

ON THE CLASSIFICATION OF HYPEROVALS

FLORIAN CAULLERY AND KAI-UWE SCHMIDT

ABSTRACT. A hyperoval in the projective plane $\mathbb{P}^2(\mathbb{F}_q)$ is a set of $q+2$ points no three of which are collinear. Hyperovals have been studied extensively since the 1950s with the ultimate goal of establishing a complete classification. It is well known that hyperovals in $\mathbb{P}^2(\mathbb{F}_q)$ are in one-to-one correspondence to polynomials with certain properties, called o-polynomials of \mathbb{F}_q . We classify o-polynomials of \mathbb{F}_q of degree less than $\frac{1}{2}q^{1/4}$. As a corollary we obtain a complete classification of exceptional o-polynomials, namely polynomials over \mathbb{F}_q that are o-polynomials of infinitely many extensions of \mathbb{F}_q .

1. INTRODUCTION AND RESULTS

An *arc* in the projective plane $\mathbb{P}^2(\mathbb{F}_q)$ is a set of points of $\mathbb{P}^2(\mathbb{F}_q)$ no three of which are collinear. It is well known that the maximum number of points in an arc in $\mathbb{P}^2(\mathbb{F}_q)$ is $q+1$ for odd q and $q+2$ for even q (see [10, Chapter 8], for example). Accordingly, an arc of size $q+1$ is called an *oval* and an arc of size $q+2$ is called a *hyperoval*. A theorem due to Segre [17] asserts that, in the case of odd q , every oval in $\mathbb{P}^2(\mathbb{F}_q)$ is a conic. This at once classifies ovals in $\mathbb{P}^2(\mathbb{F}_q)$ for odd q . In contrast, the classification of hyperovals is a major open problem in finite geometry, which has attracted sustained interest over the last sixty years. For surveys on progress toward this classification we refer to [2], [10], and [15], for example.

Throughout this paper we let q be a power of two. Hyperovals have a canonical description via polynomials over \mathbb{F}_q .

Definition 1.1. An *o-polynomial* of \mathbb{F}_q is a polynomial $f \in \mathbb{F}_q[x]$ of degree at most $q-1$ that induces a permutation on \mathbb{F}_q satisfying $f(0) = 0$ and $f(1) = 1$ and

$$(1) \quad \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ f(a) & f(b) & f(c) \end{pmatrix} \neq 0 \quad \text{for all distinct } a, b, c \in \mathbb{F}_q.$$

By a suitable choice of coordinates, we may assume without loss of generality that the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ are contained in a hyperoval. It is well known (and easily verified) that every such hyperoval

Date: 11 March 2014 (revised 30 June 2015).

2010 Mathematics Subject Classification. Primary: 05B25; Secondary: 11T06, 51E20.

in $\mathbb{P}^2(\mathbb{F}_q)$ can be written as

$$(2) \quad \{(f(c), c, 1) : c \in \mathbb{F}_q\} \cup \{(1, 0, 0), (0, 1, 0)\},$$

where f is an o-polynomial of \mathbb{F}_q . Conversely, if f is an o-polynomial of \mathbb{F}_q , then (2) is a hyperoval in $\mathbb{P}^2(\mathbb{F}_q)$.

For example, $f(x) = x^2$ is an o-polynomial of \mathbb{F}_{2^h} for all $h > 1$. There exist several other infinite families of o-polynomials and some sporadic examples. For a list of known hyperovals, as of 2003, we refer to [15]. Since 2003, no new hyperovals have been found.

O-polynomials of \mathbb{F}_{2^h} have been classified for $h \leq 5$ [7], [13], [16] and monomial o-polynomials of \mathbb{F}_{2^h} have been classified for $h \leq 30$ [6]. There is also a classification of monomial o-polynomials of a certain form, namely those of degree $2^i + 2^j$ [3] or $2^i + 2^j + 2^k$ [19]. O-polynomials of degree at most 6 are also classified [10, Theorem 8.31].

Our main result is the following classification of low-degree o-polynomials. Call two polynomials $f, g \in \mathbb{F}_q[x]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$ *equivalent* if there exists an $a \in \mathbb{F}_q$ such that

$$g(x) = \frac{f(x+a) + f(a)}{f(1+a) + f(a)}.$$

It is readily verified that this equivalence indeed defines an equivalence relation and that it preserves the property of being an o-polynomial of \mathbb{F}_q .

Theorem 1.2. *If f is an o-polynomial of \mathbb{F}_q of degree less than $\frac{1}{2}q^{1/4}$, then f is equivalent to either x^6 or x^{2^k} for a positive integer k .*

It is well known that x^6 is an o-polynomial of \mathbb{F}_{2^h} if and only if h is odd and that x^{2^k} is an o-polynomial of \mathbb{F}_{2^h} if and only if k and h are coprime.

Now consider polynomials $f \in \mathbb{F}_q[x]$ with the property that f is an o-polynomial of \mathbb{F}_{q^r} for infinitely many r ; we call such a polynomial an *exceptional* o-polynomial of \mathbb{F}_q . Exceptional o-polynomials provide a uniform construction for hyperovals in infinitely many projective planes. Theorem 1.2 gives a complete classification of exceptional o-polynomials.

Corollary 1.3. *If f is an exceptional o-polynomial of \mathbb{F}_q , then f is equivalent to either x^6 or x^{2^k} for a positive integer k .*

The specialisation of Corollary 1.3 to the case that f is a monomial was conjectured by Segre and Bartocci [18] and was recently proved by Hernando and McGuire [8] (another, much simpler, proof of this case was later given by Zieve [20]).

2. PROOF OF THEOREM 1.2

We begin with recalling several standard results, for which proofs can be found in [10, Chapter 8], for example. Our first result is an almost immediate consequence of the definition of an o-polynomial.

Lemma 2.1 ([10, Corollary 8.23]). *Every o-polynomial of \mathbb{F}_q with $q > 2$ has only terms of positive even degree.*

We need the following (easy) classification of o-polynomials of degree 6.

Lemma 2.2 ([10, Theorem 8.31]). *If f is an o-polynomial of degree 6, then f is equivalent to x^6 .*

We also need the following result, originally proved by Payne [14] and later by Hirschfeld [9] with a different method, classifying translation hyperovals.

Lemma 2.3 ([10, Theorem 8.41]). *Every o-polynomial, in which the degree of every term is a power of two, is in fact a monomial.*

Now let $f \in \mathbb{F}_q[x]$ and define the polynomial

$$\begin{aligned} \Phi_f(x, y, z) &= \frac{1}{(x+y)(x+z)(y+z)} \cdot \det \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ f(x) & f(y) & f(z) \end{pmatrix} \\ &= \frac{x(f(y) + f(z)) + y(f(x) + f(z)) + z(f(x) + f(y))}{(x+y)(x+z)(y+z)}. \end{aligned}$$

The condition (1) is equivalent to the condition that all points in $\mathbb{A}^3(\mathbb{F}_q)$ of the surface defined by

$$\Phi_f(x, y, z) = 0$$

satisfy $x = y$, $x = z$, or $y = z$. This leads us to the following result, which essentially follows from a refinement of the Lang-Weil bound [11] for the number of \mathbb{F}_q -rational points in algebraic varieties.

Proposition 2.4. *Let $f \in \mathbb{F}_q[x]$ be of degree less than $\frac{1}{2}q^{1/4}$. If Φ_f has an absolutely irreducible factor over \mathbb{F}_q , then f is not an o-polynomial of \mathbb{F}_q .*

Proof. If f has degree 0 or 1, then f is not an o-polynomial by Lemma 2.1, so assume that f has degree at least 2. We first show that Φ_f is not divisible by $x + y$, $x + z$, or $y + z$. Suppose, for a contradiction, that Φ_f is divisible by $x + y$. Then the partial derivative of

$$x(f(y) + f(z)) + y(f(x) + f(z)) + z(f(x) + f(y))$$

with respect to x is divisible by $x + y$, or equivalently,

$$f(y) + f(z) + (y + z)f'(y) = 0.$$

This forces the degree of f to be 0 or 1, contradicting our assumption. Hence, by symmetry, Φ_f is not divisible by $x + y$, $x + z$, or $y + z$. Therefore, $\Phi_f(x, y, x)$, $\Phi_f(x, y, y)$, and $\Phi_f(x, x, z)$ are nonzero polynomials, and so each has at most dq zeros in $\mathbb{A}^2(\mathbb{F}_q)$, where d is the degree of Φ_f (see [12, Theorem 6.13], for example).

Now suppose that Φ_f has an absolutely irreducible factor over \mathbb{F}_q . Then, by a refinement of the Lang-Weil bound [11] due to Ghorpade and Lachaud [5,

11.3], the number of points in $\mathbb{A}^3(\mathbb{F}_q)$ of the surface defined by $\Phi_f(x, y, z) = 0$ is at least

$$q^2 - (d-1)(d-2)q^{3/2} - 12(d+3)^4 q.$$

Hence the number of such points that are not on one of the planes $x = y$, $x = z$, or $y = z$ is at least

$$q^2 - (d-1)(d-2)q^{3/2} - 12(d+3)^4 q - 3dq,$$

which is positive since

$$0 \leq d \leq \frac{1}{2}q^{1/4} - 3.$$

Then our remarks preceding the proposition imply that f is not an o-polynomial of \mathbb{F}_q . \square

In order to prove Theorem 1.2, we first use the constraints given by Lemmas 2.1, 2.2, and 2.3 and then show that in all remaining cases, Φ_f has an absolutely irreducible factor over \mathbb{F}_q unless f is one of the polynomials in Theorem 1.2. To do so, we frequently use the polynomials

$$(3) \quad \phi_j(x, y, z) = \frac{x(y^j + z^j) + y(x^j + z^j) + z(x^j + y^j)}{(x+y)(x+z)(y+z)}.$$

Then, writing

$$f(x) = \sum_{i=0}^d a_i x^i,$$

we have

$$\Phi_f(x, y, z) = \sum_{i=0}^d a_i \phi_i(x, y, z).$$

If j is an even positive integer, not equal to 6 or a power of two, then ϕ_j has an absolutely irreducible factor over \mathbb{F}_2 (and so proves Corollary 1.3 in the case that f is a monomial). This was conjectured by Segre and Bartocci [18] and proved by Hernando and McGuire [8] (and can also be deduced with a few extra steps from an argument due to Zieve [20, Section 5]).

Lemma 2.5 ([8, Theorem 8]). *Let j be an even positive integer, not equal to 6 or a power of two. Then ϕ_j has an absolutely irreducible factor over \mathbb{F}_2 .*

If f is an o-polynomial of \mathbb{F}_q , then either $q = 2$ and f has degree 1 or $q > 2$ and f has positive even degree by Lemma 2.1. Hence to prove Theorem 1.2, we can assume that f has positive even degree. In the case that f has positive even degree that is neither 6 nor a power of two, we show that Φ_f has an absolutely irreducible factor over \mathbb{F}_q , and using Proposition 2.4 prove the statement of Theorem 1.2 in this case.

Proposition 2.6. *Let $f \in \mathbb{F}_q[x]$ be of positive even degree not equal to 6 or a power of two. Then Φ_f has an absolutely irreducible factor over \mathbb{F}_q .*

Proposition 2.6 will follow from Lemma 2.5 and the following simple observation due to Aubry, McGuire, and Rodier [1] (in which $\overline{\mathbb{F}_q}$ is the algebraic closure of \mathbb{F}_q).

Lemma 2.7 ([1, Lemma 2.1]). *Let S and P be projective surfaces in $\mathbb{P}^3(\overline{\mathbb{F}}_q)$ defined over \mathbb{F}_q . If $S \cap P$ has a reduced absolutely irreducible component defined over \mathbb{F}_q , then S has an absolutely irreducible component defined over \mathbb{F}_q .*

Proof of Proposition 2.6. Write

$$f(x) = \sum_{i=0}^d a_i x^i,$$

where $a_d \neq 0$, and consider the homogenisation of Φ_f , namely

$$\tilde{\Phi}_f(w, x, y, z) = \sum_{i=0}^d a_i \phi_i(x, y, z) w^{d-i}.$$

The intersection of the projective surface defined by $\tilde{\Phi}_f(w, x, y, z) = 0$ with the plane defined by $w = 0$ is the projective curve defined by $\phi_d(x, y, z) = 0$ and $w = 0$. By Lemma 2.5, ϕ_d has an absolutely irreducible factor over \mathbb{F}_q . Notice that ϕ_d is square-free, which follows from the fact that the partial derivative of

$$x(y^d + z^d) + y(x^d + z^d) + z(x^d + y^d)$$

with respect to x is in $\mathbb{F}_2[y, z]$ (using that d is even) and from symmetry. Therefore, Lemma 2.7 implies that $\tilde{\Phi}_f$ (and therefore Φ_f) has an absolutely irreducible factor over \mathbb{F}_q . \square

In view of Proposition 2.6 and Lemma 2.2, it remains to prove Theorem 1.2 when the degree of f is a power of two. In view of Lemmas 2.1 and 2.3, this case follows from Proposition 2.4 and the following result.

Proposition 2.8. *Let k be an integer satisfying $k \geq 2$ and let $f \in \mathbb{F}_q[x]$ be a polynomial of the form*

$$f(x) = \sum_{i=1}^{2^k-1} a_{2i} x^{2i}$$

such that $a_{2^k} \neq 0$ and such that the degree of at least one term in f is not a power of two. Then Φ_f is absolutely irreducible.

To prove Proposition 2.8, we use the following corollary to Lucas's theorem (see [4], for example).

Lemma 2.9. *The binomial coefficient $\binom{n}{m}$ is even if and only if at least one of the base-2 digits of m is greater than the corresponding digit of n .*

Proof of Proposition 2.8. Suppose, for a contradiction, that Φ_f is not absolutely irreducible. Let ϕ_j be defined by (3). Our proof relies on the following claim.

Claim. *There exists $\theta \in \mathbb{F}_{2^k} - \mathbb{F}_2$ such that for all $i \in \{1, 2, \dots, 2^{k-1}\}$, we have*

$$a_{2i} = 0 \quad \text{or} \quad x + z + \theta(y + z) \text{ divides } \phi_{2i}(x, y, z).$$

We defer the proof of the claim and first deduce the statement in the proposition from the claim. Let n be an even integer such that a_n is nonzero. By putting $x = \theta y + (\theta + 1)z$ into

$$(x + y)(x + z)(y + z)\phi_n(x, y, z),$$

we see from the claim that

$$yz^n + zy^n + (y^n + z^n)(\theta y + (\theta + 1)z) + (y + z)(\theta y + (\theta + 1)z)^n = 0,$$

which implies that

$$(\theta + \theta^n)y^n + ((\theta + 1) + (\theta + 1)^n)z^n + \sum_{m=1}^{n-1} \binom{n}{m} \theta^m y^m (\theta + 1)^{n-m} z^{n-m} = 0.$$

Comparing coefficients, we find that $\binom{n}{m}$ is even for each $m \in \{1, \dots, n-1\}$. It is then readily verified that Lemma 2.9 implies that n must be a power of two. Therefore, the degree of every term in f is a power of two, contradicting our assumption. Hence Φ_f is absolutely irreducible.

To prove the claim, we repeatedly use the identity

$$(4) \quad \phi_{2^i}(x, y, x) = \left(\frac{x^i + y^i}{x + y} \right)^2 \quad \text{for each } i \geq 1,$$

which is elementary to verify. We also use the expansion

$$\Phi_f = a_2\phi_2 + a_4\phi_4 + \dots + a_{2^k}\phi_{2^k}.$$

Since Φ_f is not absolutely irreducible by assumption, we may write

$$(5) \quad a_2\phi_2 + a_4\phi_4 + \dots + a_{2^k}\phi_{2^k} = (P_s + P_{s-1} + \dots + P_0)(Q_t + Q_{t-1} + \dots + Q_0),$$

where P_i and Q_i are zero or homogeneous polynomials of degree i , defined over the algebraic closure of \mathbb{F}_q , and $s, t > 0$ and $P_s Q_t$ is nonzero. Without loss of generality we may also assume that $s \leq t$. We have

$$(6) \quad \begin{aligned} \phi_{2^k}(x, y, z) &= \frac{(x + z)^{2^k - 1} + (y + z)^{2^k - 1}}{x + y} \\ &= \prod_{\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2} (x + z + \alpha(y + z)). \end{aligned}$$

Since $a_{2^k}\phi_{2^k} = P_s Q_t$ by (5), we find from (6) that P_s and Q_t are coprime and from (4) that

$$(7) \quad P_s(x, y, x)Q_t(x, y, x) = a_{2^k}(x + y)^{2^k - 2}.$$

From (5) we have

$$0 = P_s Q_{t-1} + P_{s-1} Q_t.$$

Since P_s and Q_t are coprime, we find that $P_s \mid P_{s-1}$, thus $P_{s-1} = 0$ by a degree argument. Similarly we have $Q_{t-1} = 0$. Let I be the smallest positive integer i such that $a_{2^k - 2^i}$ is nonzero (this I exists and satisfies $I < 2^k - 1$ by

our assumed form of f). With a simple induction, involving the preceding argument, we conclude that

$$(8) \quad P_{s-1} = \cdots = P_{s-2I+1} = 0.$$

In the next step we have from (5) that

$$a_{2^k-2I}\phi_{2^k-2I} = P_s Q_{t-2I} + P_{s-2I} Q_t,$$

which using (7) gives

$$(9) \quad a_{2^k-2I}\phi_{2^k-2I}(x, y, x) = \beta a_{2^k}(x+y)^s Q_{t-2I}(x, y, x) + \beta^{-1}(x+y)^t P_{s-2I}(x, y, x)$$

for some nonzero β in the algebraic closure of \mathbb{F}_q . Write $I = 2^\ell e$ for some nonnegative integer ℓ and some positive odd integer e . Using (4), we have

$$\phi_{2^k-2I}(x, y, x) = \left(\frac{(x^{2^{k-\ell-1}-e} + y^{2^{k-\ell-1}-e})^{2^\ell}}{x+y} \right)^2.$$

Since $2^{k-\ell-1} - e$ is odd, the polynomial

$$x^{2^{k-\ell-1}-e} + y^{2^{k-\ell-1}-e}$$

splits into distinct factors, and therefore the largest power of $x+y$ dividing $\phi_{2^k-2I}(x, y, x)$ is at most $2(2^\ell - 1)$. Hence, since $a_{2^k-2I} \neq 0$ and $s \leq t$ by assumption, we have in view of (9) that

$$s \leq 2(2^\ell - 1) \leq 2(I - 1).$$

Therefore, we find from (8) that $P_i = 0$ unless $i = s$ and then from (5) that

$$a_{2^k-2j}\phi_{2^k-2j} = P_s Q_{t-2j} \quad \text{for each } j \in \{0, 1, \dots, 2^{k-1} - 1\}.$$

This shows that P_s divides $a_{2^k-2j}\phi_{2^k-2j}$ for each $j \in \{0, 1, \dots, 2^{k-1} - 1\}$, which in view of $a_{2^k}\phi_{2^k} = P_s Q_t$ and (6) proves our claim. \square

REFERENCES

- [1] Y. Aubry, G. McGuire, and F. Rodier, *A few more functions that are not APN infinitely often*, Finite fields: theory and applications, Contemp. Math., vol. 518, Amer. Math. Soc., Providence, RI, 2010, pp. 23–31.
- [2] W. Cherowitzo, *Hyperovals in Desarguesian planes: an update*, Discrete Math. **155** (1996), no. 1-3, 31–38.
- [3] W. E. Cherowitzo and L. Storme, *α -flocks with oval herds and monomial hyperovals*, Finite Fields Appl. **4** (1998), no. 2, 185–199.
- [4] N. J. Fine, *Binomial coefficients modulo a prime*, Amer. Math. Monthly **54** (1947), 589–592.
- [5] S. R. Ghorpade and G. Lachaud, *Étale cohomology, Lefschetz theorems and number of points of singular varieties over finite fields*, Mosc. Math. J. **2** (2002), no. 3, 589–631.
- [6] D. G. Glynn, *A condition for the existence of ovals in $\text{PG}(2, q)$, q even*, Geom. Dedicata **32** (1989), no. 2, 247–252.
- [7] M. Hall, Jr., *Ovals in the Desarguesian plane of order 16*, Ann. Mat. Pura Appl. (4) **102** (1975), 159–176.

- [8] F. Hernando and G. McGuire, *Proof of a conjecture of Segre and Bartocci on monomial hyperovals in projective planes*, Des. Codes Cryptogr. **65** (2012), no. 3, 275–289.
- [9] J. W. P. Hirschfeld, *Ovals in desarguesian planes of even order*, Ann. Mat. Pura Appl. **102** (1975), 79–89.
- [10] ———, *Projective geometries over finite fields*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1998.
- [11] S. Lang and A. Weil, *Number of points of varieties in finite fields*, Amer. J. Math. **76** (1954), 819–827.
- [12] R. Lidl and H. Niederreiter, *Finite fields*, second ed., Encyclopedia of Mathematics and its Applications, vol. 20, Cambridge University Press, Cambridge, 1997.
- [13] C. M. O’Keefe and T. Penttila, *Hyperovals in PG(2, 16)*, European J. Combin. **12** (1991), no. 1, 51–59.
- [14] S. E. Payne, *A complete determination of translation ovoids in finite Desarguanian planes*, Atti Accad. Naz. Lincei **51** (1971), 328–331.
- [15] T. Penttila, *Configurations of ovals*, J. Geom. **76** (2003), no. 1-2, 233–255.
- [16] T. Penttila and G. F. Royle, *Classification of hyperovals in PG(2, 32)*, J. Geom. **50** (1994), no. 1-2, 151–158.
- [17] B. Segre, *Ovals in a finite projective plane*, Canad. J. Math. **7** (1955), 414–416.
- [18] B. Segre and U. Bartocci, *Ovali ed altre curve nei piani di Galois di caratteristica due*, Acta Arith. **18** (1971), 423–449.
- [19] T. L. Vis, *Monomial hyperovals in Desarguesian planes*, ProQuest LLC, Ann Arbor, MI, 2010, Thesis (Ph.D.)—University of Colorado at Denver.
- [20] M. E. Zieve, *Planar functions and perfect nonlinear monomials over finite fields*, Des. Codes Cryptogr. **75** (2015), no. 1, 71–80.

INSTITUT DE MATHÉMATIQUES DE LUMINY, CNRS-UPR9016, 163 AV. DE LUMINY,
CASE 907, 13288 MARSEILLE CEDEX 9, FRANCE.

E-mail address, F. Caullery: florian.caullery@etu.univ-amu.fr

FACULTY OF MATHEMATICS, OTTO-VON-GUERICKE UNIVERSITY, UNIVERSITÄTSPLATZ 2,
39106 MAGDEBURG, GERMANY.

E-mail address, K.-U. Schmidt: kaiuwe.schmidt@ovgu.de