AN EXTREMAL PROBLEM FOR POLYNOMIALS

UN PROBLÈME EXTRÉMAL POUR LES POLYNÔMES

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Abstract. We give a solution to an extremal problem for polynomials, which asks for complex numbers $\alpha_0, \ldots, \alpha_n$ of unit magnitude that minimise the largest supremum norm on the unit circle for all polynomials of degree $n$ whose $k$-th coefficient is either $\alpha_k$ or $-\alpha_k$.

Résumé. Nous donnons dans ce papier une solution à un problème extrémal sur les polynômes qui est de trouver des nombres complexes $\alpha_0, \ldots, \alpha_n$ de module égal à 1 qui minimisent, sur le cercle unité, la plus grande borne supérieure de la norme pour tous les polynômes de degrés $n$ qui ont pour $k$-ième coefficient $\alpha_k$ ou $-\alpha_k$.

1. Results

Extremal problems for polynomials are typically of the following form. Let $f(z)$ be a polynomial with coefficients restricted to be in a subset of the complex numbers (often, this set is $\{-1, 1\}$). How well can $|f(z)|$ approximate a constant function when $z$ ranges over the unit circle? This meta problem has many variations, most of which are open (see, for example, Littlewood [8], Borwein [1], and Erdélyi [2] for surveys on selected problems). To quantify the gap between $|f(z)|$ and a constant function, different norms on the unit circle have been considered. The supremum norm

$$\|f\| = \max_{|z|=1} |f(z)|$$

has received particular attention. For example, Erdős [3, Problem 22], [4], and Littlewood [7] were interested in the minimum of $\|f_n\|$, where $f_n$ is a polynomial of degree $n - 1$ with coefficients of absolute value 1. In particular, Erdős [4] conjectured that there is some $c > 0$ such that $\|f_n\|/\sqrt{n} \geq 1+c$ for all polynomials $f_n$ of degree $n - 1$ (for every $n \geq 1$) whose coefficients have absolute value 1. Kahane [5] proved that there

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is no such \( c \), but the modified conjecture where \( f_n \) is restricted to have coefficients 1 or \(-1\) only remains open.

In this paper, we study an extremal problem in which the goal is to minimise the largest supremum norm for polynomials whose \( k \)-th coefficient is either \( \alpha_k \) or \(-\alpha_k \), where the minimisation is over the complex numbers \( \alpha_0, \alpha_1, \ldots \) satisfying \( |\alpha_0| = |\alpha_1| = \cdots = 1 \). Specifically, defining

\[
B_n(\phi_0, \ldots, \phi_{n-1}) = \max_{\epsilon_0, \ldots, \epsilon_{n-1} \in \{-1, 1\}} \left\| \sum_{k=0}^{n-1} \epsilon_k e^{2\pi i \phi_k z^k} \right\|
\]

for \( \phi_0, \ldots, \phi_{n-1} \in [0, 1] \), we are interested in the minimum

\[
b(n) = \min_{\phi_0, \ldots, \phi_{n-1} \in [0, 1]} B_n(\phi_0, \ldots, \phi_{n-1}).
\]

This minimisation problem is also related to an optimisation problem in communications engineering and, in this context, variations of the problem have been studied by Tarokh and Jafarkhani [10], Litsyn and Wunder [6], and Schmidt [9], among others.

It is easy to see that \( b(n) \leq n \) and that \( b(1) = 1 \) and \( b(2) = 2 \), but the value of \( b(n) \) is unknown for all \( n \geq 3 \). One might be tempted to conjecture that \( b(n) \) is monotonically increasing and \( b(n)/n \) is monotonically decreasing, but this also remains unknown.

Our main result is the following.

**Theorem 1.** We have

\[
\lim_{n \to \infty} \frac{b(n)}{n} = \frac{2}{\pi}.
\]

We prove Theorem 1 in Propositions 2 and 3 below. Proposition 2 establishes that \( b(n)/n > 2/\pi \) for all \( n \geq 1 \) and Proposition 3 gives explicit constructions of \( \phi_0, \ldots, \phi_{n-1} \) for which \( B_n(\phi_0, \ldots, \phi_{n-1})/n \) approaches \( 2/\pi \) arbitrarily closely.

**Proposition 2.** For each \( n \geq 1 \), we have

\[
\frac{b(n)}{n} > \frac{2}{\pi}.
\]
Proof. We have
\[
B_n(\phi_0, \ldots, \phi_{n-1}) \geq \max_{\epsilon_0, \ldots, \epsilon_{n-1} \in \{-1, 1\}} \left| \sum_{k=0}^{n-1} \epsilon_k e^{2\pi i \phi_k} \right|
\]
\[
= \max_{\epsilon_0, \ldots, \epsilon_{n-1} \in \{-1, 1\}} \max_{\psi \in [0, 1]} \Re \left( \sum_{k=0}^{n-1} \epsilon_k e^{2\pi i \phi_k} e^{2\pi i \psi} \right)
\]
\[
= \max_{\psi \in [0, 1]} \left| \sum_{k=0}^{n-1} \cos(2\pi (\phi_k + \psi)) \right|.
\]
Since the sum cannot be constant for all \(\psi\), we can further bound this expression as follows
\[
B_n(\phi_0, \ldots, \phi_{n-1}) > \int_0^1 \left| \sum_{k=0}^{n-1} \cos(2\pi (\phi_k + \psi)) \right| d\psi
\]
\[
= \sum_{k=0}^{n-1} \int_0^1 \left| \cos(2\pi \psi) \right| d\psi = \frac{2n}{\pi},
\]
as required. \(\square\)

**Proposition 3.** Let \(\alpha\) be an irrational number and write \(\phi_k = \alpha k^2\). Then
\[
\lim_{n \to \infty} \frac{B_n(\phi_0, \ldots, \phi_{n-1})}{n} = \frac{2}{\pi}.
\]

Proof. We have
\[
B_n(\phi_0, \ldots, \phi_{n-1}) = \max_{\epsilon_0, \ldots, \epsilon_{n-1} \in \{-1, 1\}} \max_{\theta, \psi \in [0, 1]} \left| \sum_{k=0}^{n-1} \epsilon_k e^{2\pi i \alpha k^2} e^{2\pi i \theta k} \right|
\]
\[
= \max_{\epsilon_0, \ldots, \epsilon_{n-1} \in \{-1, 1\}} \max_{\theta, \psi \in [0, 1]} \Re \left( \sum_{k=0}^{n-1} \epsilon_k e^{2\pi i (\alpha k^2 + \theta k + \psi)} \right)
\]
\[
= \max_{\theta, \psi \in [0, 1]} \sum_{k=0}^{n-1} \left| \cos(2\pi (\alpha k^2 + \theta k + \psi)) \right|.
\]
Since \(\alpha\) is irrational, we obtain the following consequence of a celebrated result on equi-distributed sequences modulo 1 due to Weyl [11, Satz 9]
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi im(\alpha k^2 + \theta k + \psi)} = 0 \quad \text{for every integer } m \neq 0.
\]
Moreover, the convergence is uniform for all $\theta, \psi \in \mathbb{R}$. It then follows easily [11, pp. 314–315] that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \cos(2\pi(\alpha k^2 + \theta k + \psi)) \right| = \int_0^1 |\cos 2\pi x| \, dx$$

uniformly for all $\theta, \psi \in \mathbb{R}$. Hence, from (1),

$$\lim_{n \to \infty} \frac{B_n(\phi_0, \ldots, \phi_{n-1})}{n} = \lim_{n \to \infty} \max_{\theta, \psi \in [0,1]} \frac{1}{n} \sum_{k=0}^{n-1} \left| \cos((2\pi(\alpha k^2 + \theta k + \psi)) \right|$$

$$= \max_{\theta, \psi \in [0,1]} \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \cos((2\pi(\alpha k^2 + \theta k + \psi)) \right|$$

$$= \max_{\theta, \psi \in [0,1]} \int_0^1 |\cos 2\pi x| \, dx = \frac{2}{\pi},$$

as required. \qed

REFERENCES


