AN EXTREMAL PROBLEM FOR POLYNOMIALS

UN PROBLÈME EXTRÉMAL POUR LES POLYNÔMES

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Abstract. We give a solution to an extremal problem for polynomials, which asks for complex numbers $\alpha_0, \dots, \alpha_n$ of unit magnitude that minimise the largest supremum norm on the unit circle for all polynomials of degree n whose k-th coefficient is either α_k or $-\alpha_k$.

Résumé. Nous donnons dans ce papier une solution à un problème extrémal sur les polynômes qui est de trouver des nombres complexes $\alpha_0, \dots, \alpha_n$ de module égal à 1 qui minimisent, sur le cercle unité, la plus grande borne supérieure de la norme pour tous les polynômes de degres n qui ont pour k-ième coefficient α_k ou $-\alpha_k$.

1. Results

Extremal problems for polynomials are typically of the following form. Let $f(z)$ be a polynomial with coefficients restricted to be in a subset of the complex numbers (often, this set is $\{-1, 1\}$). How well can $|f(z)|$ approximate a constant function when z ranges over the unit circle? This meta problem has many variations, most of which are open (see, for example, Littlewood [8], Borwein [1], and Erdélyi [2] for surveys on selected problems). To quantify the gap between $|f(z)|$ and a constant function, different norms on the unit circle have been considered. The supremum norm

$$\|f\| = \max_{|z|=1} |f(z)|$$

has received particular attention. For example, Erdős [3, Problem 22], [4], and Littlewood [7] were interested in the minimum of $\|f_n\|$, where f_n is a polynomial of degree $n-1$ with coefficients of absolute value 1. In particular, Erdős [4] conjectured that there is some $c > 0$ such that $\|f_n\|/\sqrt{n} \geq 1+c$ for all polynomials f_n of degree $n-1$ (for every $n \geq 1$) whose coefficients have absolute value 1. Kahane [5] proved that there

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is no such c, but the modified conjecture where f_n is restricted to have coefficients 1 or -1 only remains open.

In this paper, we study an extremal problem in which the goal is to minimise the largest supremum norm for polynomials whose k-th coefficient is either α_k or $-\alpha_k$, where the minimisation is over the complex numbers $\alpha_0, \alpha_1, \dots$ satisfying $|\alpha_0| = |\alpha_1| = \cdots = 1$. Specifically, defining

$$B_n(\phi_0, \dots, \phi_{n-1}) = \max_{\epsilon_0, \dots, \epsilon_{n-1} \in \{-1,1\}} \left\| \sum_{k=0}^{n-1} \epsilon_k e^{2\pi i \phi_k} z^k \right\|$$

for $\phi_0, \dots, \phi_{n-1} \in [0,1]$, we are interested in the minimum

$$b(n) = \min_{\phi_0, \dots, \phi_{n-1} \in [0,1]} B_n(\phi_0, \dots, \phi_{n-1}).$$

This minimisation problem is also related to an optimisation problem in communications engineering and, in this context, variations of the problem have been studied by Tarokh and Jafarkhani [10], Litsyn and Wunder [6], and Schmidt [9], among others.

It is easy to see that $b(n) \le n$ and that $b(1) = 1$ and $b(2) = 2$, but the value of $b(n)$ is unknown for all $n \ge 3$. One might be tempted to conjecture that $b(n)$ is monotonically increasing and $b(n)/n$ is monotonically decreasing, but this also remains unknown.

Our main result is the following.

Theorem 1. *We have*

$$\lim_{n \to \infty} \frac{b(n)}{n} = \frac{2}{\pi}.$$

We prove Theorem 1 in Propositions 2 and 3 below. Proposition 2 establishes that $b(n)/n > 2/\pi$ for all $n \ge 1$ and Proposition 3 gives explicit constructions of $\phi_0, \dots, \phi_{n-1}$ for which $B_n(\phi_0, \dots, \phi_{n-1})/n$ approaches $2/\pi$ arbitrarily closely.

Proposition 2. *For each $n \ge 1$, we have*

$$\frac{b(n)}{n} > \frac{2}{\pi}.$$

Proof. We have

$$\begin{aligned}
B_n(\phi_0,\dots,\phi_{n-1}) &\geq \max_{\epsilon_0,\dots,\epsilon_{n-1}\in\{-1,1\}} \left|\sum_{k=0}^{n-1}\epsilon_k e^{2\pi i\phi_k}\right| \\
&= \max_{\epsilon_0,\dots,\epsilon_{n-1}\in\{-1,1\}} \max_{\psi\in[0,1]} \operatorname{Re}\left(\sum_{k=0}^{n-1}\epsilon_k e^{2\pi i\phi_k} e^{2\pi i\psi}\right) \\
&= \max_{\psi\in[0,1]} \sum_{k=0}^{n-1} \big|\cos(2\pi(\phi_k+\psi))\big|.
\end{aligned}$$

Since the sum cannot be constant for all ψ, we can further bound this expression as follows

$$\begin{aligned}
B_n(\phi_0,\dots,\phi_{n-1}) &> \int_0^1 \sum_{k=0}^{n-1} \big|\cos(2\pi(\phi_k+\psi))\big|\,d\psi \\
&= \sum_{k=0}^{n-1} \int_0^1 \big|\cos(2\pi\psi)\big|\,d\psi = \frac{2n}{\pi},
\end{aligned}$$

as required. □

Proposition 3. *Let α be an irrational number and write $\phi_k = \alpha k^2$. Then*

$$\lim_{n\to\infty}\frac{B_n(\phi_0,\dots,\phi_{n-1})}{n} = \frac{2}{\pi}.$$

Proof. We have

$$\begin{aligned}
B_n(\phi_0,\dots,\phi_{n-1}) &= \max_{\epsilon_0,\dots,\epsilon_{n-1}\in\{-1,1\}} \max_{\theta\in[0,1]} \left|\sum_{k=0}^{n-1}\epsilon_k e^{2\pi i\alpha k^2} e^{2\pi i\theta k}\right| \\
&= \max_{\epsilon_0,\dots,\epsilon_{n-1}\in\{-1,1\}} \max_{\theta,\psi\in[0,1]} \operatorname{Re}\left(\sum_{k=0}^{n-1}\epsilon_k e^{2\pi i(\alpha k^2+\theta k+\psi)}\right) \\
&= \max_{\theta,\psi\in[0,1]} \sum_{k=0}^{n-1} \big|\cos(2\pi(\alpha k^2+\theta k+\psi))\big|. \qquad (1)
\end{aligned}$$

Since α is irrational, we obtain the following consequence of a celebrated result on equi-distributed sequences modulo 1 due to Weyl [11, Satz 9]

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi i m(\alpha k^2+\theta k+\psi)} = 0 \quad \text{for every integer } m\neq 0.$$

Moreover, the convergence is uniform for all $\theta, \psi \in \mathbb{R}$. It then follows easily [11, pp. 314–315] that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \big|\cos(2\pi(\alpha k^2 + \theta k + \psi))\big| = \int_0^1 |\cos 2\pi x|\, dx$$

uniformly for all $\theta, \psi \in \mathbb{R}$. Hence, from (1),

$$\begin{aligned}
\lim_{n\to\infty} \frac{B_n(\phi_0, \dots, \phi_{n-1})}{n} &= \lim_{n\to\infty} \max_{\theta,\psi\in[0,1]} \frac{1}{n} \sum_{k=0}^{n-1} \big|\cos((2\pi(\alpha k^2 + \theta k + \psi))\big| \\
&= \max_{\theta,\psi\in[0,1]} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \big|\cos((2\pi(\alpha k^2 + \theta k + \psi))\big| \\
&= \max_{\theta,\psi\in[0,1]} \int_0^1 |\cos 2\pi x|\, dx = \frac{2}{\pi},
\end{aligned}$$

as required. □

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