ON A PROBLEM DUE TO LITTLEWOOD CONCERNING POLYNOMIALS WITH UNIMODULAR COEFFICIENTS

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Abstract. Littlewood raised the question of how slowly $\|f_n\|_4^4 - \|f_n\|_2^4$ (where $\|.\|_r$ denotes the L^r norm on the unit circle) can grow for a sequence of polynomials f_n with unimodular coefficients and increasing degree. The results of this paper are the following. For

$$g_n(z) = \sum_{k=0}^{n-1} e^{\pi i k^2/n}\, z^k$$

the limit of $(\|g_n\|_4^4 - \|g_n\|_2^4)/\|g_n\|_2^3$ is $2/\pi$, which resolves a mystery due to Littlewood. This is however not the best answer to Littlewood's question: for the polynomials

$$h_n(z) = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} e^{2\pi i jk/n}\, z^{nj+k}$$

the limit of $(\|h_n\|_4^4 - \|h_n\|_2^4)/\|h_n\|_2^3$ is shown to be $4/\pi^2$. No sequence of polynomials with unimodular coefficients is known that gives a better answer to Littlewood's question. It is an open question as to whether such a sequence of polynomials exists.

1. Introduction

For real $r \geq 1$, the L^r norm of a polynomial $f \in \mathbb{C}[z]$ on the unit circle is

$$\|f\|_r = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(e^{i\theta})\right|^r d\theta\right)^{1/r}.$$

There is sustained interest in the L^r norm of polynomials with restricted coefficients (see, for example, Littlewood [14], Borwein [2], and Erdélyi [5] for surveys on selected problems). Littlewood raised the question of how slowly $\|f_n\|_4^4 - \|f_n\|_2^4$ can grow for a sequence of polynomials f_n with restricted coefficients and increasing degree. This problem is also of interest in the theory of communications, because $\|f\|_4^4$ equals the sum of squares of the aperiodic autocorrelations of the sequence formed from the coefficients of f [2, p. 122]; in this context one considers the *merit factor* $\|f\|_2^4/(\|f\|_4^4 - \|f\|_2^4)$. Much work on Littlewood's question has been done when the coefficients are -1 or 1; see [8] for recent advances. In the situation where the coefficients are

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restricted to have unit magnitude, the polynomials

$$g_n(z) = \sum_{k=0}^{n-1} e^{\pi i k^2/n}\, z^k \quad \text{for integral } n \ge 1$$

are of particular interest [11], [12], [13], [14].[1] These polynomials are also the main ingredient in Kahane's celebrated semi-probabilistic construction of ultra-flat polynomials [9], which disproves a conjecture due to Erdős [6]. Write

$$\alpha_n = \frac{\|g_n\|_4^4 - \|g_n\|_2^4}{\|g_n\|_2^3}$$

(note that $\|f\|_2 = \sqrt{n}$ for every polynomial f of degree $n-1$ with unimodular coefficients). Based on the work in [11] and [12] and calculations carried out by Swinnerton-Dyer, Littlewood concluded in [13] that

$$\lim_{n\to\infty} \alpha_n = \sqrt{2} - \frac{2}{\pi}(\sqrt{2}-1) = 1.15051\dots, \tag{1}$$

but expressed doubt in his own conclusion. He knew that

$$0.604 \le \alpha_n \le 0.656 \quad \text{for } 18 \le n \le 41 \tag{2}$$

and noted [13, Appendix] "There is a considerable mystery here. I have checked my calculations at least six times, and they have been checked also in great detail by Dr. Flett." Littlewood raised this issue again in his book [14, p. 27] and asked for a resolution of this puzzle.

Borwein and Choi [3] conjectured

$$\|g_n\|_4^4 = n^2 + \frac{2}{\pi} n^{3/2} + \delta_n n^{1/2} + O(n^{-1/2}),$$

where $\delta_n = -2$ for $n \equiv 0, 1 \pmod 4$ and $\delta_n = 1$ for $n \equiv 2, 3 \pmod 4$ (this was not stated explicitly as a conjecture in [3], but was confirmed by the authors [4] to be a tentative conclusion based on numerical evidence). This conjecture implies in particular

$$\lim_{n\to\infty} \alpha_n = \frac{2}{\pi} = 0.63661\dots. \tag{3}$$

Independently, Antweiler and Bömer [1] made observations similar to (2), while Stańczak and Boche [17] and Mercer [15] derived bounds for α_n. In particular, Mercer [15] showed that

$$\limsup_{n\to\infty} \alpha_n < \frac{16}{3\pi^{3/2}} = 0.95779\dots,$$

and thereby confirming Littlewood's suspicion (although Mercer was apparently unaware of Littlewood's work).

We shall resolve Littlewood's puzzle by proving that (1) is incorrect and the conjecture (3) is true.

[1] Some authors consider $g_n(e^{\pm\pi i/n} z)$, which however has the same L^r norm as $g_n(z)$.

Theorem 1. *We have*

$$\lim_{n\to\infty}\frac{\|g_n\|_4^4-\|g_n\|_2^4}{\|g_n\|_2^3}=\frac{2}{\pi}.$$

We shall also show that this is not the best possible answer to Littlewood's question. To do so, we consider the polynomials

$$h_n(z)=\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}e^{2\pi ijk/n}\,z^{nj+k}\quad\text{for integral } n\ge 1$$

of degree n^2-1, which have been studied by Turyn [18], among others.

Theorem 2. *We have*

$$\lim_{n\to\infty}\frac{\|h_n\|_4^4-\|h_n\|_2^4}{\|h_n\|_2^3}=\frac{4}{\pi^2}.$$

This is the best known answer to Littlewood's question: there is no sequence of polynomials f_n with unimodular coefficients for which the limit of $(\|f_n\|_4^4-\|f_n\|_2^4)/\|f_n\|_2^3$ is known to be less than $4/\pi^2$. It is an open question as to whether such a sequence of polynomials exists.

In the radar literature [10, Ch. 6], the sequences formed from the coefficients of g_n and h_n are called *Chu* and *Frank sequences*, respectively. Our results show that their merit factors grow like $(\pi/2)\sqrt{n}$ and $(\pi^2/4)\sqrt{n}$, respectively, which explains numerical results reported in [1].

2. Proof of Theorem 1

We begin with summarising known results (see [13, p. 371], for example). For a polynomial $f\in\mathbb{C}[z]$ with $f(z)=\sum_{k=0}^{d-1}a_kz^k$, we readily verify that

$$f(z)\overline{f(z^{-1})}=\sum_{u=-(d-1)}^{d-1}c_uz^u,$$

where

$$c_u=\sum_{0\le j,j+u<d}a_j\overline{a_{j+u}}.\tag{4}$$

The numbers c_u satisfy $c_u=\overline{c_{-u}}$. Hence

$$\|f\|_4^4=\frac{1}{2\pi}\int_0^{2\pi}\Big(f(e^{i\theta})\overline{f(e^{i\theta})}\Big)^2\,d\theta=c_0^2+2\sum_{u=1}^{d-1}|c_u|^2.\tag{5}$$

Lemma 3. *For each $n\ge 1$, we have*

$$\|g_n\|_4^4=n^2-\epsilon_n+4\sum_{1\le u\le n/2}\left(\frac{\sin(\pi u^2/n)}{\sin(\pi u/n)}\right)^2,\tag{6}$$

where $\epsilon_n=2$ for $n\equiv 2\pmod 4$ and $\epsilon_n=0$ otherwise.

Proof. For $f = g_n$, elementary manipulations reveal that the numbers c_u in (4) satisfy

$$|c_u| = \left|\frac{\sin(\pi u^2/n)}{\sin(\pi u/n)}\right|$$

for $1 \le u \le n-1$. The desired result then follows from (5) after noting that $c_0 = n$ and $|c_u| = |c_{n-u}|$ for $1 \le u \le n-1$ and $2|c_{n/2}| = \epsilon_n$ for even n. $\square$

We now prove Theorem 1 by finding an asymptotic evaluation of the sum on the right hand side of (6).

Let x be a real number satisfying $0 < x \le \pi/2$. From the inequality $x - x^3/6 \le \sin x \le x$ we see that

$$0 < \frac{1}{(\sin x)^2} - \frac{1}{x^2} < 1,$$

and therefore

$$\left|\sum_{1\le u\le n/2}\left(\frac{\sin(\pi u^2/n)}{\sin(\pi u/n)}\right)^2 - \sum_{1\le u\le n/2}\left(\frac{\sin(\pi u^2/n)}{\pi u/n}\right)^2\right| < \frac{n}{2}.$$

Thus, defining the function $r : \mathbb{R} \to \mathbb{R}$ by

$$r(x) = \left(\frac{\sin(\pi x^2/n)}{\pi x/n}\right)^2,$$

the theorem is proved by showing that

$$\lim_{n\to\infty}\frac{1}{n^{3/2}}\sum_{1\le u\le n/2} r(u) = \frac{1}{2\pi}. \tag{7}$$

It is consequence of the Euler-Maclaurin formula [16, Theorem B.5] that, for real numbers a and b with $a < b$, the expression

$$\left|\sum_{a<u\le b} r(u) - \int_a^b r(x)\,dx\right|$$

is at most

$$\frac{1}{2}\Big(|r(a)| + |r(b)|\Big) + \frac{1}{12}\left(|r'(a)| + |r'(b)| + \int_a^b \big|r''(x)\big|\,dx\right).$$

We take $b = n/2$ and let a tend to zero. Elementary calculus shows that

$$|r(n/2)| \le \frac{4}{\pi^2}, \quad |r'(n/2)| \le \frac{8}{\pi} + \frac{16}{n\pi^2}, \quad \lim_{a\to 0} r(a) = \lim_{a\to 0} r'(a) = 0,$$

and $|r''(x)| \le 34$ for all real x. Therefore

$$\left|\sum_{1\le u\le n/2} r(u) - \int_0^{n/2} r(x)\,dx\right| \le \frac{2}{\pi^2} + \frac{2}{3\pi} + \frac{4}{3n\pi^2} + \frac{17n}{12},$$

and so

$$\lim_{n\to\infty} \frac{1}{n^{3/2}} \sum_{1\le u\le n/2} r(u) = \lim_{n\to\infty} \frac{1}{n^{3/2}} \int_0^{n/2} r(x)\,dx,$$

provided that both limits exist. Substituting $y = \pi x^2/n$, we see that this last expression equals

$$\lim_{n\to\infty} \frac{1}{2\pi^{3/2}} \int_0^{\pi n/4} \frac{(\sin y)^2}{y^{3/2}}\,dy = \frac{1}{2\pi^{3/2}} \int_0^{\infty} \frac{(\sin y)^2}{y^{3/2}}\,dy.$$

This establishes (7), and so completes the proof, since

$$\int_0^{\infty} \frac{(\sin y)^2}{y^{3/2}}\,dy = \sqrt{\pi} \tag{8}$$

(see Gradshteyn and Ryzhik [7, 3.823]).

For completeness, we sketch a proof of the identity (8). To do so, we readily verify that

$$\frac{\Gamma(3/2)}{y^{3/2}} = \int_0^{\infty} e^{-yt}\sqrt{t}\,dt \quad \text{for } y > 0,$$

which together with $\Gamma(3/2) = \sqrt{\pi}/2$ yields

$$\int_0^{\infty} \frac{(\sin y)^2}{y^{3/2}}\,dy = \frac{2}{\sqrt{\pi}} \int_0^{\infty}\int_0^{\infty} e^{-yt}\sqrt{t}\,(\sin y)^2\,dt\,dy.$$

Since the integrand on the right hand side is nonnegative, we can interchange the order of integration by Tonelli's theorem. The integral therefore equals

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} \sqrt{t} \int_0^{\infty} e^{-yt}\,(\sin y)^2\,dy\,dt = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{2\sqrt{t}}{t^3+4t}\,dt = \sqrt{\pi}.$$

The inner integral on the left hand side is just the Laplace transform of $(\sin y)^2$, while the integral on the right hand side can be evaluated by first substituting $t = x^2$ (which makes the integrand rational) and then using standard techniques.

3. Proof of Theorem 2

We begin with proving a counterpart of Lemma 3 for the polynomials h_n.

Lemma 4. *For each $n \ge 1$, we have*

$$\|h_n\|_4^4 = n^4 - \gamma_n + 8n \sum_{1\le v\le n/2} \sum_{1\le k\le v} \left(\frac{\sin(\pi k/n)}{\sin(\pi v/n)}\right)^2,$$

where

$$\gamma_n = \begin{cases} 3n^2 & \text{for even } n \\ 2n^2 - 2n & \text{for odd } n. \end{cases}$$

Proof. Write $\zeta = e^{2\pi i/n}$. Then, for $f = h_n$, the numbers c_u in (4) are given by (see also Turyn [18])

$$c_{nu+v} = \sum_{j=0}^{n-u-1}\sum_{k=0}^{n-v-1} \zeta^{jk-(j+u)(k+v)} + \sum_{j=0}^{n-u-2}\sum_{k=n-v}^{n-1} \zeta^{jk-(j+u+1)(k+v)}$$

for $0 \le u, v < n$. Rearrange and use $\sum_{k=0}^{n-1} \zeta^{k(u+1)} = 0$ for $n \nmid u+1$ (note that the second term is zero for $u+1 = n$) to see that

$$\overline{c_{nu+v}} = \zeta^{uv} \sum_{k=0}^{n-v-1} \zeta^{ku} \sum_{j=0}^{n-u-1} \zeta^{jv} - \zeta^{(u+1)v} \sum_{k=0}^{n-v-1} \zeta^{k(u+1)} \sum_{j=0}^{n-u-2} \zeta^{jv} \tag{9}$$

for $0 \le u, v < n$. Evaluation of the sums over j gives, for $0 \le u < n$ and $0 < v < n$,

$$\begin{aligned}\overline{c_{nu+v}} &= \frac{1}{\zeta^v - 1} \sum_{k=0}^{n-v-1} \left(\zeta^{ku}(1-\zeta^{uv}) - \zeta^{k(u+1)}(1-\zeta^{(u+1)v})\right) \\ &= \frac{1}{\zeta^v - 1} \sum_{k=0}^{n-v-1} \left[\zeta^{(k+v)u}(\zeta^{k+v}-1) - \zeta^{ku}(\zeta^k - 1)\right].\end{aligned}$$

We can write this as

$$\left(\sum_{k=v}^{n-1} - \sum_{k=0}^{n-v-1}\right)\zeta^{ku}\, \frac{\zeta^k - 1}{\zeta^v - 1},$$

from which we see that

$$\sum_{u=0}^{n-1} |c_{nu+v}|^2 = n\left(\sum_{k=v}^{n-1} + \sum_{k=0}^{n-v-1} - \sum_{k=v}^{n-v-1} - \sum_{k=v}^{n-v-1}\right)\left|\frac{\zeta^k - 1}{\zeta^v - 1}\right|^2 \tag{10}$$

for $0 < v < n$. For $0 < v < n/2$ all of these sums are nonempty, so that after grouping them together we have, for $0 < v < n/2$,

$$\begin{aligned}\sum_{u=0}^{n-1} |c_{nu+v}|^2 &= n\left(\sum_{k=n-v}^{n-1} + \sum_{k=0}^{v-1}\right)\left|\frac{\zeta^k - 1}{\zeta^v - 1}\right|^2 \\ &= 2n \sum_{k=0}^{v} \left|\frac{\zeta^k - 1}{\zeta^v - 1}\right|^2 - n \\ &= 2n \sum_{k=1}^{v} \left(\frac{\sin(\pi k/n)}{\sin(\pi v/n)}\right)^2 - n. \end{aligned} \tag{11}$$

Using (9) we readily verify that $c_{nu} = 0$ for $u \ne 0$. Therefore, since $c_0 = n^2$ trivially, we have from (5)

$$\|h_n\|_4^4 = n^4 + 2\sum_{v=1}^{n-1}\sum_{u=0}^{n-1} |c_{nu+v}|^2. \tag{12}$$

We also have

$$c_{nu+v}=-\zeta^v\,c_{nu+n-v}\quad\text{for }(u,v)\neq(0,0), \tag{13}$$

which also follows from (9) using the identities

$$\sum_{k=0}^{v-1}\zeta^{kw}=-\zeta^{wv}\sum_{k=0}^{n-v-1}\zeta^{kw}$$

for integers w and v satisfying $n\nmid w$ and $0\le v<n$ and

$$\sum_{j=0}^{n-w-1}\zeta^{-jv}=\zeta^{(w+1)v}\sum_{j=0}^{n-w-1}\zeta^{jv}$$

for integers w and v.

Now, for odd n, we have from (12) and (13)

$$\|h_n\|_4^4=n^4+4\sum_{v=1}^{(n-1)/2}\sum_{u=0}^{n-1}|c_{nu+v}|^2$$

and the desired result follows from (11). Similarly, for even n, we have

$$\|h_n\|_4^4=n^4+4\sum_{v=1}^{n/2-1}\sum_{u=0}^{n-1}|c_{nu+v}|^2+2\sum_{u=0}^{n-1}|c_{nu+n/2}|^2.$$

Using (10), we find that

$$2\sum_{u=0}^{n-1}|c_{nu+n/2}|^2=\frac{n}{2}\sum_{k=0}^{n-1}|\zeta^k-1|^2=n^2,$$

and therefore, by (11),

$$\|h_n\|_4^4=n^4-n^2+4n+8n\sum_{v=1}^{n/2-1}\sum_{k=1}^{v}\left(\frac{\sin(\pi k/n)}{\sin(\pi v/n)}\right)^2.$$

To obtain the desired expression in the lemma for even n, we extend the summation over v to $n/2$ and subtract the correction term

$$8n\sum_{k=1}^{n/2}\big(\sin(\pi k/n)\big)^2=n\sum_{k=0}^{n-1}|\zeta^k-1|^2+4n=2n^2+4n. \qquad\square$$

In order to prove Theorem 2, we invoke Lemma 4 and show that

$$8n\sum_{1\le v\le n/2}\;\sum_{1\le k\le v}\left(\frac{\sin(\pi k/n)}{\sin(\pi v/n)}\right)^2=\frac{4}{\pi^2}n^3+O(n^2). \tag{14}$$

To do so, we make repeated use of the following elementary bound, which is also a simple consequence of the Euler-Maclaurin formula [16, Theorem B.5].

Let $r:\mathbb{R}\to\mathbb{R}$ be a differentiable function and let a and b be real numbers with $a<b$. Then

$$\left|\sum_{a<k\le b} r(k)-\int_a^b r(x)\,dx\right|\le\frac{1}{2}\bigg(|r(a)|+|r(b)|+\int_a^b\big|r'(x)\big|\,dx\bigg). \tag{15}$$

We first take $r(x)=(\sin(\pi x/n))^2$ and $(a,b)=(0,v)$, so that for $1\le v\le n/2$, we have

$$\begin{aligned}\sum_{k=1}^{v}(\sin(\pi k/n))^2&=\int_0^v(\sin(\pi x/n))^2dx+O(1)\\&=\frac{n}{\pi}\int_0^{\pi v/n}(\sin y)^2\,dy+O(1)\\&=\frac{n}{2\pi}\Big(\pi v/n-\sin(\pi v/n)\cos(\pi v/n)\Big)+O(1).\end{aligned}$$

Letting

$$p(y)=\frac{y-\sin y\cos y}{(\sin y)^2},$$

we then have

$$\sum_{1\le v\le n/2}\;\sum_{1\le k\le v}\left(\frac{\sin(\pi k/n)}{\sin(\pi v/n)}\right)^2=\frac{n}{2\pi}\sum_{1\le v\le n/2}p(\pi v/n)+O(n). \tag{16}$$

We now apply (15) with $r(x)=p(\pi x/n)$ and $b=n/2$ and let a tend to zero. We have

$$p'(y)=2-\frac{2(y-\sin y\cos y)\cos y}{(\sin y)^3}$$

from which, using $x-x^3/6\le\sin x\le x$ and $1-x^2/2\le\cos x\le 1$ together with elementary calculus, we find that

$$-3<p'(y)\le 2\quad\text{for } 0<y\le\pi/2.$$

Hence $|r'(x)|<3\pi/n$ for $0<x\le n/2$. Since we also have $r(n/2)=\pi/2$ and $\lim_{a\to 0}r(a)=0$, we find from (15) that (16) equals

$$\frac{n}{2\pi}\int_0^{n/2}p(\pi x/n)dx+O(n)=\frac{n^2}{2\pi^2}\int_0^{\pi/2}p(y)dy+O(n).$$

The desired result (14) is then established by showing that

$$\int_0^{\pi/2}p(y)dy=1. \tag{17}$$

By differentiation we readily verify that

$$\int\frac{y-\sin y\cos y}{(\sin y)^2}\,dy=-\frac{y}{\tan y}+C$$

for some arbitrary constant C and (17) follows by application of l'Hôpital's rule.

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