A Recursive Recomputation Approach to Smoothing in Nonlinear and Bayesian State-Space Modeling

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Contents of the talk

- In this talk, we deal with a computational issue in numerical smoothing in nonlinear and Bayesian state-space models.
- Link between optimization and today’s talk: Automatic differentiation.
State Space Model

- Basic model for analyzing events with one dimensional structure, e.g., time-series, signal, DNA sequence etc.
- Wide applications in statistics, machine learning, signal processing, bioinformatics etc.
State Space Model

\[ x_{t+1} = f(x_t) + \varepsilon_t \]
\[ y_t = g(x_t) + \eta_t \]

\[ \eta_t, \varepsilon_t : \text{(Non Gaussian) white noize} \]
State Space Model

- \( f(x), g(x) \) Vector functions of appropriate dimensions
- \( x_t \) : State vector
- \( y_t \) : Observation
State Space Model

- $Y_0, \ldots, Y_{T-1}$: Observation
  (Considered as realization of $y_t$.)
- $T$: Length of data
- Notation

$$s_{t_1:t_2} \equiv (s_{t_1}, \ldots, s_{t_2})$$
State Space Model

- **Major Problem:**
  Computing conditional distribution of \( x(t) \) given observation \( Y(\cdot) \).

\[
\begin{align*}
  p(x_{t+1} | Y_{0:t}) & : \text{Predictive distribution} \\
  p(x_t | Y_{0:t}) & : \text{Filtering distribution} \\
  p(x_t | Y_{0:T}) & : \text{Smoothing distribution}
\end{align*}
\]
Applications of Smoothing

- Change point problems
- Estimation of a posteriori probability of models
- Target tracking
- Image processing
- Robotics
- Speech recognition etc. etc.
Scheme for computing filtering and predictive distribution

\[ p(x_t | Y_{1:t}) \] Filtering Distribution
\[ \downarrow \]
\[ p(x_{t+1} | Y_{1:t}) \] Innovation of the State
\[ \downarrow \]
\[ p(x_{t+1} | Y_{1:t+1}) \] Predictive Distribution
\[ \downarrow \]
\[ Y_{t+1} \] Filtering Distribution

Bayes Theorem
Recursive formulas for predictive and filtering distributions

(Predictive distribution)

\[ P(x_{t+1}|Y_{1:t}) = \int P(x_{t+1}|x_t)P(x_t|Y_{1:t})\,dx_t \]

(Filtering distribution)

\[ P(x_{t+1}|Y_{t+1}) = \frac{P(Y_{t+1}|x_{t+1})P(x_{t+1}|Y_{1:t})}{\int P(Y_{t+1}|x_{t+1})P(x_{t+1}|Y_{1:t})\,dx_{t+1}} \]
Scheme for computing smoothing distributions

\[
p(x_{t-2}|Y_{1:t-2}) \quad \leftarrow \quad p(x_t|Y_{1:T}) \quad \leftarrow \quad p(x_{t-1}|Y_{1:t-1}) \quad \text{Filtering distribution}
\]

\[
p(x_{t-1}|Y_{1:T}) \quad \leftarrow \quad p(x_t|Y_{1:t}) \quad \text{Smoothing distribution}
\]

\[
p(x_t|Y_{1:T}) \quad \text{Smoothing distribution}
\]
Difficulty in computing smoothing distributions

- Recursion is executed in the reverse chronicle order
- Need for preserving all filtering distributions!
- Storage is proportional to T
Difficulty in computing smoothing distributions

- Let $M$ be the storage necessary for storing the filtering distribution at each time step.
- Then computation of smoothing requires $O(MT)$ storage.
- $M$ is very large (hi-dimensional state space case)
Representation of State Distribution

- Typically, state distribution is approximately represented as the set of huge number of particles in the state space. (Particle filter; major innovation in the last decade)
A Remedy: Recomputation Scheme

- **Snapshot at t**: the set of all variables necessary to reproduce all the computational process after t.
- **Idea**: Take snapshots at appropriate time points and then recompute filtering distributions as they are needed.
Elementary Recursive Recomputation Scheme

We explain the method with $T=81$.

Ordinary, the storage of 81M is required to compute Smoothing distributions.
Perform computation of filtering distributions once up to $t=72$
Taking snapshots at $t=0, 9, 18, \ldots , 72$
Compute the filtering distributions from $t=72$ to 80 and back. (Use 8M workspace which is going to be reused.)
Compute the filtering distributions from $t=63$ to 71 and back. (8M workspace is reused.)
Snapshots

[0, 8] [9, 17] [18, 26] [27, 35] [36, 44] [45, 53] [54, 62] [63, 71] [72, 80]

Workspace; Reused

8M

R \[\uparrow\] Q \[\uparrow\] P \[\uparrow\] M \[\uparrow\] L \[\uparrow\] K \[\uparrow\] H \[\uparrow\] G \[\uparrow\] F \[\uparrow\]
Snapshots

Workspace; Reused

8M
After all, we are able to compute smoothing distributions with $9 + 8 = 17$ M storage.
Elementary Recursive Recomputation Scheme

- Generally, if one takes a snapshot every $\sqrt{T}$ step, then we may reduce storage down to $2\sqrt{T}M$ by computing twice the whole filtering distributions.
Recursive Recomputation Scheme

- Reduces further storage to $O((\log T)M)$ by applying the idea of recomputation recursively, at the cost of $O(\log T)$ times recomputation of whole filtering distributions.

- We explain the method by applying to Smooth the series with $T=81$. ($t=0\sim80$)
[0, 80]

[0, 26]

[27, 53]

[54, 80]

N 9 18

[0, 8]

[9, 17]

[18, 26]

[27, 35][36, 44][45, 53][54, 62][63, 71][72, 80]

8 17 26 35 44 53 62 71 80
Recursive Recomputation Scheme

- We are able to compute the smoothing distributions with $5 + 8 = 13$ M storage (instead of $81$ or $17$).
- The number of whole filtering computation is $3$ (or a bit less).
Recursive Recomputation Scheme (Summary)

- We can reduce the space complexity for smoothing from $O(MT)$ to $O(M \log T)$.
  (at the cost of $O(\log T)$ times computation of whole filtering distributions.)
Background (In connection with optimization)

- The idea of saving storage by recomputation was developed in automatic differentiation.
- Specifically, the idea of recursive recomputation was introduced by Griewank in 1992.
Easy Implementation

(Initialization)
- Execute a forward sweep taking snapshots under a certain rule (explained later).
- At the end of the sweep (t=T-1), the set of snapshots snap(T-1) is available.
Easy Implementation

(Reverse sweep with partial forward sweeps)

- If $t = 0$ then stop
- If not, execute the following procedure.
  
  (We assume that the filtering distribution $F(t)$ has been computed, and the set of snapshots $\text{snap}(t)$ is available. We compute $F(t-1)$ and $\text{snap}(t-1)$ below.)
Execute partial forward sweep from the snapshot which is before t-1 and closest to t-1 to compute F(t-1). On the way, we take snapshots to construct snap(t-1).
Easy Implementation:
Rule for snapshots:

- Provide areas snapshot(0), snapshot(1), …, snapshot(K) to store snapshots.
- If k lower bits are zero in the binary representation of t, then we store the snapshot F(t) at snapshot(k).
t=15 (After the forward sweep)

- Snapshot(0): \( t=(1111)_{2} = 15 \)
- Snapshot(1): \( t=(1110)_{2} = 14 \)
- Snapshot(2): \( t=(1100)_{2} = 12 \)
- Snapshot(3): \( t=(1000)_{2} = 8 \)
- Snapshot(*): \( t=(0000)_{2} = 0 \)
t=14

- Snapshot(0):
- Snapshot(1): \( t=(1110)_2 = 14 \)
- Snapshot(2): \( t=(1100)_2 = 12 \)
- Snapshot(3): \( t=(1000)_2 = 8 \)
- Snapshot(*): \( t=(0000)_2 = 0 \)
t=13 (Recomp. from t=12)

- Snapshot(0): \( t=(1001)_2 = 13 \)
- Snapshot(1): 
- Snapshot(2): \( t=(1100)_2 = 12 \)
- Snapshot(3): \( t=(1000)_2 = 8 \)
- Snapshot(*): \( t=(0000)_2 = 0 \)
t=12

- Snapshot(0): 
- Snapshot(1): 
- Snapshot(2): $t=(1100)_2 = 12$
- Snapshot(3): $t=(1000)_2 = 8$
- Snapshot(*): $t=(0000)_2 = 0$
t=11 (Recomp. from t=8)

- Snapshot(0):
- Snapshot(1):
- Snapshot(2):
- Snapshot(3): $t=(1000)_2 = 8$
- Snapshot(*): $t=(0000)_2 = 0$
t=11 (Recomp. from t=8)

- Snapshot(0): $t=(1001)_2 = 9$
- Snapshot(1):
- Snapshot(2):
- Snapshot(3): $t=(1000)_2 = 8$
- Snapshot(*): $t=(0000)_2 = 0$
t=11 (Recomp. from t=8)

- Snapshot(0): $t=(1001)_2 = 9$
- Snapshot(1): $t=(1010)_2 = 10$
- Snapshot(2):
- Snapshot(3): $t=(1000)_2 = 8$
- Snapshot(*): $t=(0000)_2 = 0$
t=11 (Recomp. from t=8)

- Snapshot(0): $t=(1011)_2 = 11$
- Snapshot(1): $t=(1010)_2 = 10$
- Snapshot(2):
- Snapshot(3): $t=(1000)_2 = 8$
- Snapshot(*): $t=(0000)_2 = 0$
t=10

- Snapshot(0):
- Snapshot(1): $t=(1010)_2 = 10$
- Snapshot(2):
- Snapshot(3): $t=(1000)_2 = 8$
- Snapshot(*): $t=(0000)_2 = 0$
t=9 (Recomp. from t=8)

- Snapshot(0): $t=(1001)_2 = 9$
- Snapshot(1):
- Snapshot(2):
- Snapshot(3): $t=(1000)_2 = 8$
- Snapshot(*): $t=(0000)_2 = 0$
t=8

- Snapshot(0):
- Snapshot(1):
- Snapshot(2):
- Snapshot(3): \( t=(1000)_2 = 8 \)
- Snapshot(*): \( t=(0000)_2 = 0 \)
t=7 (Recomp. from t=0)

- Snapshot(0): t=(0111)\_2 = 7
- Snapshot(1): t=(0110)\_2 = 6
- Snapshot(2): t=(0100)\_2 = 4
- Snapshot(3):
- Snapshot(*): t=(0000)\_2 = 0
Application

- We applied the new method to smooth the Nikkei 225 stock price data with a stochastic volatility model with particle filters.
- We used the path-sampling smoothing algorithm of the particle filter by Kitagawa.
Path-sampling smoother

- Let N be the number of particles.
State Space Model

\[ x_{t+1} = f(x_t) + \varepsilon_t \]
\[ y_t = g(x_t) + \eta_t \]

\( \eta_t, \varepsilon_t \) : (Non Gaussian) white noise
Path-sampling Smoother

- Traces the parents-children relation of particles.
- $O(N)$ operations per time step.
- $N$ can be millions (in view of computation time), but the limit of $N$ comes from storage.
- Suffers from degeneracy.
Path-sampling smoother

- What is degeneracy?
  The number of particles representing the smoothing distribution reduces quickly as $t$ is traced back.
  (if you start with numerous and diverse particles, the particles can easily reduces 10 – 20 after a while.)
Nikkei 225 Data (From Jan. 87 to Aug. 90)

The Black Monday (Oct. 87)

Bubble Crash (Jan 90)
The model is given by:

\[ x_{t+1} = -2x_t + x_{t-1} + \varepsilon_t, \]

\[ y_t = x_t + \eta_t, \]

\[ \varepsilon_t \sim \text{Cauchy}(0, \tau_1) \]

\[ \eta_t \sim \text{Normal}(0, \exp(p_t)) \]
Model (II)

\[ p_{t+1} = -2p_t + p_{t-1} + \delta_t, \]

\[ \delta_t \sim \text{Normal}(0, \tau_2) \]
Smoothed Trend (median)
Residual (Volatility is time-varying variance of the sequence)
Smoothing with the Particle Filter

- Computer SGI Altix3700
  (256CPU, Intel Itanium2 1.3GHz, Main Memory 1920GB)
- 1 CPU, 5GB
- 150,000 particles can be used with smoothing by ordinary implementation.
  (Comparable with high-end PC environment)
The smoothing distributions of volatility (150,000 particles)

Each curve shows 2.3%, 15.9%, 50%(solid line), 84.1%, 97.7% points.
Smoothing with the particle filter

- 3,000,000 particles (20 times more than the ordinary implementation) can be used for smoothing with the recursive recomputation scheme.
Smoothing distributions of volatility (3 mil particles)

Each curve shows 2.3%, 15.9%, 50% (solid line), 84.1%, 97.7% points.
Comparison with other “standard particle smoothers”

- Forward-backward smoother
- Two-filter smoother
- These methods require $O(N^3)$ operations per time step. We cannot increase $N$ more than 5000. (Since $N$ is modest, there is no need for saving storage.)
Comparison with other “standard particle smoothers”

- Path-sampling Method is $O(N)$ per step but suffers from degeneracy.
- In the following we compare the smoothing distributions computed in 7 hours with $N=5000$ (for the two $O(N^2)$ methods) and $N=3000000$ (for path-sampling method).
Forward-Backward Method

Each curve shows 2.3%, 50%(solid line), 97.7% points.
2 Filter Formulae

Each curve shows 2.3%, 50% (solid line), 97.7% points.
The New Method

Each curve shows 2.3%, 15.9%, 50%(solid line), 84.1%, 97.7% points.
Conclusion

- The path-sampling smoother with numerous particles performs the best. Using numerous particles helps.
- It is possible to extend the technique to fixed-lag smoothers.