A Recursive Recomputation Approach to Smoothing in Nonlinear and Bayesian State-Space Modeling

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(joint work with Kazuyuki Nakamura
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Modification of Slide Presented at Workshop on Advances in Optimization (Tokyo Inst. Tech.), Apr20, 2007

Contents of the talk

- In this talk, we deal with a computational issue in numerical smoothing in nonlinear and Bayesian state-space models.
- Link between optimization and today's talk: Automatic differentiation.

Paper

Kazuyuki Nakamura and Takashi Tsuchiya:

A Recursive Recomputation Approach for Smoothing in Nonlinear State Space Modeling --- An Attempt for Reducing Space Complexity ---

(Appeared in IEEE Transactions on Signal Processing)

State Space Model

- Basic model for analyzing events with one dimensional structure, e.g., time-series, signal, DNA sequence etc.
- Wide applications in statistics, machine learning, signal processing, bioinformatics etc.

State Space Model

Innovation of state

$$
\begin{aligned}
x_{t+1} &= f(x_t) + \varepsilon_t \\
y_t &= g(x_t) + \eta_t
\end{aligned}
$$

Observation

η_t, ε_t : (Non Gaussian) white noize

State Space Model

- $f(x),\ g(x)$ Vector functions of appropriate dimensions
- x_t : State vector
- y_t : Observation

State Space Model

- $Y_0, \ldots, Y_{T-1}$: Observation (Considered as realization of y_t .)
- T : Length of data
- Notation

$$s_{t_1:t_2} \equiv (s_{t_1}, \ldots, s_{t_2})$$

State Space Model

- Major Problem：
  Computing conditional distribution of x(t) given observation Y(・).

$p(x_{t+1}|Y_{0:t})$ ： Predictive distribution

$p(x_t|Y_{0:t})$ ： Filtering distribution

$p(x_t|Y_{0:T})$ ： Smoothing distribution

Applications of Smoothing

- Change point problems
- Estimation of a posteriori probability of models
- Target tracking
- Image processing
- Robotics
- Speech recognition etc. etc.

Scheme for computing filtering and predictive distribution

↓ ← Y_t Bayes Theorem

$p(x_t|Y_{1:t})$ Filtering Distribution

↓ Innovation of the State

$p(x_{t+1}|Y_{1:t})$ Predictive Distribution

↓ ← Y_{t+1} Bayes Theorem

$p(x_{t+1}|Y_{1:t+1})$ Filtering Distribution

Recursive formulas for predictive and filtering distributions

(Predictive distribution)

$$P(x_{t+1}|Y_{1:t}) = \int P(x_{t+1}|x_t)P(x_t|Y_{1:t})dx_t$$

(Filtering distribution)

$$P(x_{t+1}|Y_{t+1}) = \frac{P(Y_{t+1}|x_{t+1})P(x_{t+1}|Y_{1:t})}{\int P(Y_{t+1}|x_{t+1})P(x_{t+1}|Y_{1:t})dx_{t+1}}$$

Scheme for computing smoothing distributions

$$p(x_{t-2}|Y_{1:t-2})$$

$$p(x_{t-1}|Y_{1:T}) \leftarrow p(x_{t-1}|Y_{1:t-1})$$

$$p(x_t|Y_{1:T}) \leftarrow p(x_t|Y_{1:t})$$

$$p(x_{t+1}|Y_{1:T})$$

Filtering distribution

Smoothing distribution

Difficulty in computing smoothing distributions

- Recursion is executed in the reverse chronicle order

↓

- Need for preserving all filtering distributions !
- Storage is proportional to T

Difficulty in computing smoothing distributions

- Let M be the storage necessary for storing the filtering distribution at each time step.
- Then computation of smoothing requires O(MT) storage.
- M is very large (hi-dimensional state space case)

Representation of State Distribution

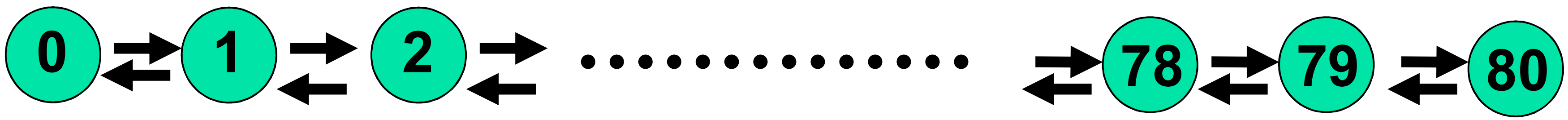
- Typically, state distribution is approximately represented as the set of huge number of particles in the state space. (Particle filter; major innovation in the last decade)

A Remedy: Recomputation Scheme

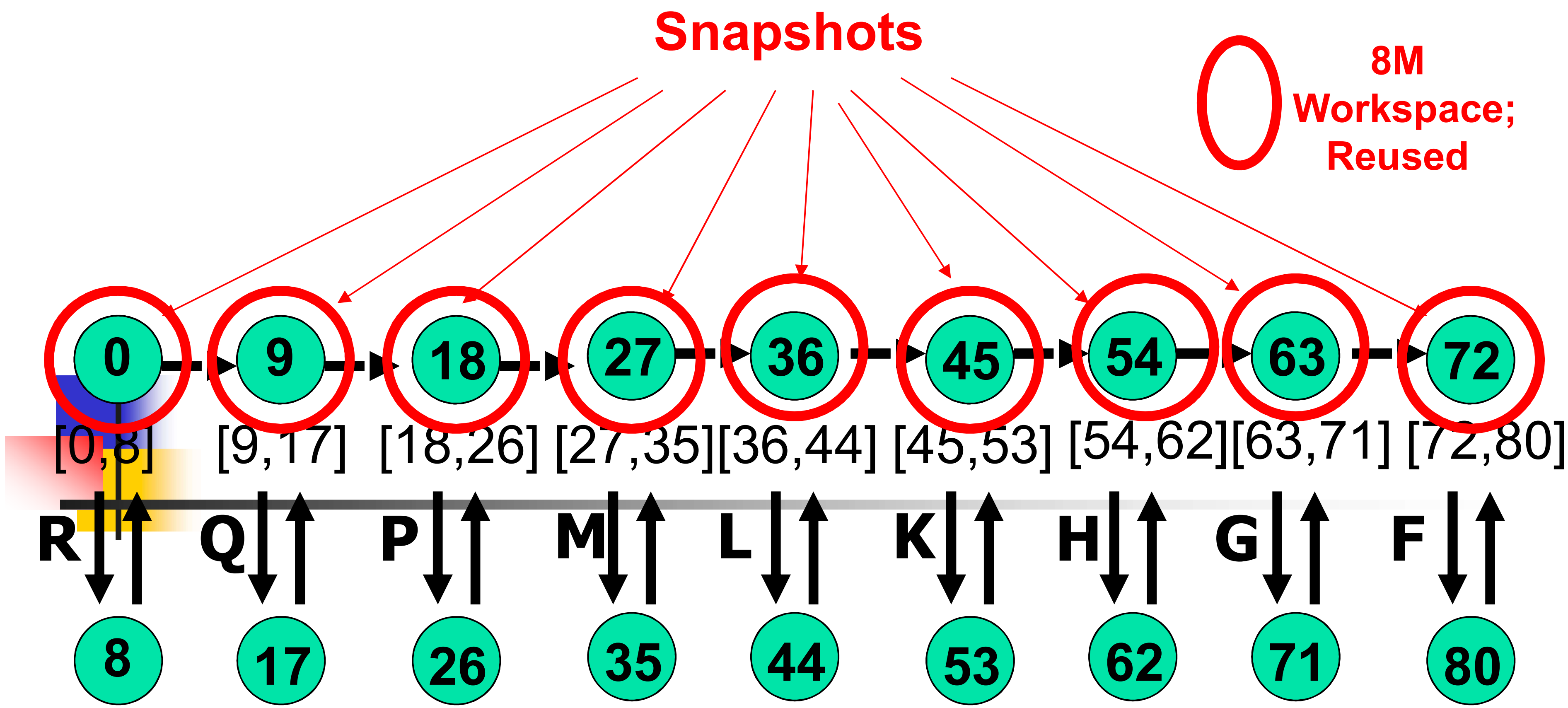
- Snapshot at t : the set of all variables necessary to reproduce all the computational process after t.
- Idea: Take snapshots at appropriate time points and then recompute filtering distributions as they are needed.

Elementary Recursive Recomputation Scheme

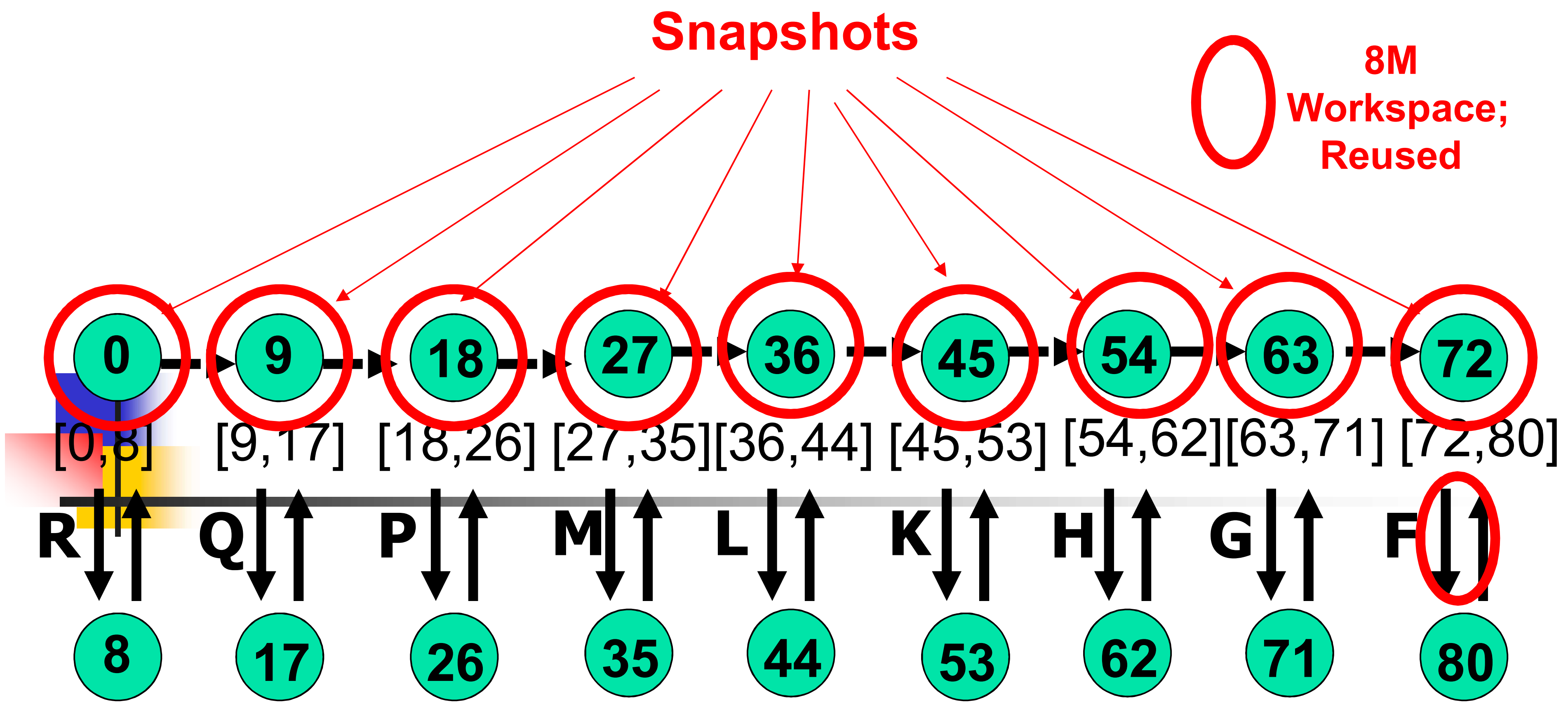
- We explain the method with T=81.



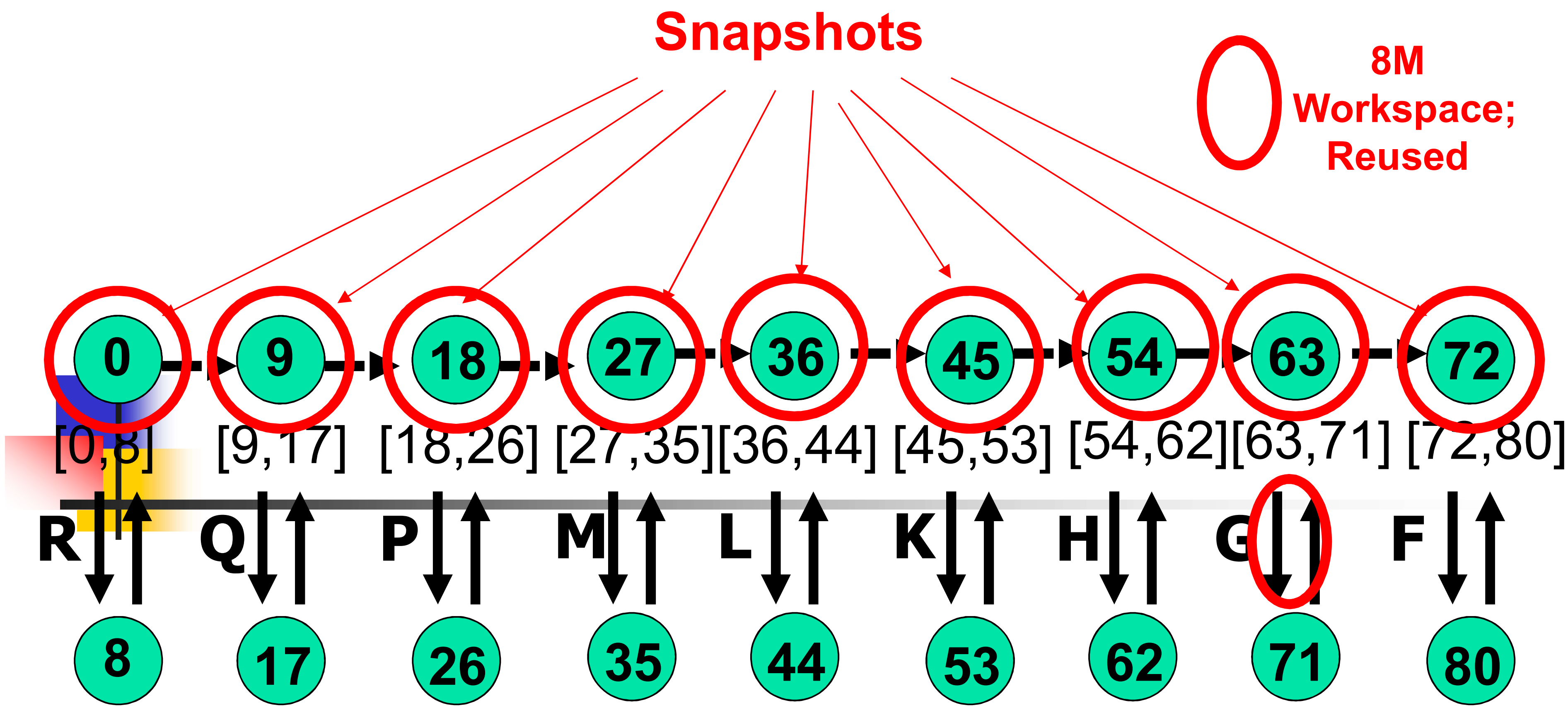
Ordinary, the storage of 81M is required to compute Smoothing distributions.



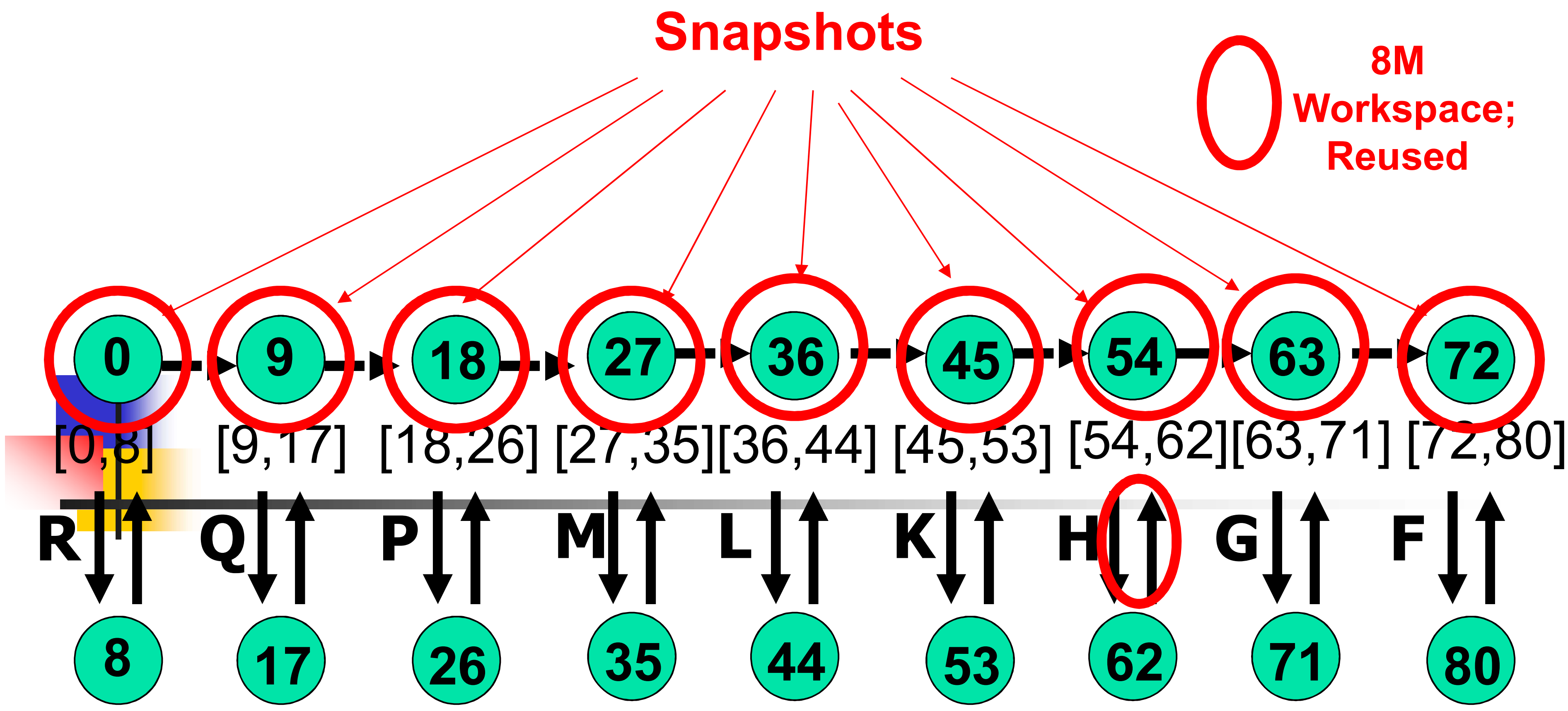
Perform computation of filtering distributions once up to t=72
Taking snapshots at t=0, 9, 18, … ,72



Compute the filtering distributions from t=72 to 80 and back. (Use 8M workspace which is going to be reused.)



Compute the filtering distributions from t=63 to 71 and back. (8M workspace is reused.)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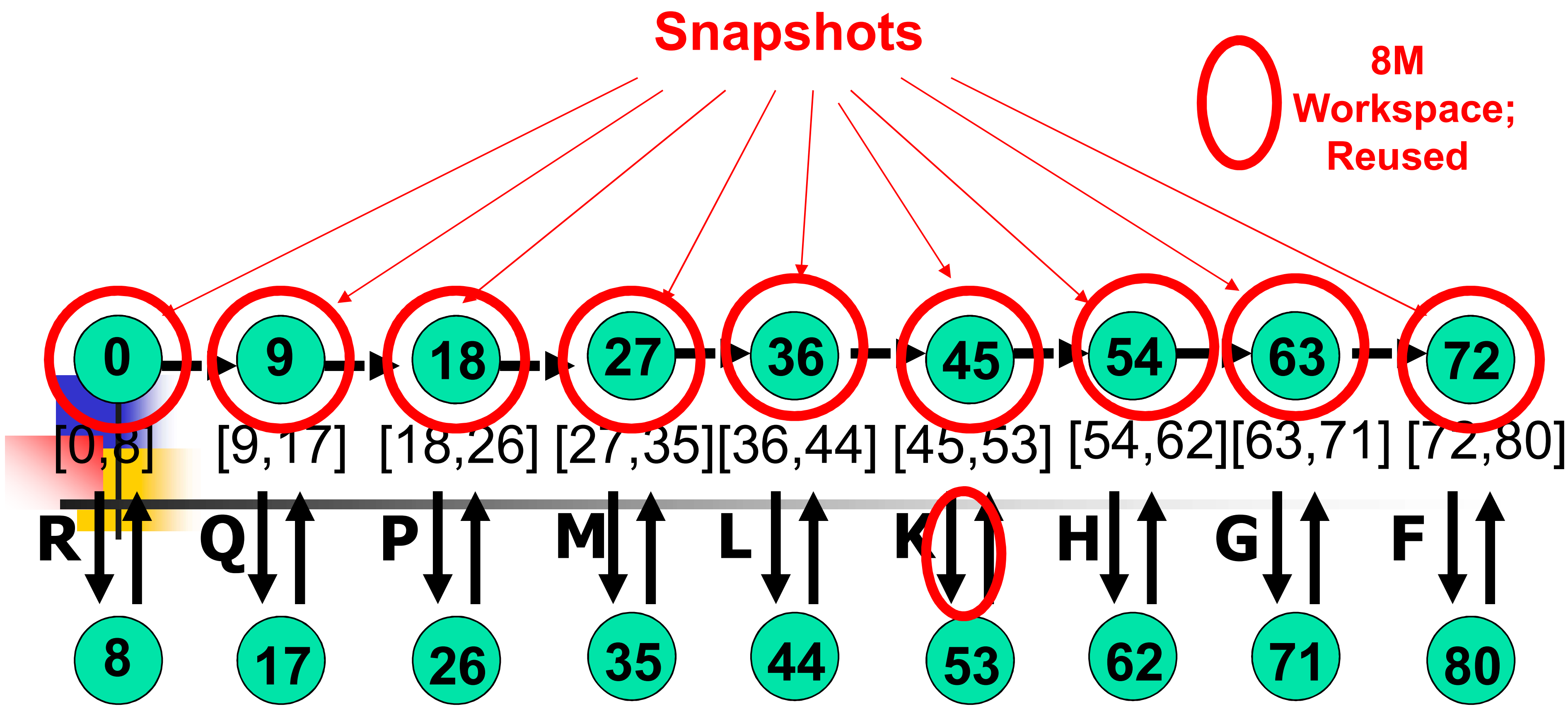
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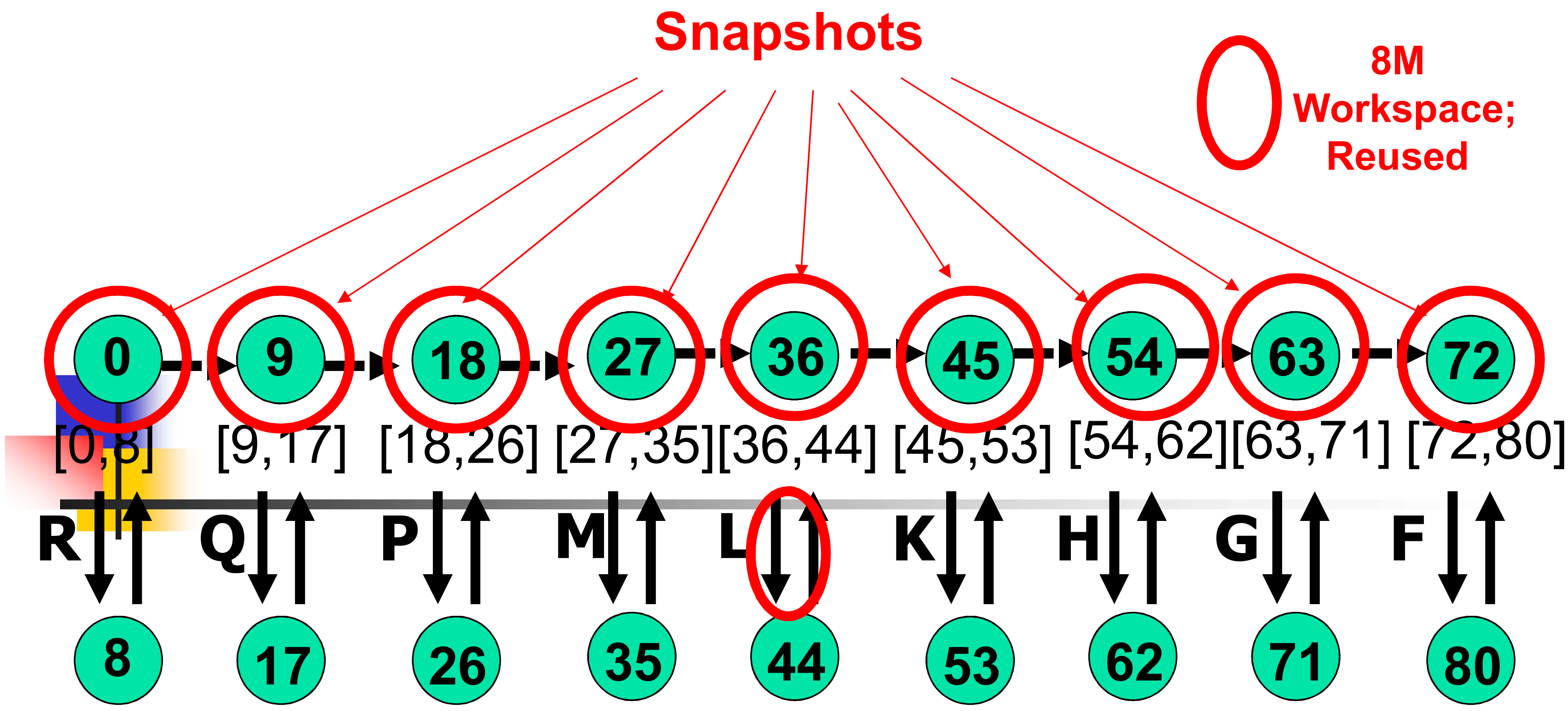
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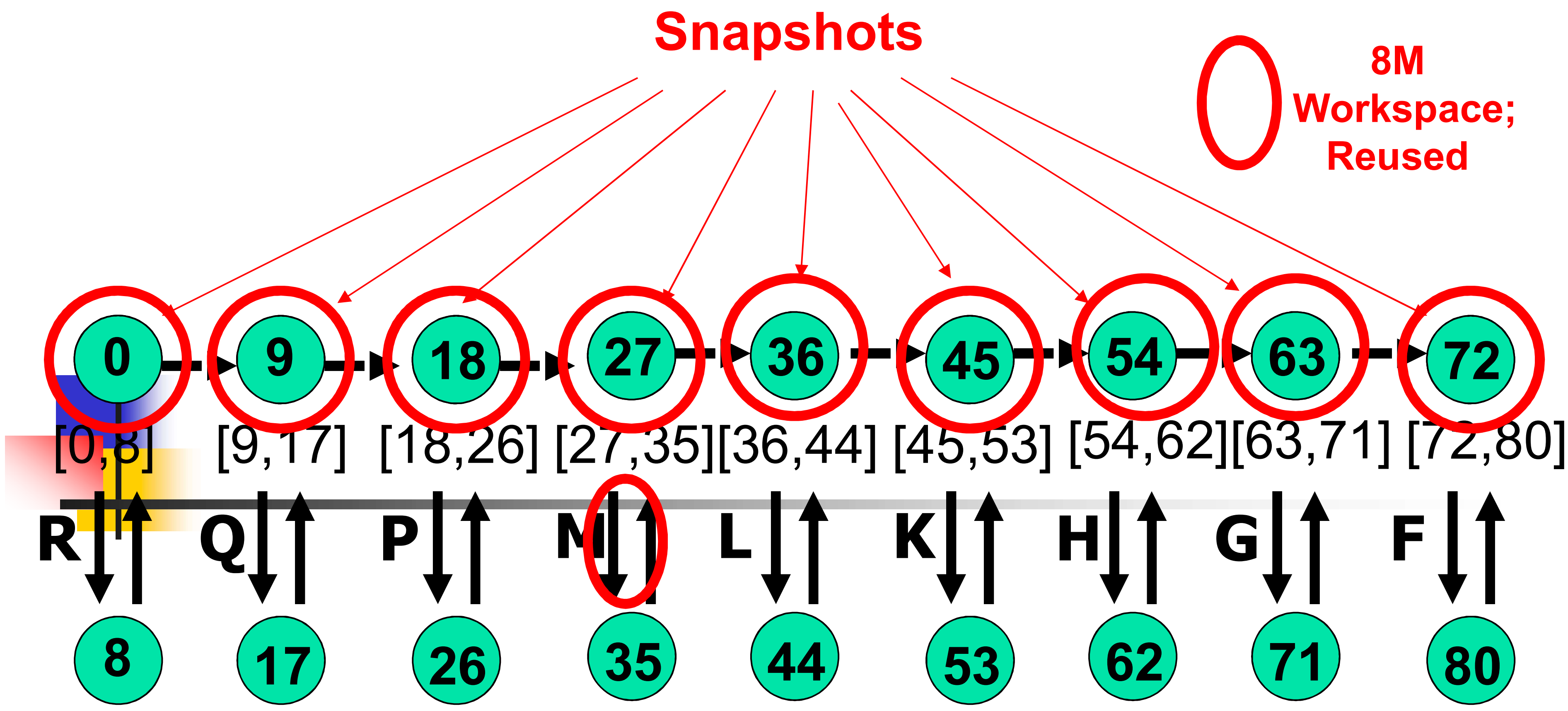

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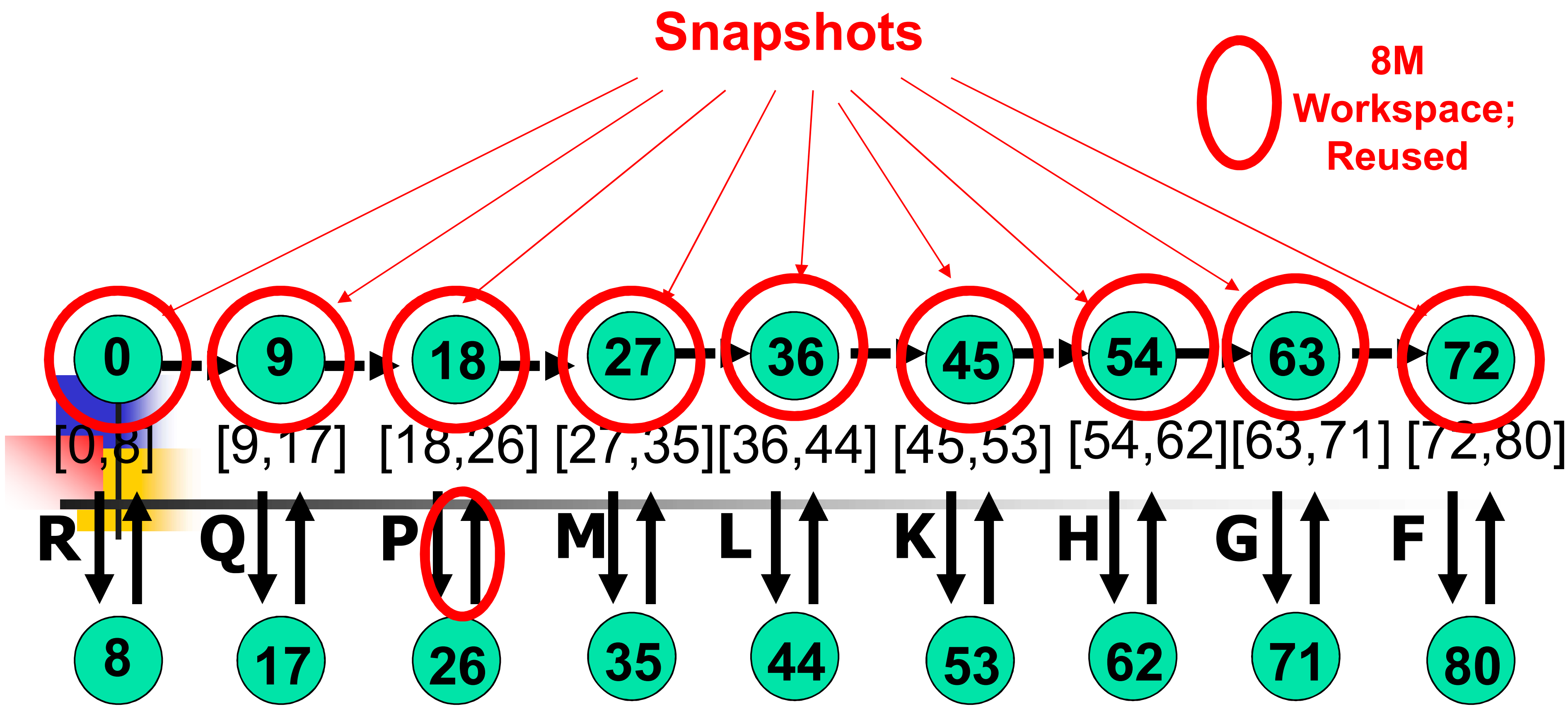
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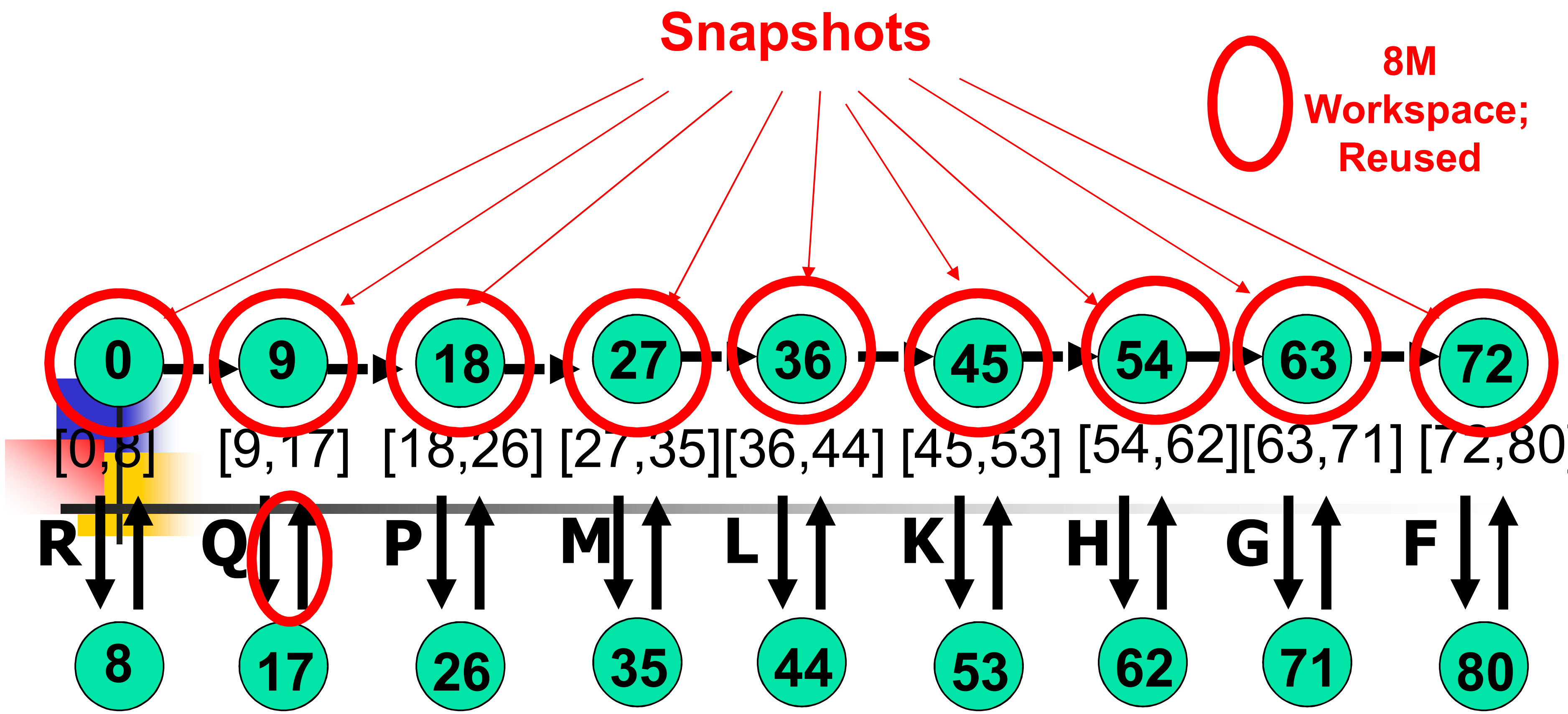
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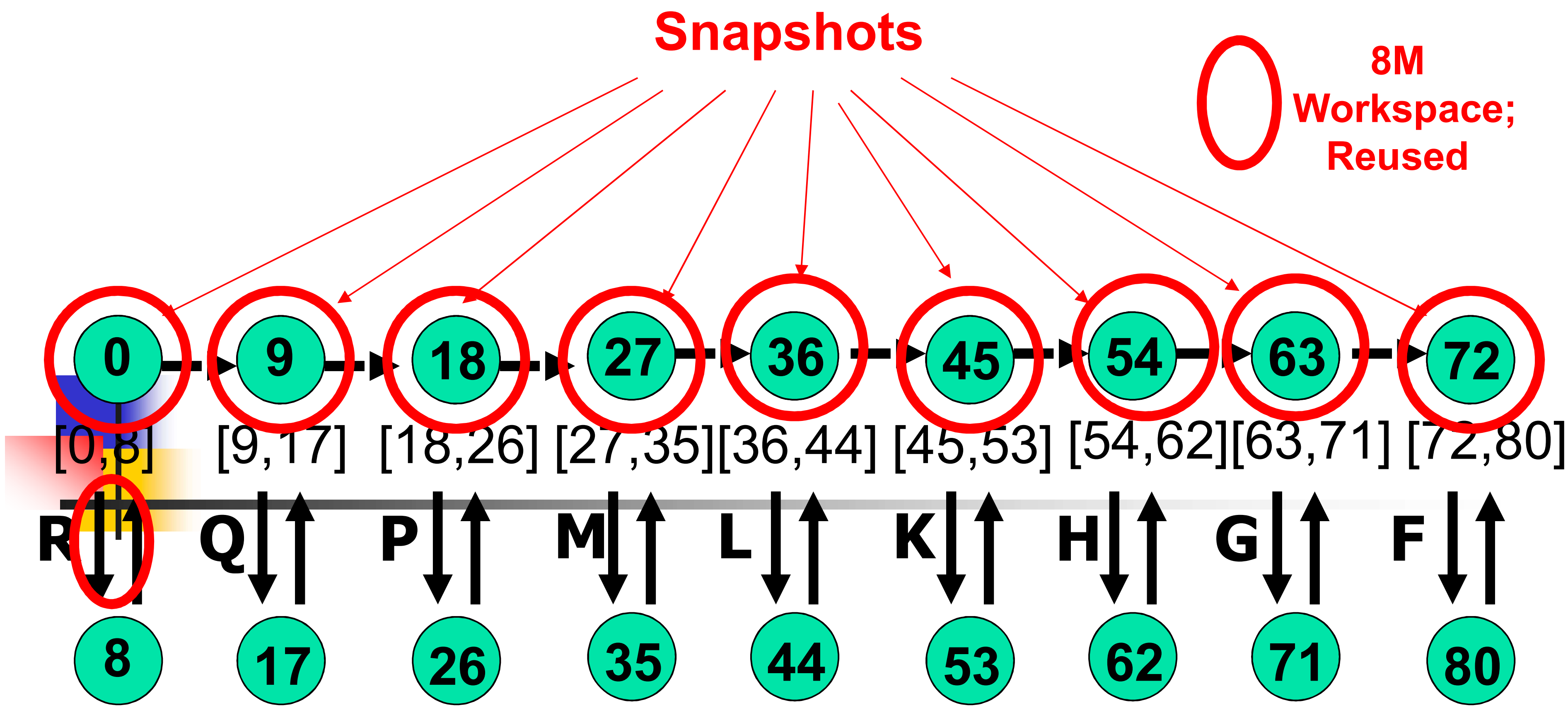


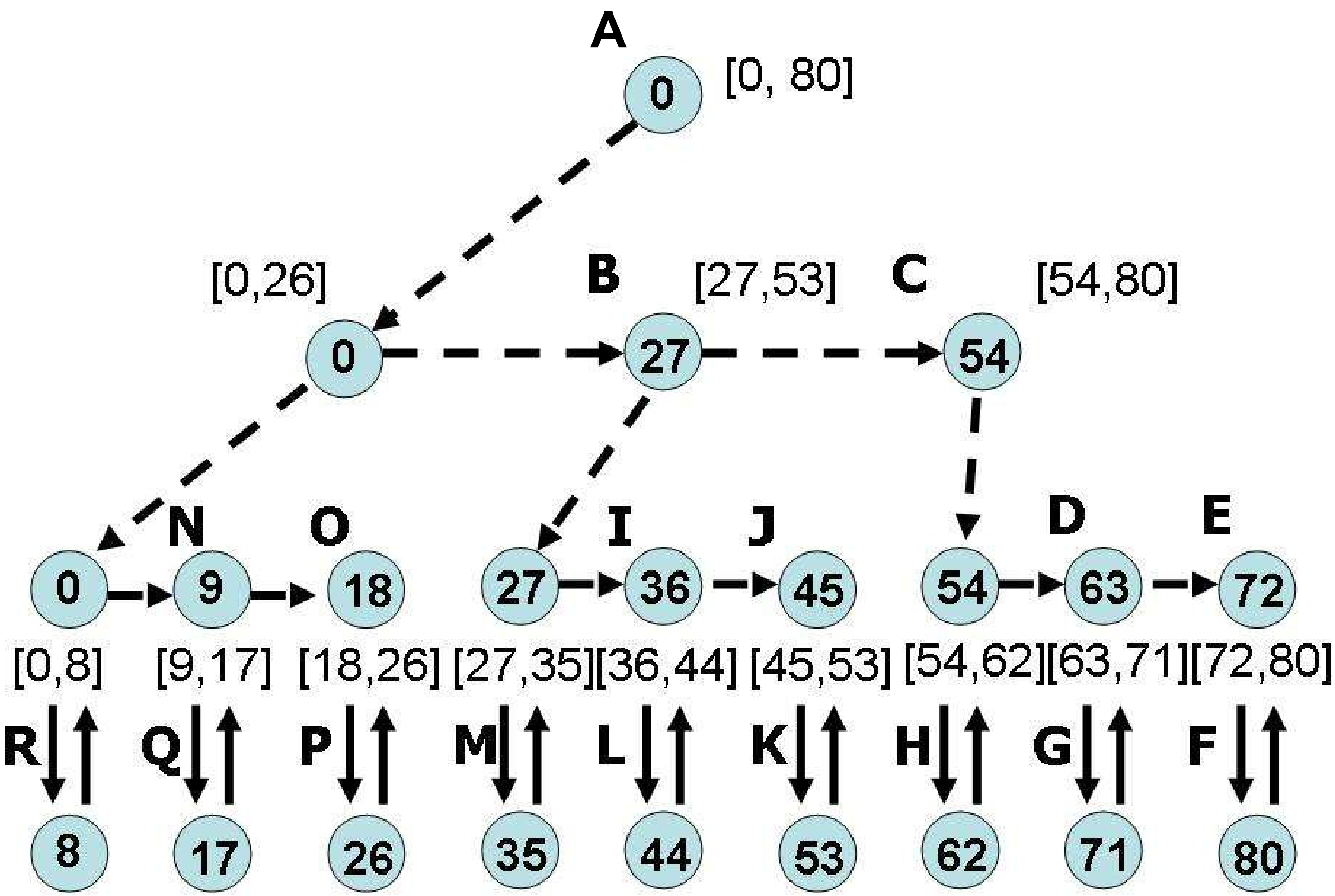
After all, we are able to compute smoothing distributions with 9+8 = 17 M storage.

Elementary Recursive Recomputation Scheme

- Generally, if one takes a snapshot every $\sqrt{T}$ step, then we may reduce storage down to $2\sqrt{T}M$ by computing twice the whole filtering distributions.

Recursive Recomputation Scheme

- Reduces further storage to O((log T)M) by applying the idea of recomputaiton recursively, at the cost of O(log T) times recomputation of whole filtering distributions
- We explain the method by applying to Smooth the series with T=81. (t=0~80)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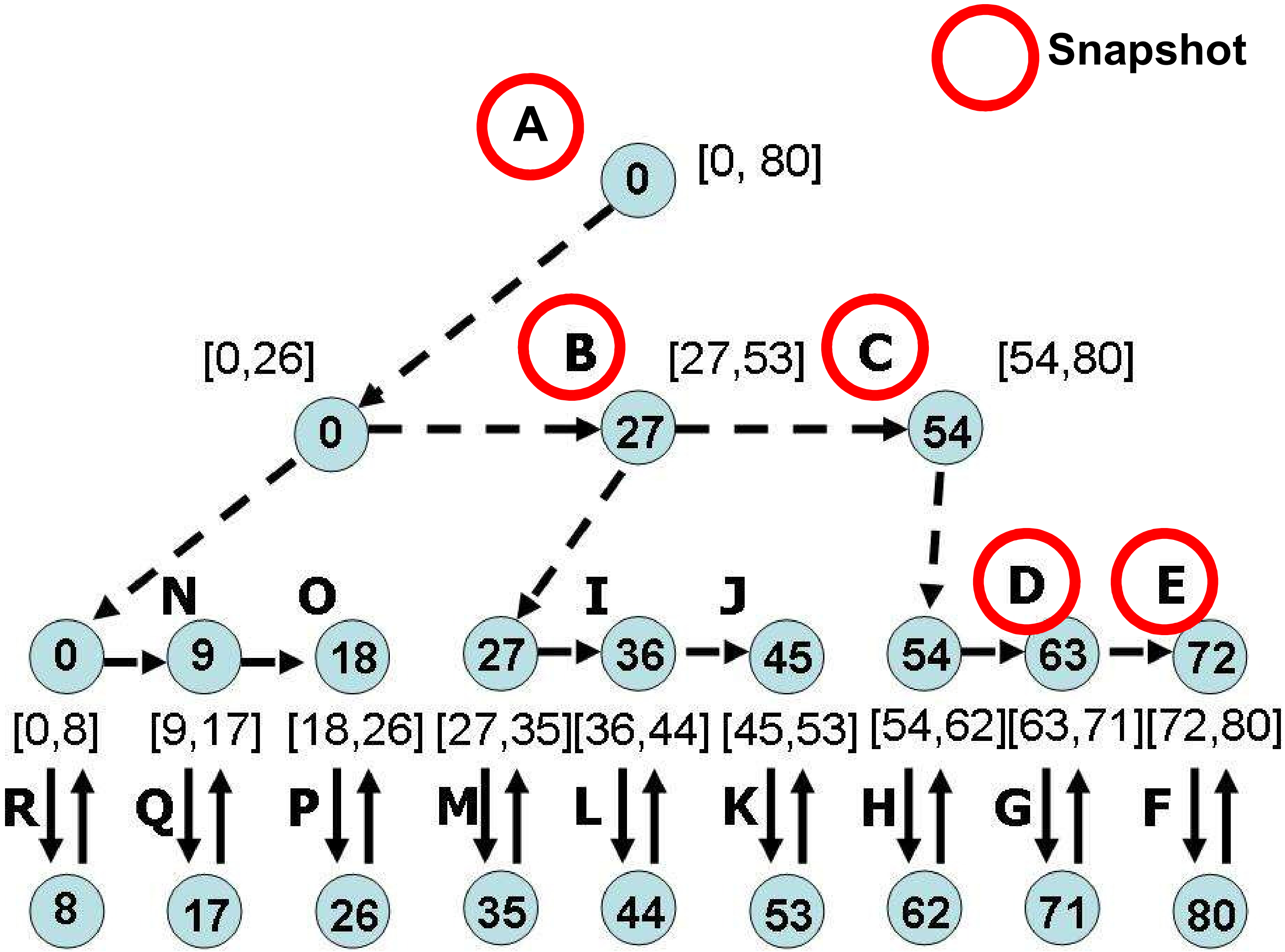
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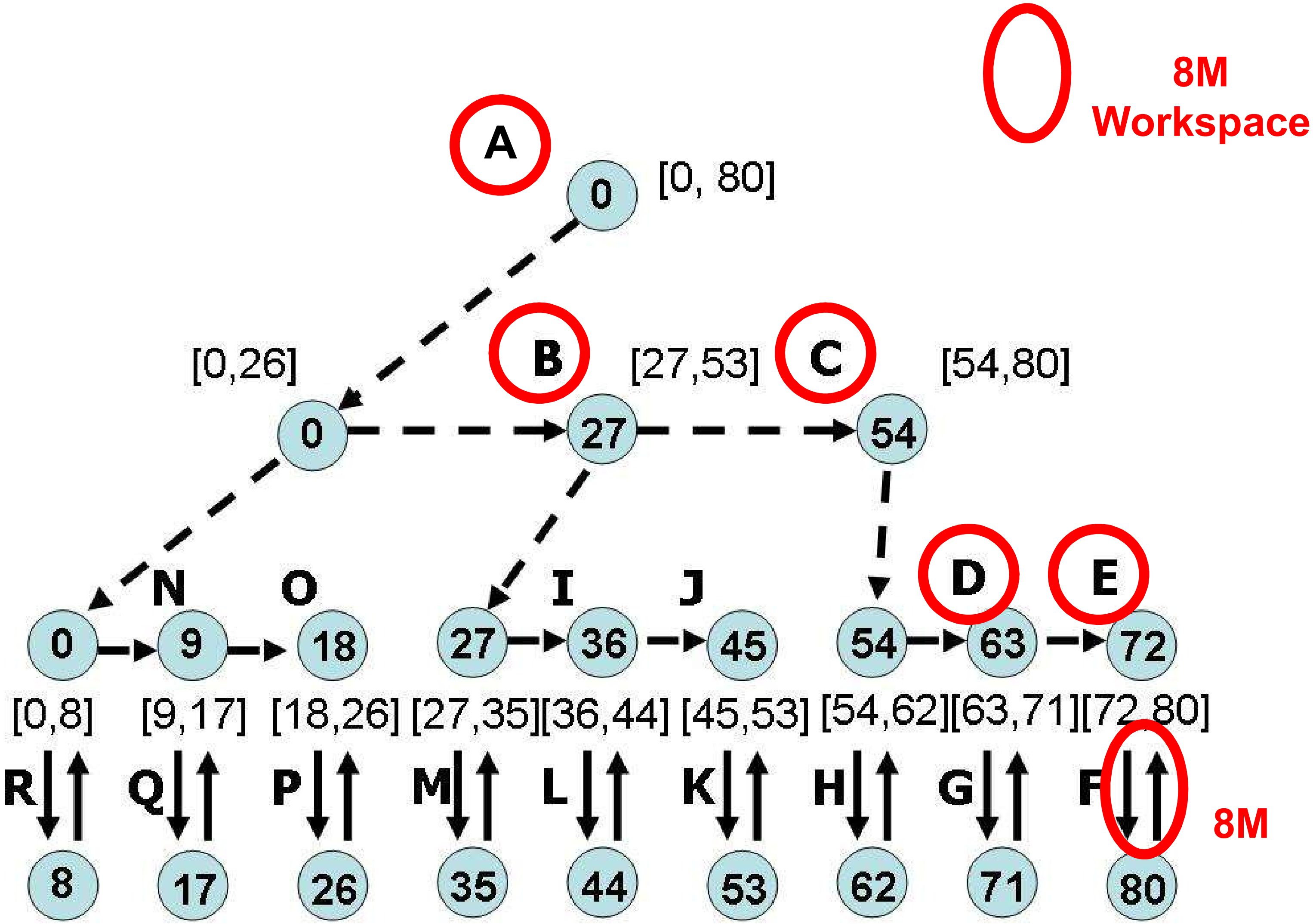
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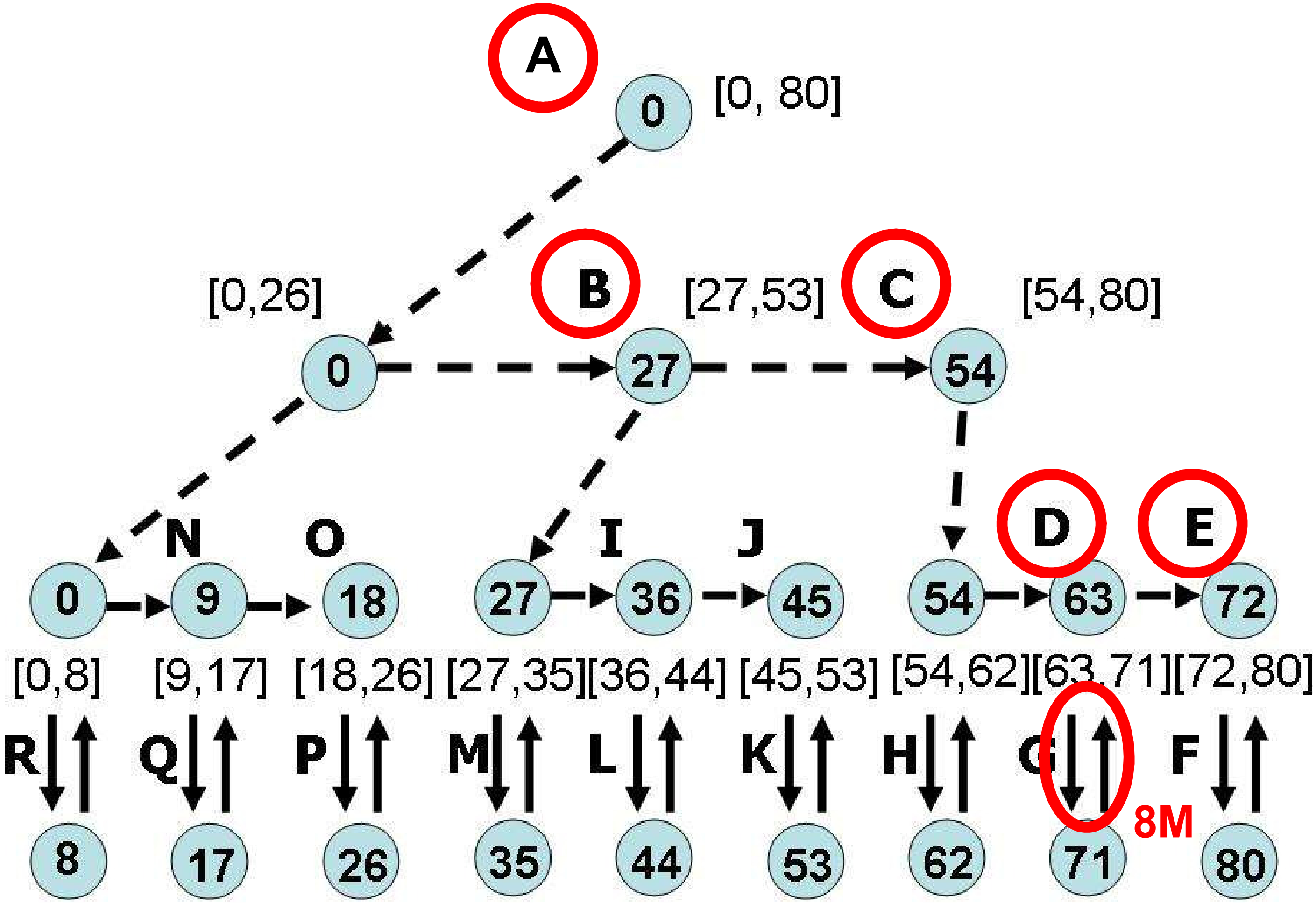

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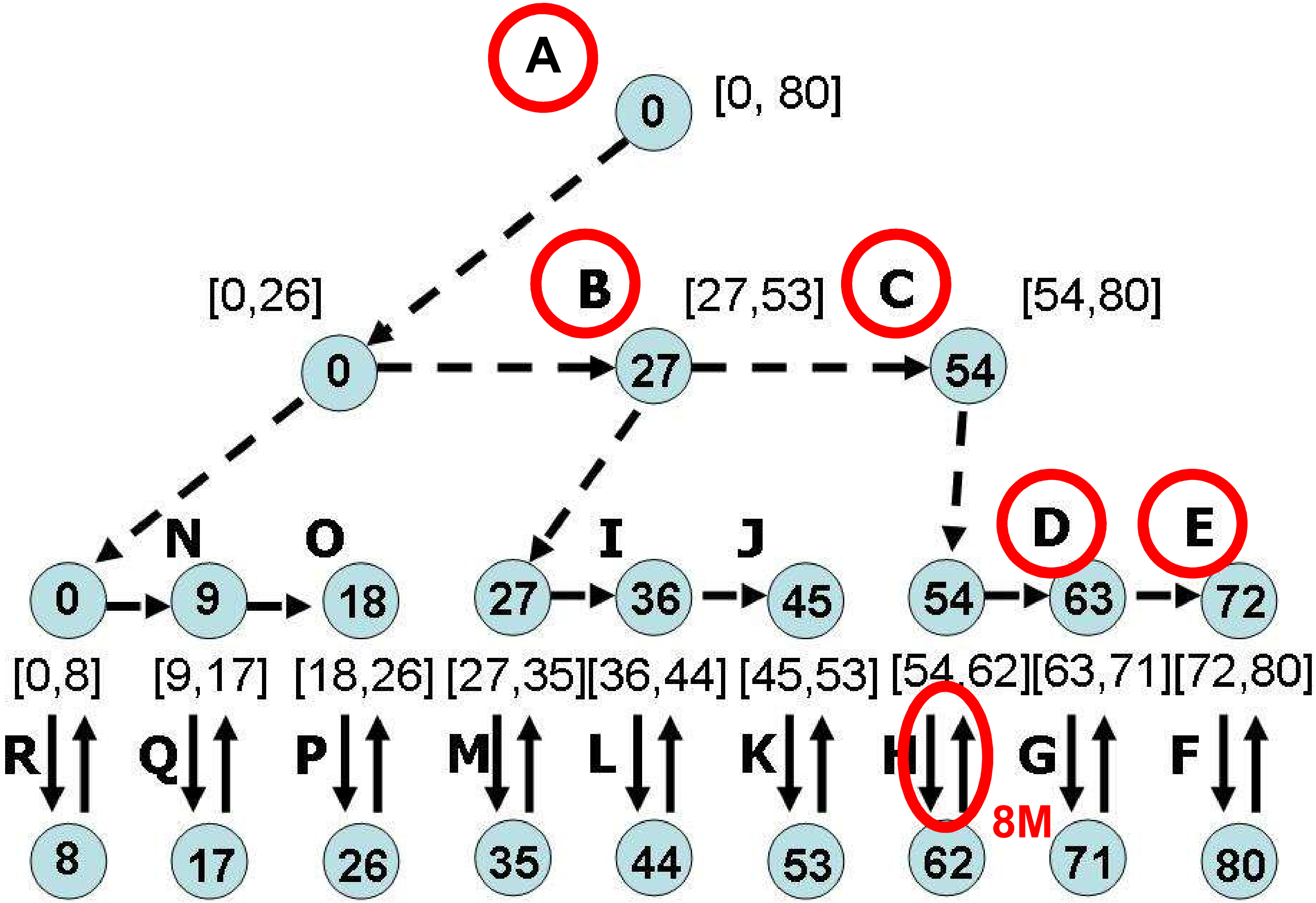
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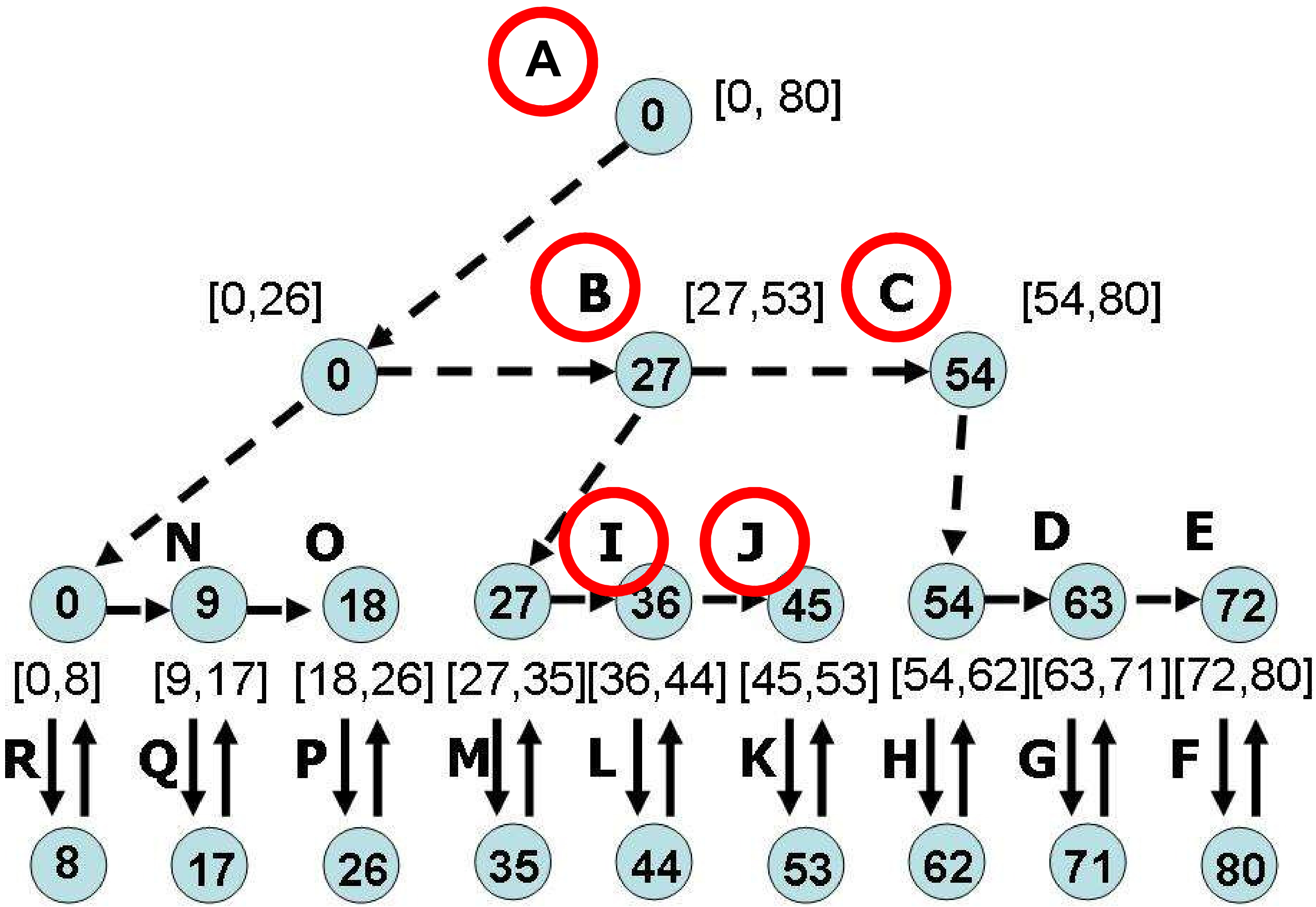

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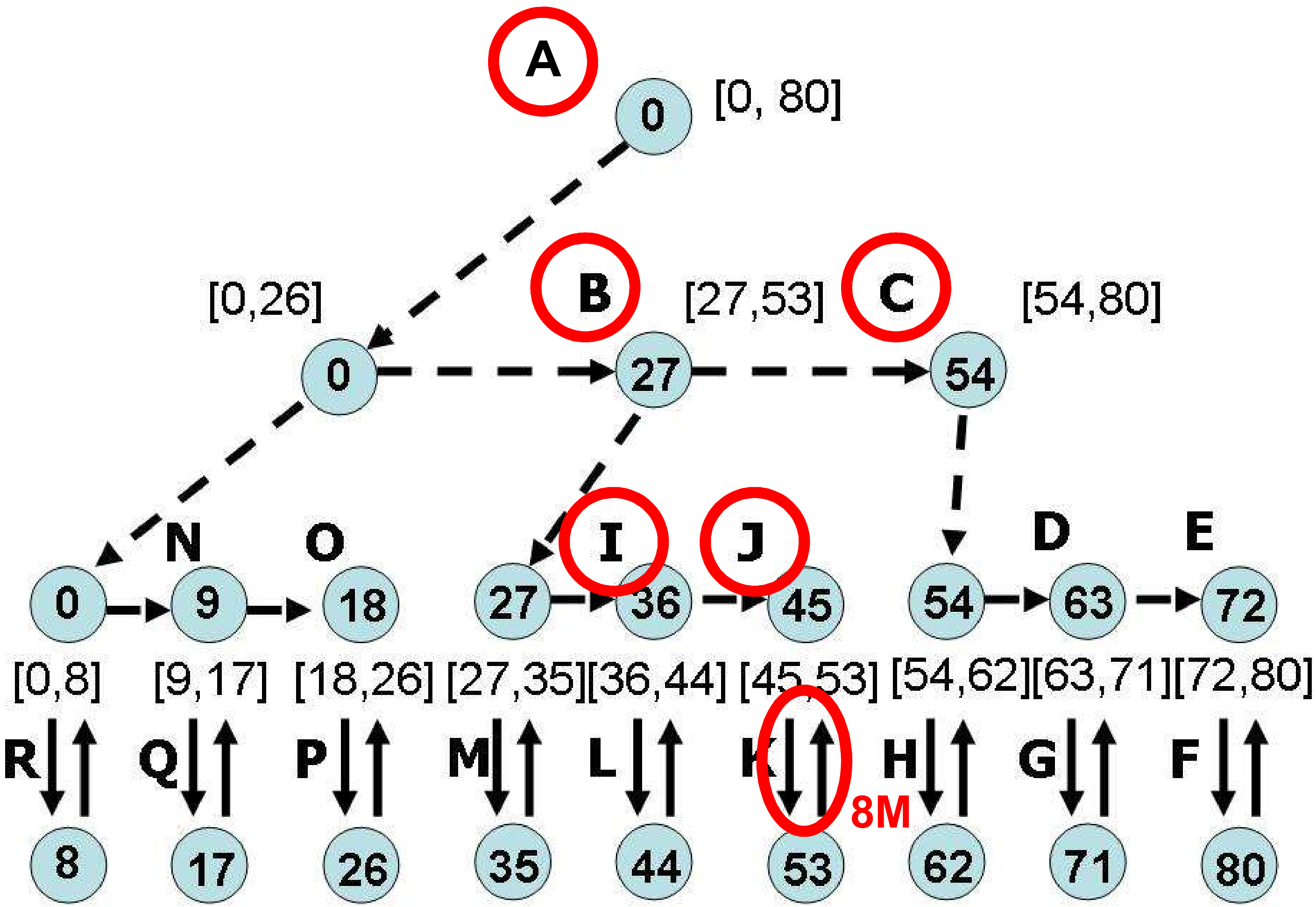
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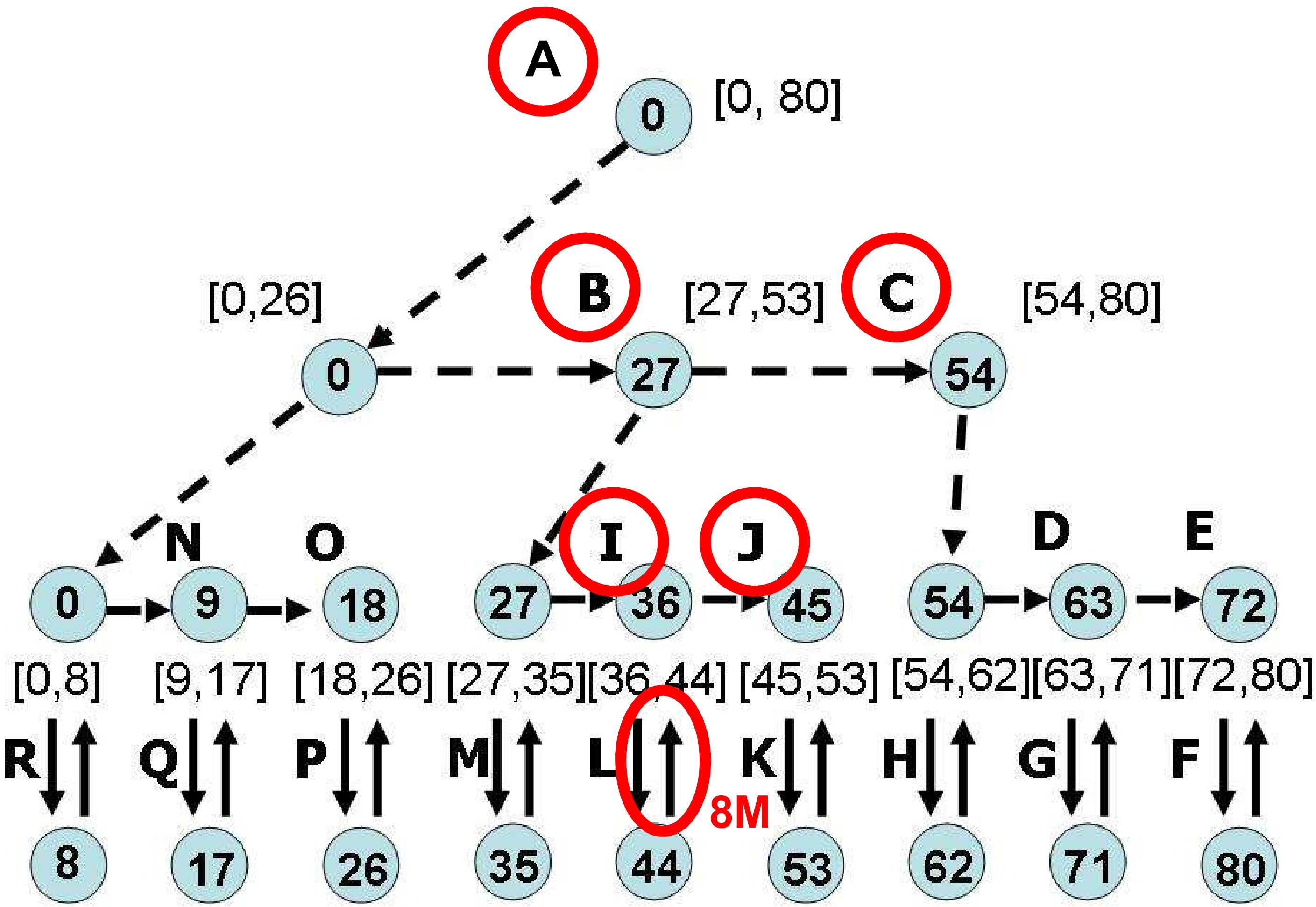

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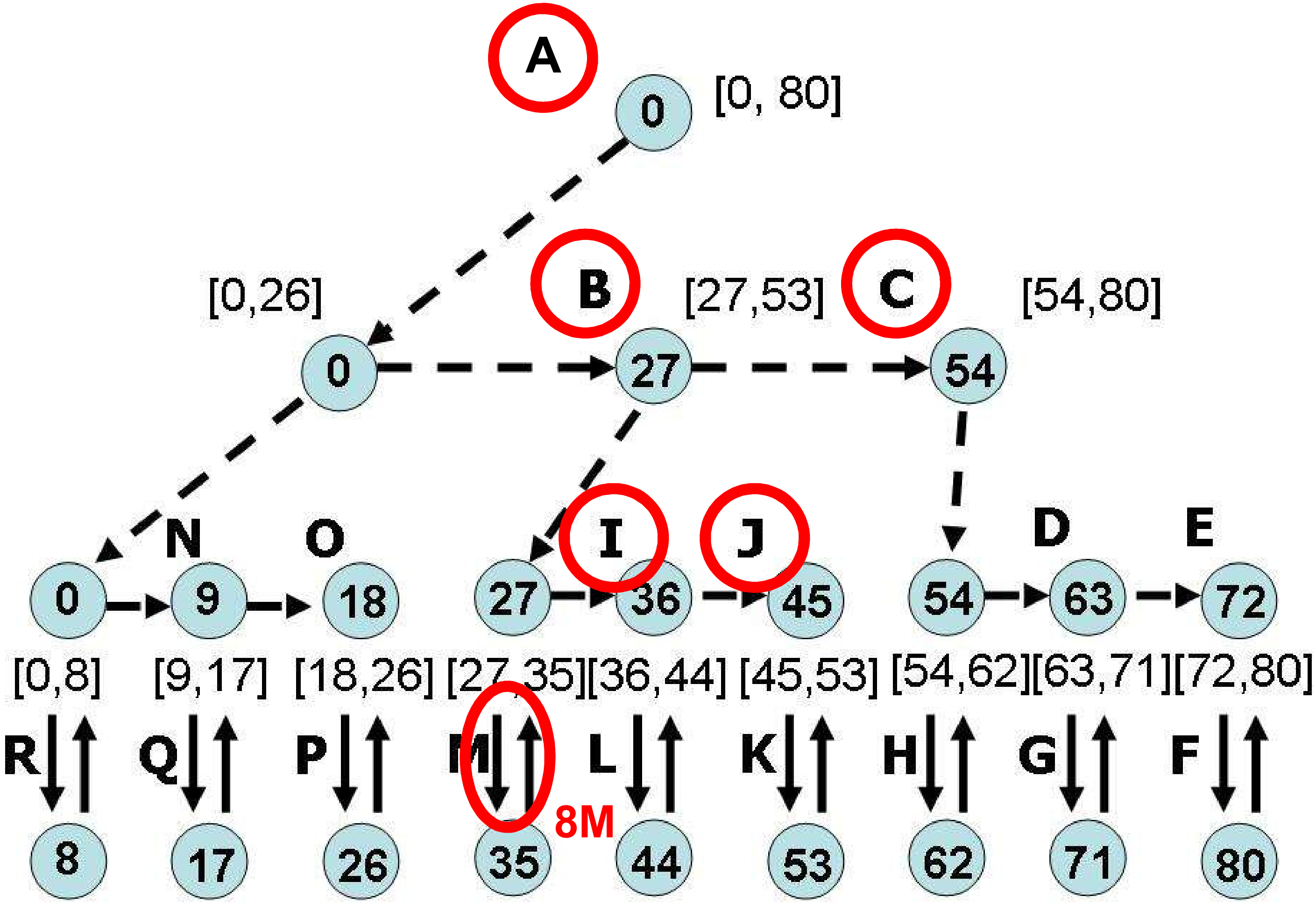
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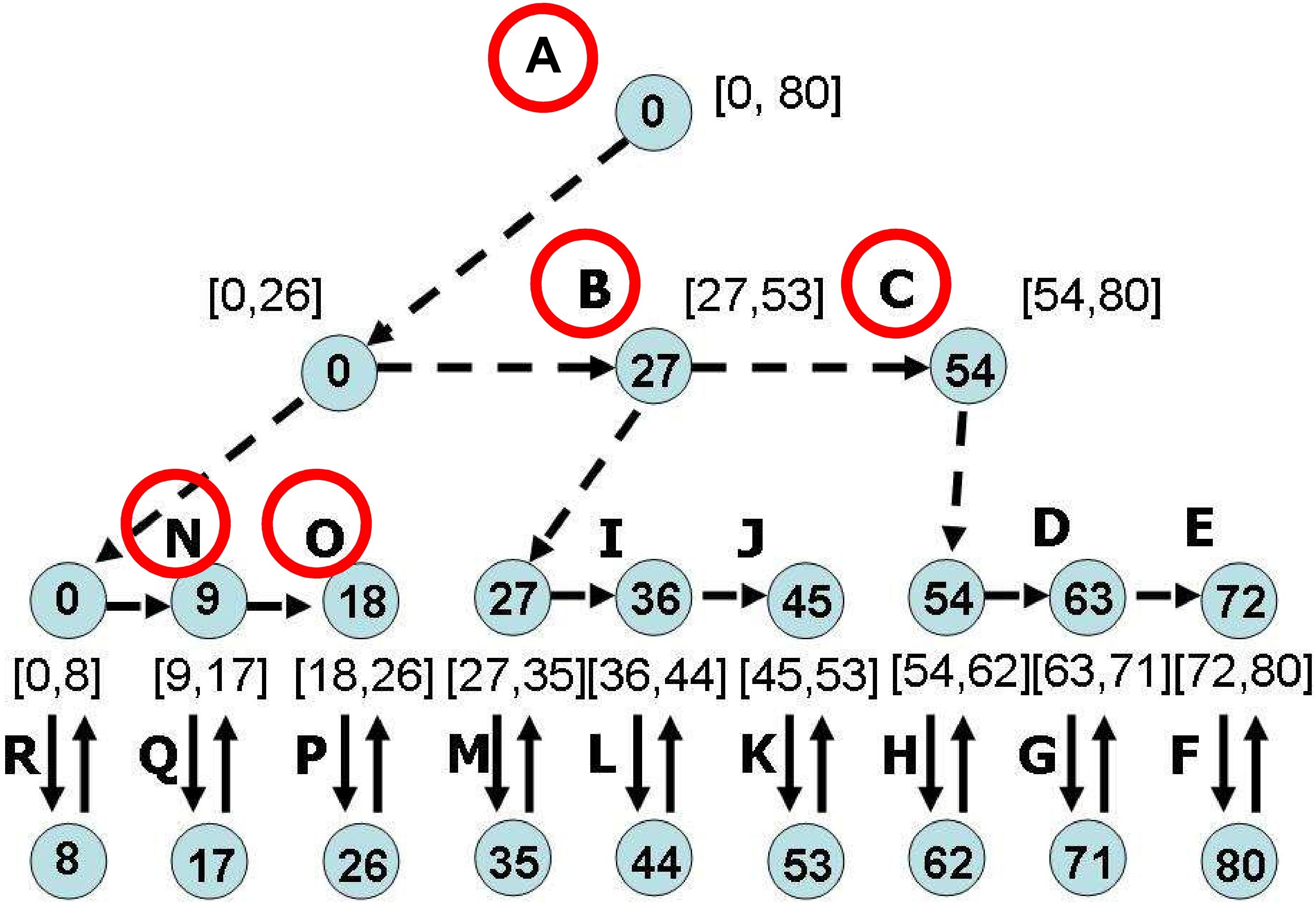
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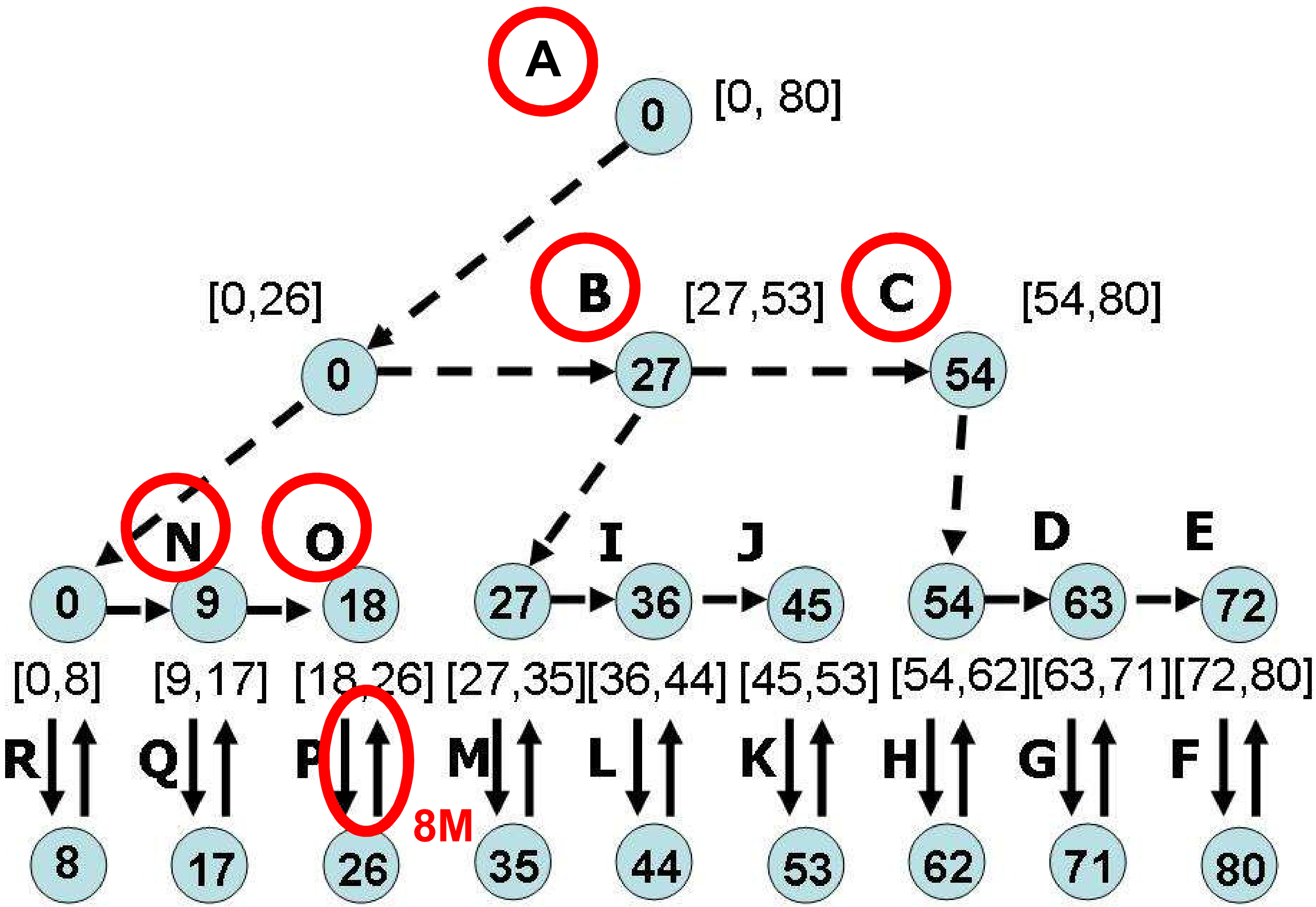
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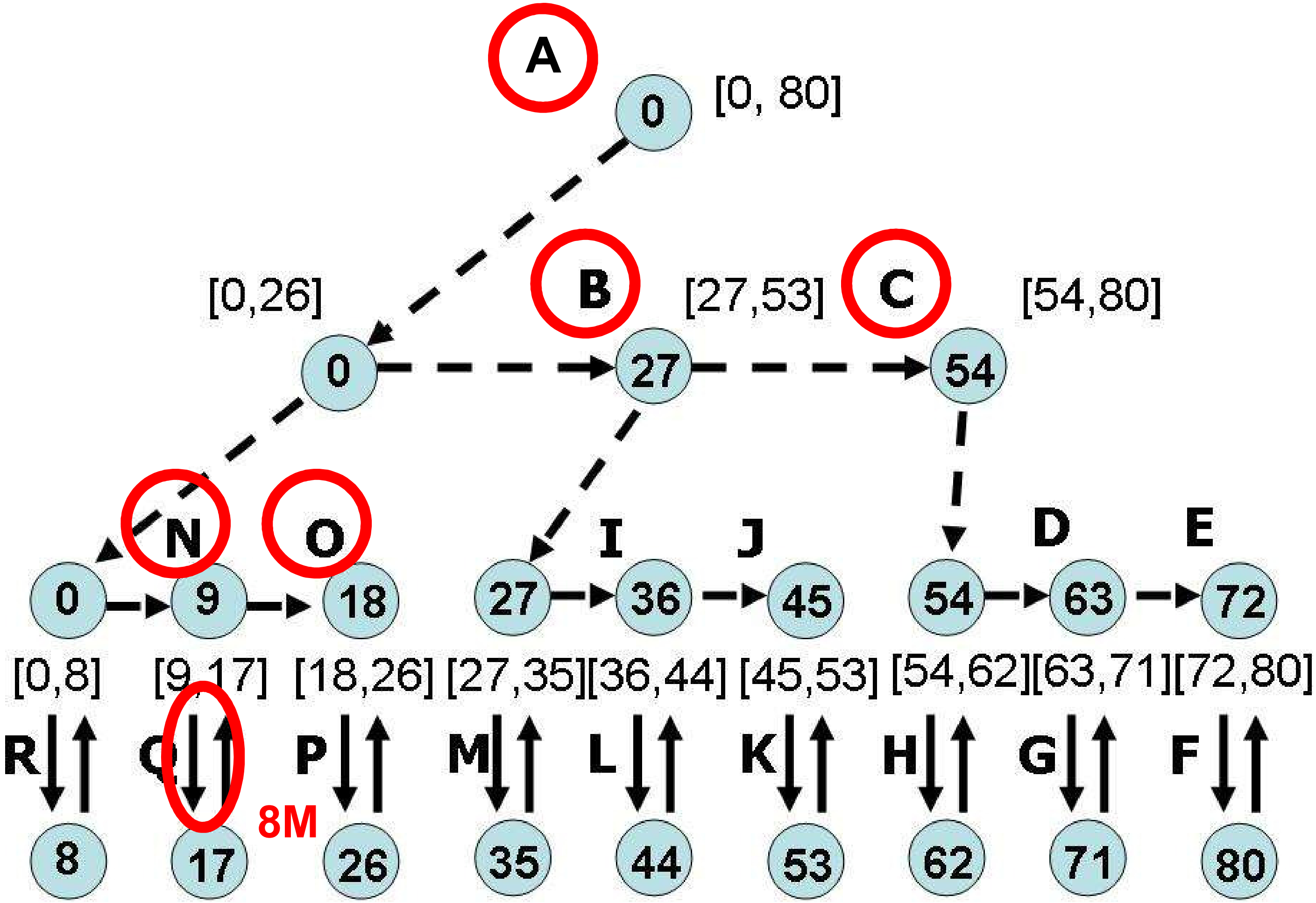
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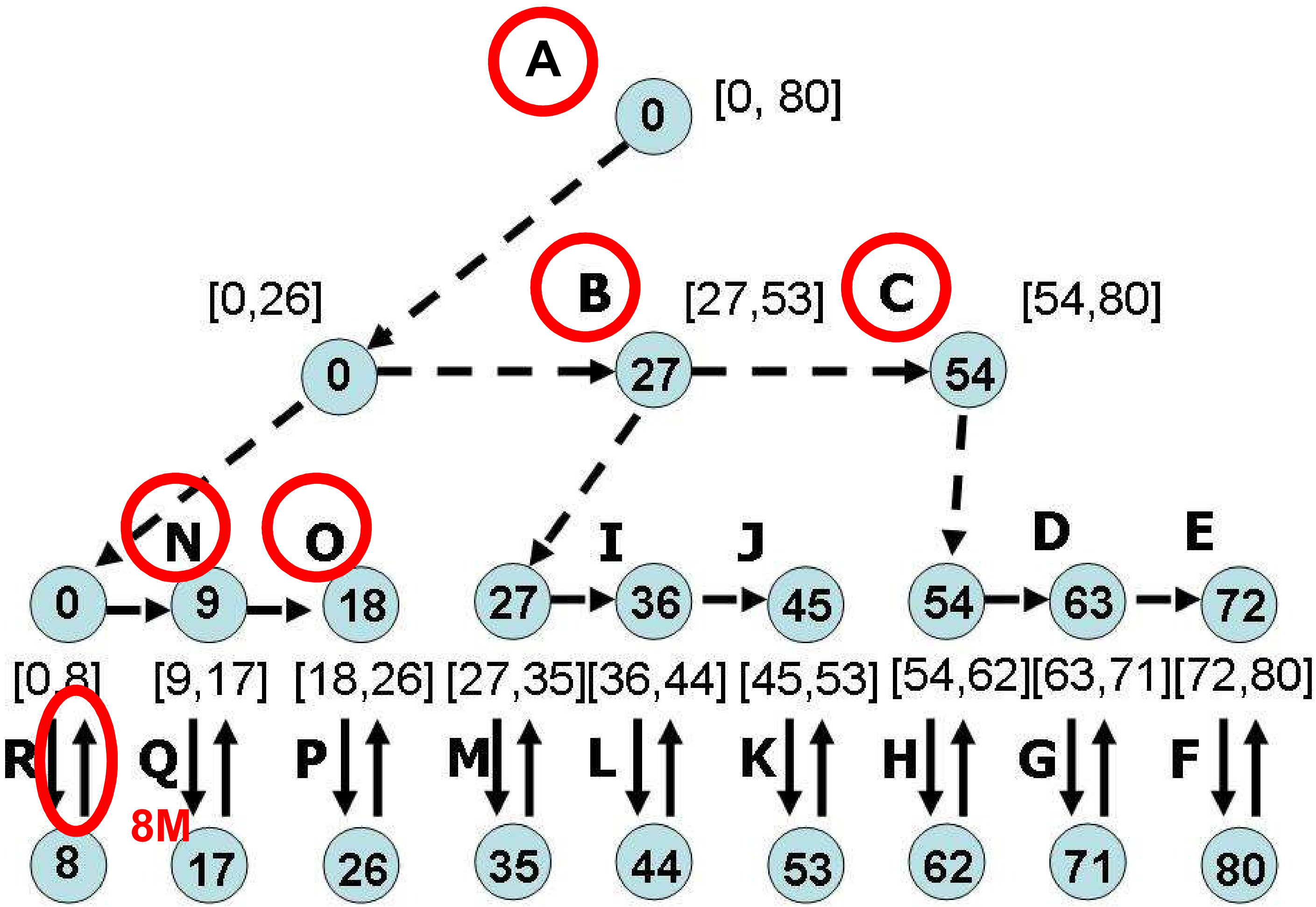
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Recursive Recomputation Scheme

- We are able to compute the smoothing distributions with 5+8 = 13 M storage (instead of 81 or 17).
- The number of whole filtering computation is 3 (or a bit less).

Recursive Recomputation Scheme (Summary)

- We can reduce the space complexity for smoothing from $O(MT)$ to $O(M \log T)$.

  (at the cost of $O(\log T)$ times computation of whole filtering distributions.)

Background (In connection with optimization)

- The idea of saving storage by recomputation was developed in automatic differentiation.
- Specifically, the idea of recursive recomputation was introduced by Griewank in 1992.

Easy Implementation

(Initialization)

- Execute a forward sweep taking snapshots under a certain rule (explained later).
- At the end of the sweep (t=T-1), the set of snapshots snap(T-1) is available.

Easy Implementation

(Reverse sweep with partial forward sweeps)

- If t = 0 then stop
- If not, execute the following procedure.

（We assume that the filtering distribution F(t) has been computed, and the set of snapshots snap(t) is available. We compute F(t-1) and snap(t-1) below.）

Easy Implementation

(To compute F(t-1))

- Execute partial forward sweep from the snapshot which is before t-1 and closest to t-1 to compute F(t-1). On the way, we take snapshots to construct snap(t-1).

Easy Implementation: Rule for snapshots:

- Provide areas snapshot(0), snapshot(1), …, snapshot(K) to store snapshots.
- If k lower bits are zero in the binary representation of t, then we store the snapshot F(t) at snapshot(k).

t=15 (After the forward sweep)

- Snapshot(0): t=(1111)_2 = 15 ←
- Snapshot(1): t=(1110)_2 = 14
- Snapshot(2): t=(1100)_2 = 12
- Snapshot(3): t=(1000)_2 = 8
- Snapshot(*): t=(0000)_2 = 0

t=14

- Snapshot(0):
- Snapshot(1): t=(1110)_2 = 14
- Snapshot(2): t=(1100)_2 = 12
- Snapshot(3): t=(1000)_2 = 8
- Snapshot(*): t=(0000)_2 = 0

t=13 (Recomp. from t=12)

- Snapshot(0): t=(1001)_2 = 13 ⇦
- Snapshot(1):
- Snapshot(2): t=(1100)_2 = 12
- Snapshot(3): t=(1000)_2 = 8
- Snapshot(*): t=(0000)_2 = 0

t=12

- Snapshot(0):
- Snapshot(1):
- Snapshot(2): t=(1100)_2 = 12
- Snapshot(3): t=(1000)_2 = 8
- Snapshot(*): t=(0000)_2 = 0

t=11 (Recomp. from t=8)

- Snapshot(0):
- Snapshot(1):
- Snapshot(2):
- Snapshot(3): t=(1000)_2 = 8
- Snapshot(*): t=(0000)_2 = 0

t=11 (Recomp. from t=8)

- Snapshot(0): t=(1001)_2 =   9
- Snapshot(1):
- Snapshot(2):
- Snapshot(3): t=(1000)_2 =   8
- Snapshot(*): t=(0000)_2 =   0

t=11 (Recomp. from t=8)

- Snapshot(0): t=(1001)_2 =   9
- Snapshot(1): t=(1010)_2 = 10
- Snapshot(2):
- Snapshot(3): t=(1000)_2 =   8
- Snapshot(*): t=(0000)_2 =   0

t=11 (Recomp. from t=8)

- Snapshot(0): t=(1011)_2 = 11
- Snapshot(1): t=(1010)_2 = 10
- Snapshot(2):
- Snapshot(3): t=(1000)_2 = 8
- Snapshot(*): t=(0000)_2 = 0

t=10

- Snapshot(0):
- Snapshot(1): t=(1010)_2 = 10
- Snapshot(2):
- Snapshot(3): t=(1000)_2 = 8
- Snapshot(*): t=(0000)_2 = 0

t=9 (Recomp. from t=8)

- Snapshot(0): t=(1001)_2 = 9
- Snapshot(1):
- Snapshot(2):
- Snapshot(3): t=(1000)_2 = 8
- Snapshot(*): t=(0000)_2 = 0

t=8

- Snapshot(0):
- Snapshot(1):
- Snapshot(2):
- Snapshot(3): t=(1000)_2 = 8 ⇐
- Snapshot(*): t=(0000)_2 = 0

t=7 (Recomp. from t=0)

- Snapshot(0): t=(0111)_2 = 7 ⇐
- Snapshot(1): t=(0110)_2 = 6
- Snapshot(2): t=(0100)_2 = 4
- Snapshot(3):
- Snapshot(*): t=(0000)_2 = 0

Application

- We applied the new method to smooth the Nikkei 225 stock price data with a stochastic volatility model with particle filters.
- We used the path-sampling smoothing algorithm of the particle filter by Kitagawa.

Path-sampling smoother

- Let N be the number of particles.

State Space Model

Innovation of state

$$x_{t+1} = f(x_t) + \varepsilon_t$$

$$y_t = g(x_t) + \eta_t$$

Observation

η_t, ε_t : (Non Gaussian) white noize

Path-sampling Smoother

- Traces the parents-children relation of particles.
- O(N) operations per time step.
- N can be millions (in view of computation time), but the limit of N comes from storage.
- Suffers from degeneracy.

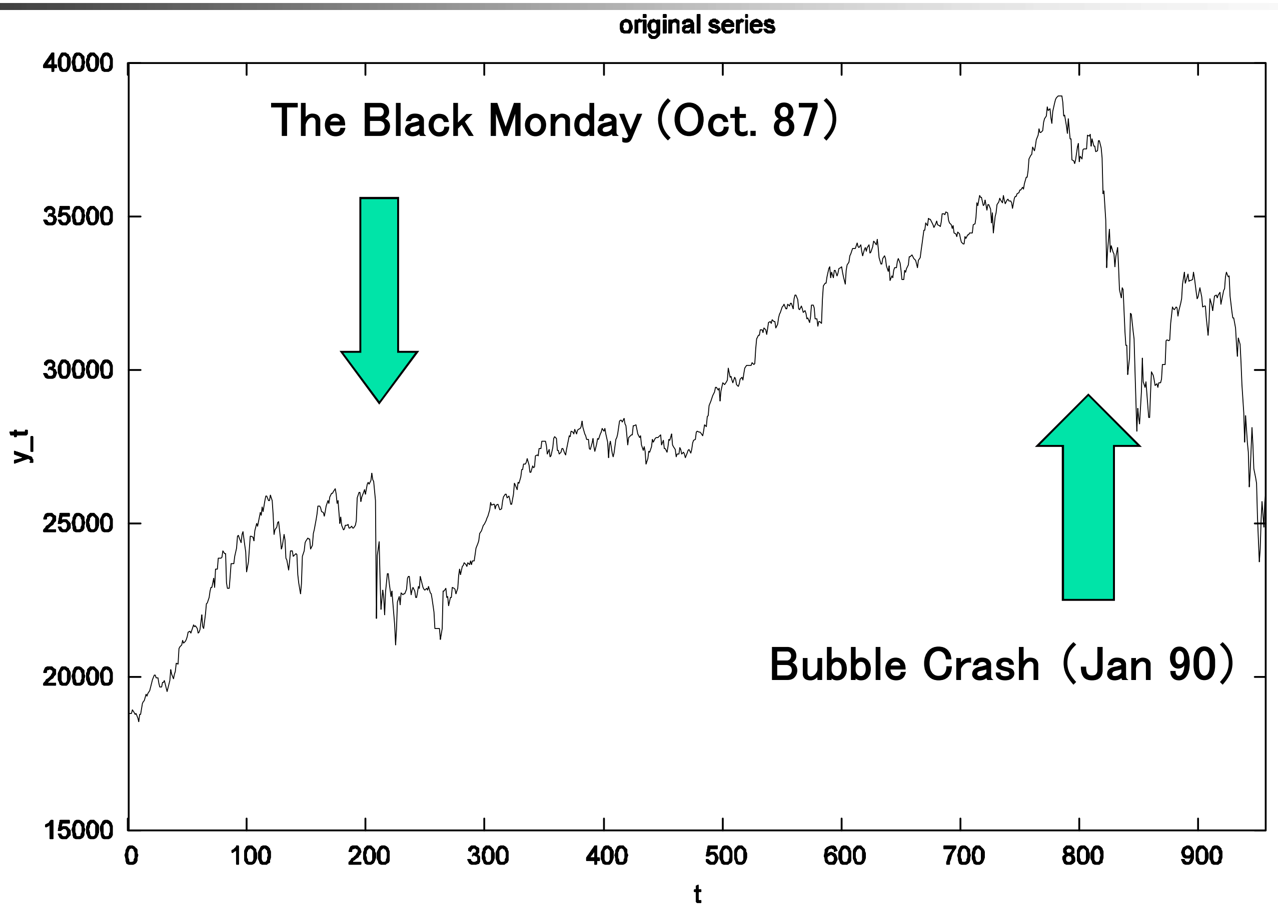
Path-sampling smoother

- What is degeneracy?

  The number of particles representing the smoothing distribution reduces quickly as t is traced back.

  (if you start with numerous and diverse particles, the particles can easily reduces 10 – 20 after a while.)

Nikkei 225 Data (From Jan. 87 to Aug. 90)



Model

Trend

$$x_{t+1} = -2x_t + x_{t-1} + \varepsilon_t,$$

$$y_t = x_t + \eta_t,$$

Stock price

Volatility

$$\varepsilon_t \sim \mathrm{Cauchy}(0, \tau_1)$$

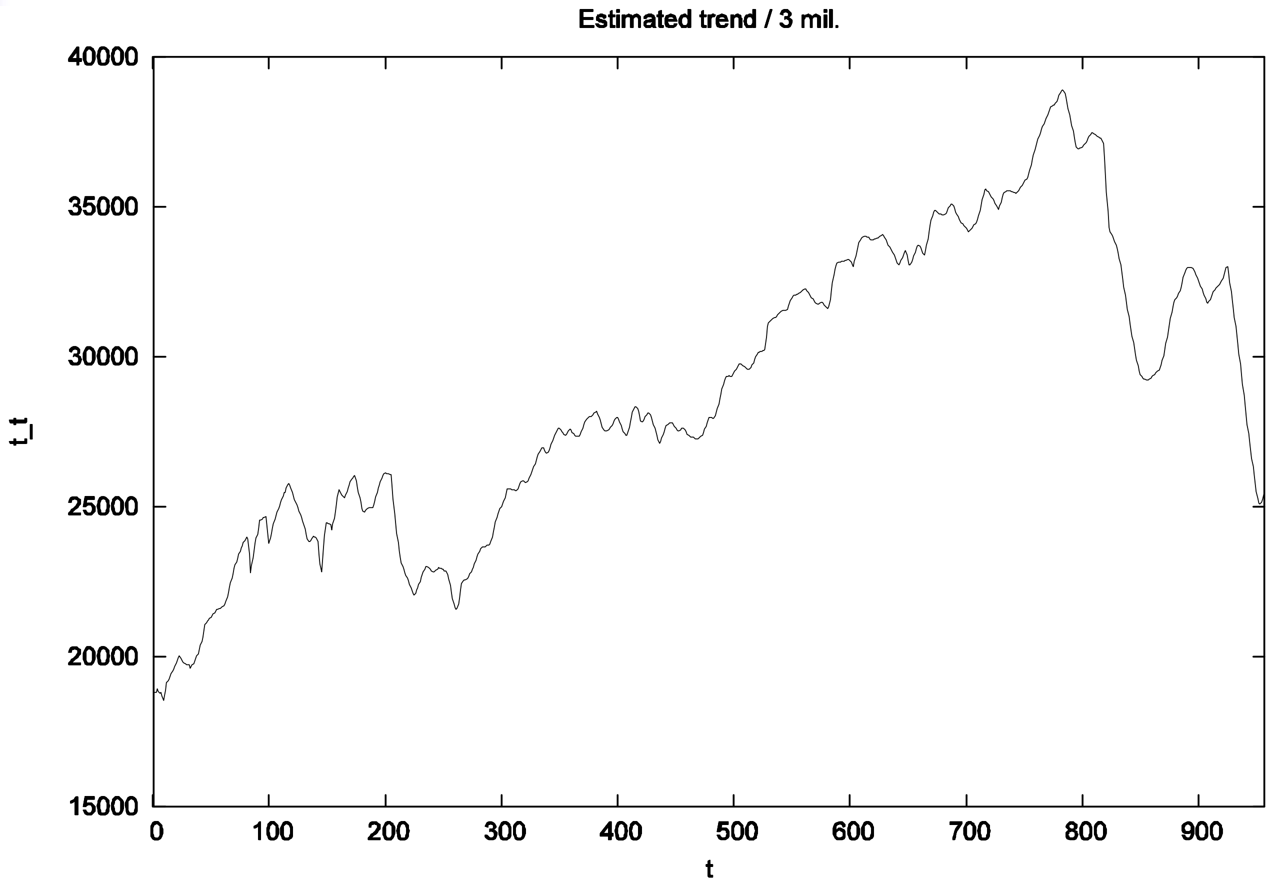
$$\eta_t \sim \mathrm{Normal}(0, \exp(p_t))$$

Model (II)

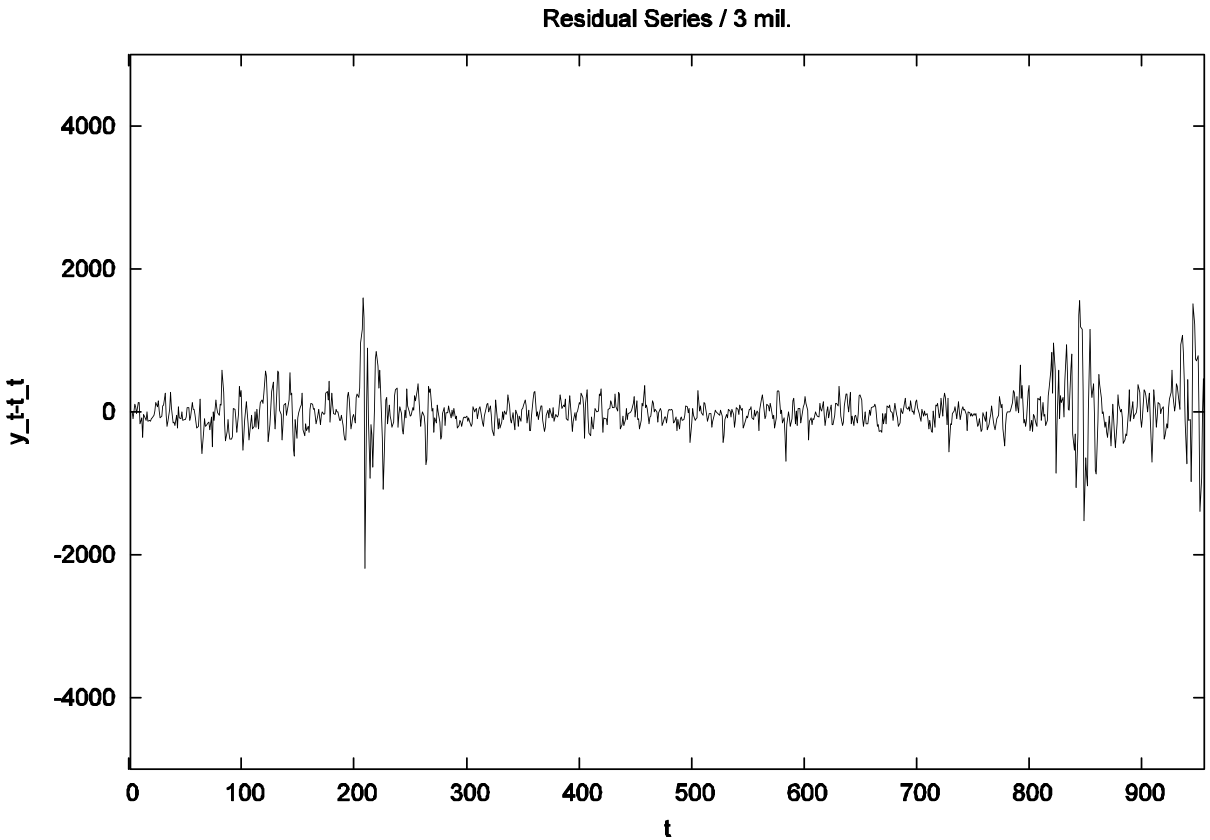
$$p_{t+1} = -2p_t + p_{t-1} + \delta_t,$$

$$\delta_t \sim \mathrm{Normal}(0, \tau_2)$$

Smoothed Trend (median)



Residual (Volatility is time-varying variance of the sequence)

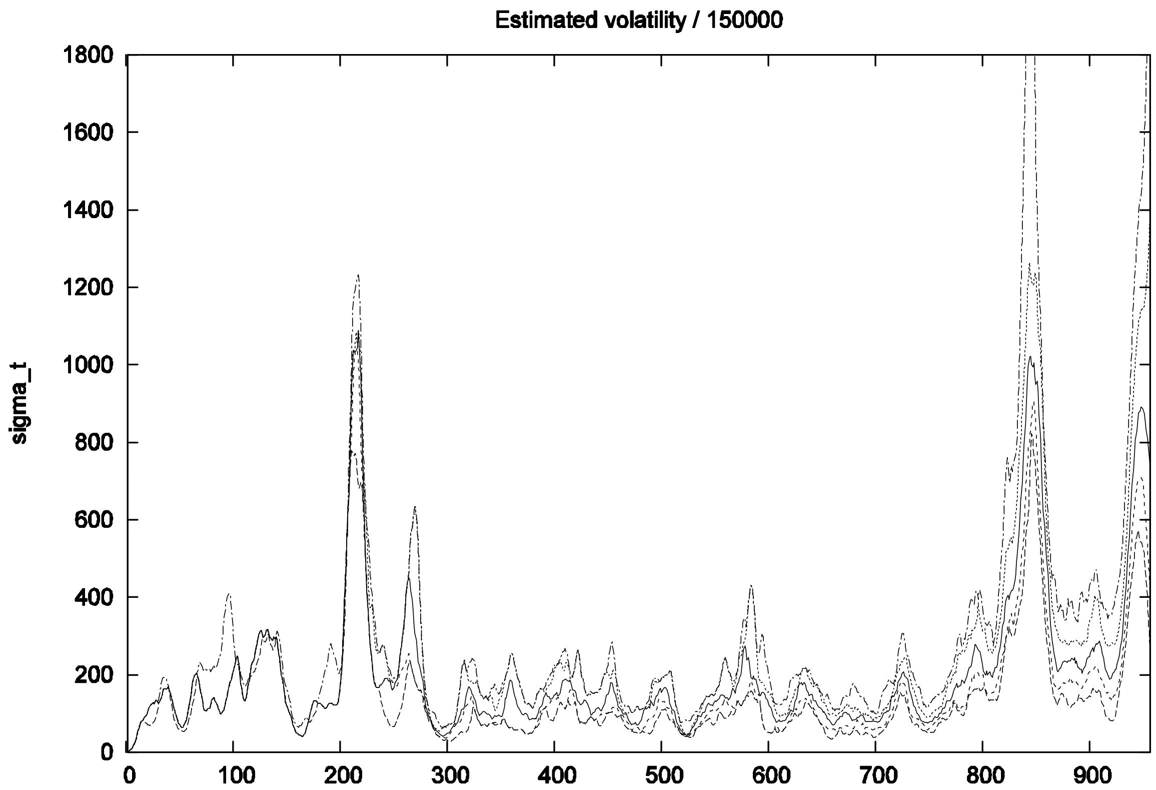


Smoothing with the Particle Filter

- Computer SGI Altix3700 (256CPU,Intel Itanium2 1.3GHz, Main Memory 1920GB)
- 1CPU, 5GB
- 150,000 particles can be used with smoothing by ordinary implementation.

(Comparable with high-end PC environment)

The smoothing distributions of volatility (150,000 particles)

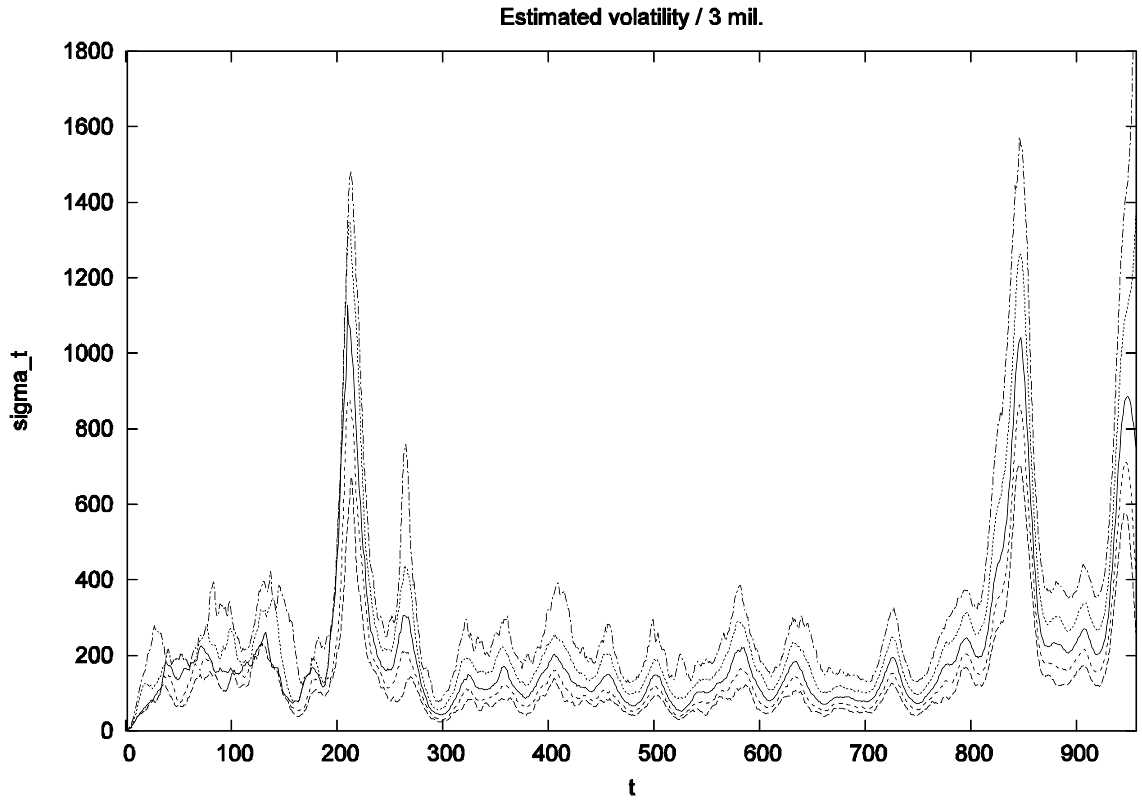


Each curve shows 2.3%, 15.9%, 50%(solid line), 84.1%, 97.7% points.

Smoothing with the particle filter

- 3,000,000 particles (20 times more than the ordinary implementation) can be used for smoothing with the recursive recomputation scheme.

Smoothing distributions of volatility (3 mil particles)



Each curve shows 2.3%, 15.9%, 50%(solid line), 84.1%, 97.7% points.

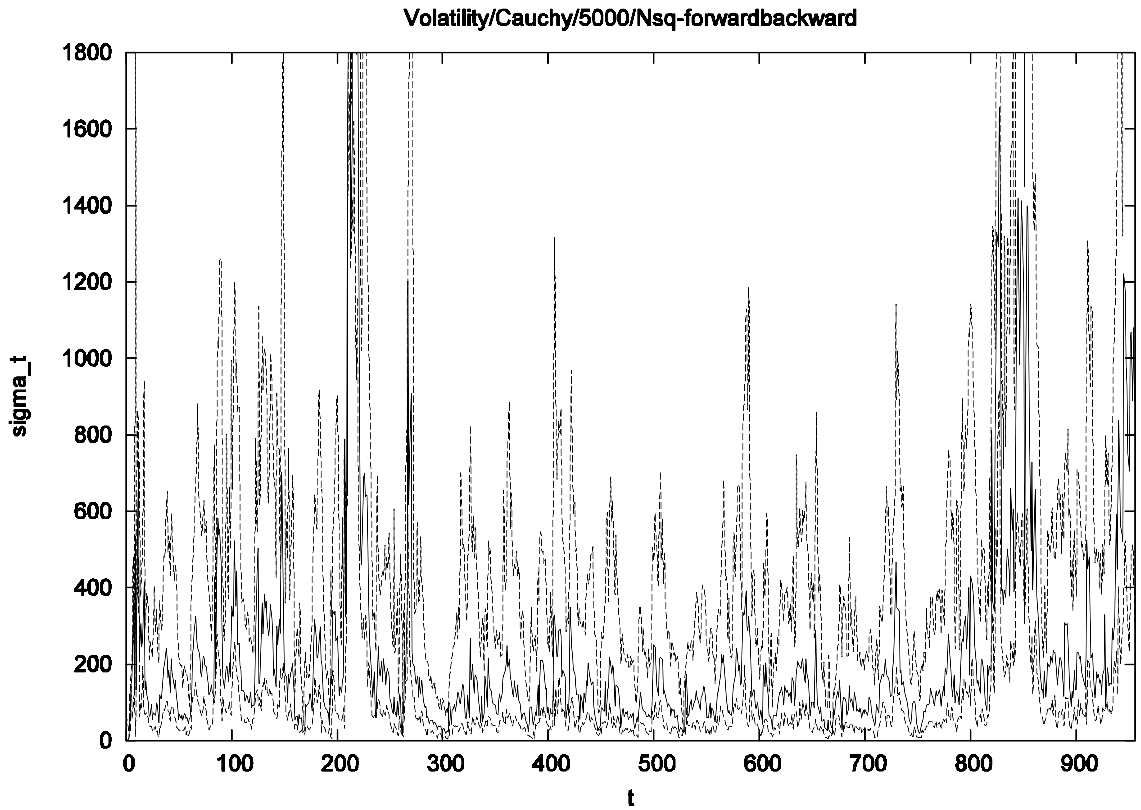
Comparison with other "standard particle smoothers"

- Forward-backward smoother
- Two-filter smoother
- These methods require $O(N^2)$ operations per time step. We cannot increase N more than 5000. (Since N is modest, there is no need for saving storage.)

Comparison with other "standard particle smoothers"

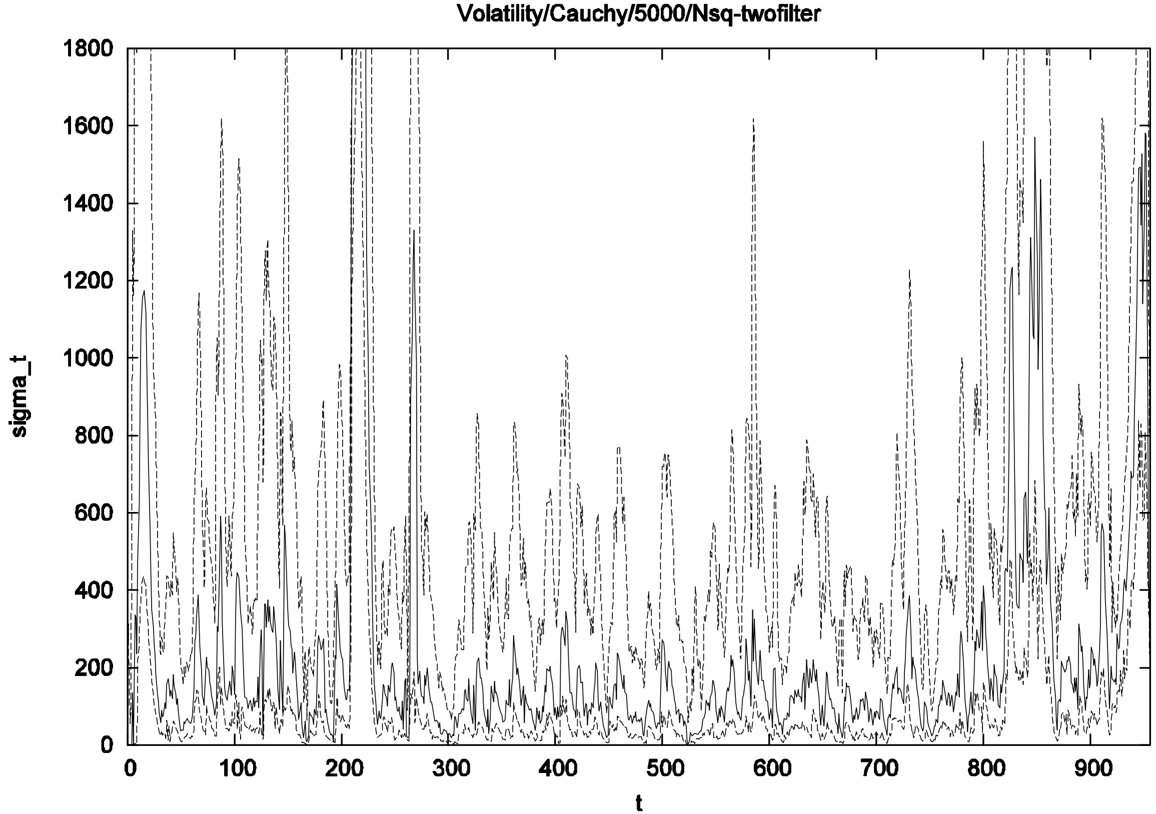
- Path-sampling Method is O(N) per step but suffers from degeneracy.
- In the following we compare the smoothing distributions computed in 7hours with N= 5000 (for the two $O(N^2)$ methods) and N=3000000 (for path-sampling method).

Forward-Backward Method



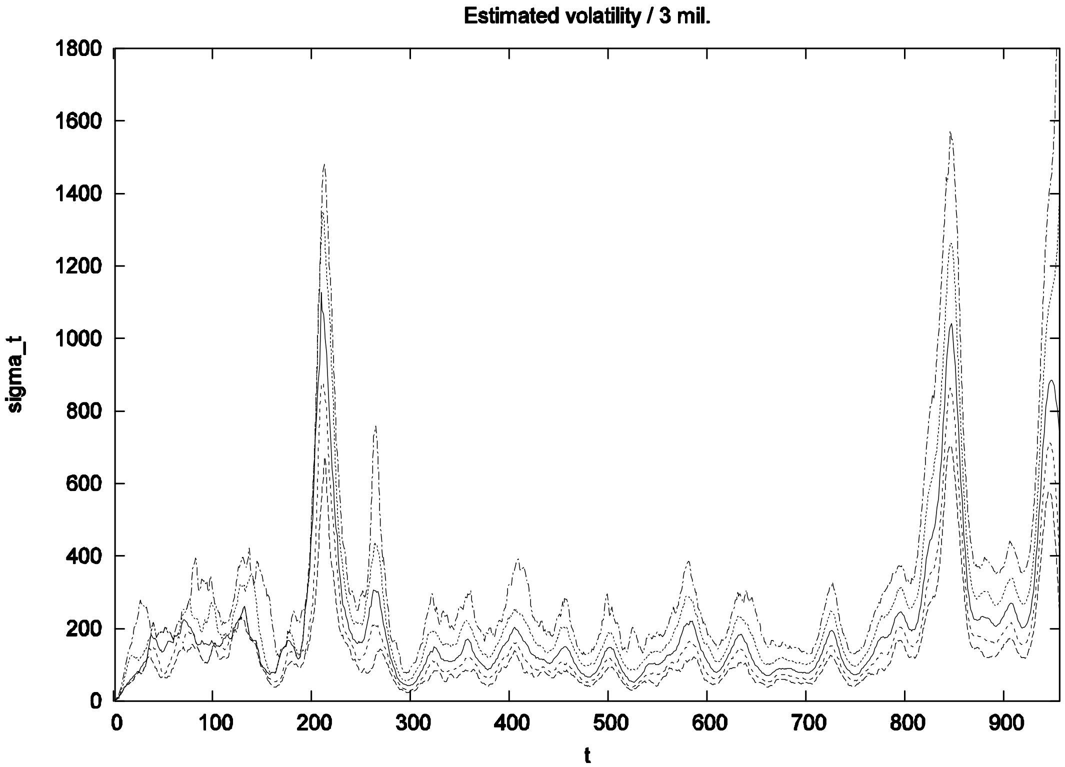
Each curve shows 2.3%, 50%(solid line), 97.7% points.

2 Filter Formulae



Each curve shows 2.3%, 50%(solid line), 97.7% points.

The New Method



Each curve shows 2.3%, 15.9%, 50%(solid line), 84.1%, 97.7% points.

Conclusion

- The path-sampling smoother with numerous particles performs the best. Using numerous particles helps.
- It is possible to extend the technique to fixed-lag smoothers.