf{y} \cdot \mathbf{x}^H\}.$$

We now state our decoding algorithm. Then we show that this algorithm always outputs the ML codeword of \mathbf{y} .

Algorithm 2: Soft-decision maximum-likelihood decoder for $\text{RM}_q(1, m)$.

- 1) Input a vector $\mathbf{y} = (y_0 \ y_1 \ \cdots \ y_{2^m-1}) \in \mathbb{C}^{2^m}$.
- 2) If $m = 1$, output the hard decisions $\hat{\mathbf{x}}$, the estimated information symbols $\hat{\mathbf{u}}$, and $\operatorname{Re}\{\mathbf{y} \cdot \hat{\mathbf{x}}^H\}$. Stop.

3) For $i = 0, 1, \dots, q - 1$ calculate

$$\mathbf{z}(i) = (z_0(i) z_1(i) \cdots z_{2^{m-1}-1}(i))$$

with

$$z_k(i) = y_k + y_{k+2^{m-1}} \xi^{-i},$$

and use the algorithm to decode $\mathbf{z}(i)$ to $\hat{\mathbf{z}}(i)$, to get the estimated information symbols $\hat{\mathbf{u}}'(i)$, and to calculate $p(i) = \text{Re}\{\mathbf{z}(i) \cdot \hat{\mathbf{z}}^H(i)\}$.

4) Determine

$$\hat{i} = \arg \max_{i \in \mathbb{Z}_q} p(i).$$

Output the decoded codeword $\hat{\mathbf{x}} = (\hat{\mathbf{z}}(\hat{i}) | \hat{\mathbf{z}}(\hat{i}) \xi^{\hat{i}})$, the estimated information symbols $\hat{\mathbf{u}} = (\hat{\mathbf{u}}'(\hat{i}) | \hat{i})$, and $p(\hat{i})$.

Theorem 3: Algorithm 2 is a soft-decision maximum-likelihood decoder for $\text{RM}_q(1, m)$.

Proof: We will show that, if Algorithm 2 is an ML decoder for $\text{RM}_q(1, m - 1)$, then it is also an ML decoder for $\text{RM}_q(1, m)$. The case $m = 1$ serves as the induction anchor, since it is obvious that Algorithm 2 is an ML decoder for $\text{RM}_q(1, 1)$. Moreover it is clear that, if $m = 1$, Algorithm 2 outputs $\text{Re}\{\mathbf{y} \cdot \hat{\mathbf{x}}^H\}$ correctly.

Now let $m > 1$. We have to show that the algorithm always outputs the ML codeword $\hat{\mathbf{x}}$ and $\text{Re}\{\mathbf{y} \cdot \hat{\mathbf{x}}^H\}$, which is potentially required for the higher stage of the decoder. From the discussion in Section II we know that a polyphase codeword $\mathbf{x} \in \text{RM}_q(1, m)$ may be expressed as $\mathbf{x} = (\mathbf{w} | \mathbf{w} \xi^i)$, where $\mathbf{w} \in \text{RM}_q(1, m - 1)$ and $i \in \mathbb{Z}_q$. By computing

$$(\hat{\mathbf{w}}, \hat{i}) = \arg \max_{\mathbf{w} \in \text{RM}_q(1, m-1), i \in \mathbb{Z}_q} \text{Re}\{\mathbf{y} \cdot (\mathbf{w} | \mathbf{w} \xi^i)^H\}, \quad (1)$$

we can obviously find $\hat{\mathbf{x}} = (\hat{\mathbf{w}} | \hat{\mathbf{w}} \xi^{\hat{i}})$. Let us inspect the real part of the dot product in (1)

$$\text{Re}\{\mathbf{y} \cdot (\mathbf{w} | \mathbf{w} \xi^i)^H\} = \sum_{k=0}^{2^{m-1}-1} \text{Re}\{(y_k + y_{k+2^{m-1}} \xi^{-i}) w_k^*\} = \text{Re}\{\mathbf{z}(i) \cdot \mathbf{w}^H\}, \quad (2)$$

where $()^*$ denotes complex conjugation and $\mathbf{z}(i)$ is as calculated in Step 3 of Algorithm 2. By hypothesis Algorithm 2 is an ML decoder for $\text{RM}_q(1, m - 1)$, which is now used to get $\hat{\mathbf{z}}(i)$ according to

$$\hat{\mathbf{z}}(i) = \arg \max_{\mathbf{w} \in \text{RM}_q(1, m-1)} \text{Re}\{\mathbf{z}(i) \cdot \mathbf{w}^H\}. \quad (3)$$

Moreover the algorithm provides $p(i) = \text{Re}\{\mathbf{z}(i) \cdot (\hat{\mathbf{z}}(i))^H\}$. In order to find $\hat{\mathbf{w}}$, and thus also $\hat{\mathbf{x}}$, it remains to compute $\hat{i} = \arg \max_{i \in \mathbb{Z}_q} p(i)$, which is done in Step 4 of Algorithm 2. Then we have $\hat{\mathbf{w}} = \hat{\mathbf{z}}(\hat{i})$ and \hat{i} , from which $\hat{\mathbf{x}}$ can be obtained. We also have, by (2) and (3), $p(\hat{i}) = \text{Re}\{\hat{\mathbf{z}}(\hat{i}) \cdot (\hat{\mathbf{z}}(\hat{i}))^H\} = \text{Re}\{\mathbf{y} \cdot \hat{\mathbf{x}}^H\}$. ■

We close this section with a brief discussion. The code $\text{RM}_q(1, m)$ may be interpreted as a union of q cosets of the code $\{(\mathbf{w} | \mathbf{w}) \mid \mathbf{w} \in \text{RM}_q(1, m - 1)\}$ with coset representatives $\{i \cdot (\mathbf{0}_{2^{m-1}} | \mathbf{1}_{2^{m-1}}) \mid i \in \mathbb{Z}_q\}$. Then Algorithm 2 basically performs the steps of the supercode decoding principle, as stated in [3] for binary codes. We also mention the relationship of Algorithm 2 and the list-decoding scheme, proposed in [7]. In this reference it was shown that an ML decoder for $\text{RM}_2(r, m)$ can be obtained when the list length is equal to 2. The basic idea behind Algorithm 2 can therefore also be interpreted as a generalization of the list-decoding method for nonbinary codes, where the list length is set to q .

IV. COMPLEXITY ANALYSIS AND COMPARISONS

We next analyze the complexity of Algorithm 2 and make comparisons with existing approaches. In practice it is likely that q is a power of 2. So we restrict our analysis to the case $q = 2^h$. We consider the number of complex additions $N^+(2^h, m)$ and the number of complex multiplications $N^\times(2^h, m)$ as complexity measurements. Notice that we regard one real addition as half a complex addition. Multiplications with $1, j, -1, -j$ are not counted, since they are trivial and can be implemented by sign bit changes and swapping of the real and imaginary part.

We start with $m = 1$. Then, in Step 2, we need to determine the hard decisions and the dot product of two vectors of length 2. The hard decision for the k th symbol is $\hat{x}_k = \xi^{\hat{u}_k}$, where

$$\hat{u}_k = \arg \max_{i \in \mathbb{Z}_{2^h}} \operatorname{Re}\{y_k \cdot \xi^{-i}\}. \quad (4)$$

For $h \leq 2$ the above calculation is a trivial task and can be accomplished with simple sign logic. For $h > 2$ we may implement this maximum calculation as follows. First determine the quadrant in which the received value lies. This leaves $2^{h-2} + 1$ candidates, over which the maximum in (4) has to be calculated. For each candidate one complex multiplication is required. Since one of the possible signal points lies on the real and another on the imaginary axis (where the multiplication is trivial), we need for each of the two hard decisions $2^{h-2} - 1$ complex multiplications. Hence

$$N^\times(2^h, 1) = \begin{cases} 0, & h \leq 2 \\ 2^{h-1} - 2, & h > 2. \end{cases}$$

Having $\operatorname{Re}\{y_k \xi^{-\hat{u}_k}\}$ for $k = 0, 1$, one real addition is required to calculate the dot product in Step 2. Hence

$$N^+(2^h, 1) = 1/2.$$

Now let $m > 1$. In Step 3 we require $2^h 2^{m-1}$ complex multiplications to calculate $z_k(i)$ for $k = 0, \dots, 2^{m-1} - 1$ and $i = 0, \dots, 2^h - 1$. If $h \leq 2$, these are trivial multiplications. For $h > 2$, 4 out of the 2^h complex multiplications for each k are trivial. Moreover, if we have computed $y_k \xi^{-i}$ for $i = 0, 1, \dots, 2^{h-1} - 1$, by sign bit changes, the values $y_k \xi^{-i}$ for $i = 2^{h-1}, \dots, 2^h - 1$ can be easily derived. After having the products $y_k \xi^{-i}$, there are $2^h 2^{m-1}$ additions required in Step 3. These are real additions if $h = 1$ and complex additions if $h > 1$. Finally we need to perform 2^h decodings of $\operatorname{RM}_q(1, m - 1)$. Hence in total we have

$$\begin{aligned} N^\times(2^h, m) &= \begin{cases} 0, & h \leq 2 \\ (2^{h-1} - 2)2^{m-1} + 2^h N^\times(h, m - 1), & h > 2 \end{cases} \\ &= \begin{cases} 0, & h \leq 2 \\ \frac{(2^{h-1}-2)(2^{hm}-2^m)}{2^{h-2}}, & h > 2, \end{cases} \end{aligned}$$

and

$$\begin{aligned} N^+(2^h, m) &= \begin{cases} 2(2^{m-2} + N^+(1, m - 1)) & h = 1 \\ 2^h(2^{m-1} + N^+(h, m - 1)) & h > 1 \end{cases} \\ &= \begin{cases} (2m - 1)2^{m-2} & h = 1 \\ \frac{2^{h(m-1)} + 2^{h+m} - 5 \cdot 2^{hm-1}}{2^{h-2}} & h > 1. \end{cases} \end{aligned}$$

Table I lists the number of complex operations required to decode $\operatorname{RM}_q(1, m)$ for different m and $q = 2^h$. It also compares the complexity of our algorithm with the complexity of [12, Algorithm 1]. Notice that for $h = 1$ only real additions are required, and its number is equal to half the indicated value. It is apparent that Algorithm 2 has lower complexity than [12, Algorithm 1]. To be precise, our

TABLE I
NUMBER OF COMPLEX OPERATIONS REQUIRED TO DECODE $\text{RM}_q(1, m)$

		Algorithm 2		[12, Algorithm 1]	
m	q	N^+	N^\times	N^+	N^\times
4	2	28	0	32	0
	4	256	0	480	0
	8	1600	1360	5440	2720
	16	11392	28080	74880	56160
5	2	72	0	80	0
	4	1088	0	1984	0
	8	12928	10912	43648	21824
	16	182528	449376	1198336	898752
6	2	176	0	192	0
	4	4480	0	8064	0
	8	103680	87360	349440	174720
	16	2920960	7190208	19173888	14380416

algorithm requires exactly one half of the multiplications required for [12, Algorithm 1]. For $h > 1$, the ratio of the number of complex additions required for Algorithm 2 and [12, Algorithm 1] is approximately $(2^{h+1} + 2^{h-1} - 1)/2^{2h}$ and converges to this value as m approaches infinity. We finally remark that the ML decoder for $\text{RM}_4(1, m)$ in [11] requires $(m + 1)2^{2m+1}$ real additions. Hence it has higher complexity than Algorithm 2.

V. CONCLUSION

We have presented a simple maximum-likelihood decoder for the first-order generalized Reed–Muller code $\text{RM}_q(1, m)$ that enables efficient decoding of certain OFDM codes with strictly bounded PMPR. It was shown that the presented algorithm has lower complexity than previously proposed maximum-likelihood decoding schemes for $\text{RM}_q(1, m)$.

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