Abs-Linearization for Piecewise Smooth Optimization

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Outline

Piecewise Smooth (PS) Functions

Definition (Piecewise Smoothness, Piecewise Linearity)

Let $\mathcal{D} \subseteq \mathbb{R}^n$ be open and $f_i : \mathcal{D} \to \mathbb{R}^m, i = 1, \ldots, k$ with $k \in \mathbb{N}$ be given.

- $f : \mathcal{D} \to \mathbb{R}^m$ is called a continuous selection of the collection $f_1, \ldots, f_k$ on the set $U \subseteq \mathcal{D}$ if f is continuous on U and
$$f(x) \in \{f_1(x), \ldots, f_k(x)\} \qquad \forall x \in U.$$

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- $f : \mathcal{D} \to \mathbb{R}^m$ is called PC^r-function with $r \in \mathbb{N} \cup \{\infty\}$ if for every $x \in \mathcal{D}$ there exists an open neighboorhood $U \subseteq \mathcal{D}$ and a finite number of C^r-functions $f_i : U \to \mathbb{R}^m$, $i = 1, \ldots, k$, such that f is a continuous selection of $f_1, \ldots, f_k$ on U.

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- A PC^r-function with $r \geq 1$ is also called piecewise smooth.
- A continuous selection $f : U \to \mathbb{R}^m$ is called piecewise linear if all elements of the collection $f_1, \ldots, f_k$ are affine functions.

S. Scholtes: Introduction to Piecewise Differentiable Equations, Springer, 2012

Piecewise Smooth Example Problems

Exact ℓ_1 penalty functions
Constrained optimization problem

$$\min_x f(x) \quad \text{s.t.} \quad c_i(x) = 0,\ i \in \mathcal{E}, \quad c_i(x) \geq 0,\ i \in \mathcal{I}$$

equivalent to unconstrained optimization problem with ℓ_1-penalty

$$\phi(x;\mu) = f(x) + \mu \sum_{i\in\mathcal{E}} |c_i(x)| + \mu \sum_{i\in\mathcal{I}} \max\{0, -c_i(x)\}$$

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Robust Optimization

Often formulated as min-max problems

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Train timetabling

yields piecewise linear optimization problem

F. Fischer, C. Helmberg: Dynamic Graph Generation and Dynamic Rolling Horizon Techniques in Large Scale Train Timetabling, 2010

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(together with Eyke Hüllermeier, Uni Pb)
= model class for classification and regression in machine learning

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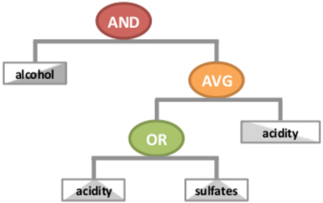

Application: Determine wine quality

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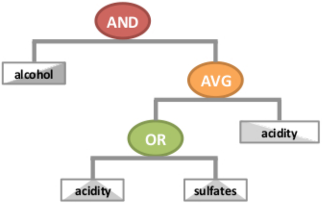

Application: Determine wine quality via a target function defined by

$$(\theta^*, \gamma^*, \sigma^*, c^*) = \text{argmin}_{\theta,\gamma,\sigma,c} \sum_{i=1}^{N} (F_{\theta,\gamma,\sigma,c}(\mathbf{x}_i) - y_i)^2 \quad \text{with}$$

$$F_{\theta,\gamma,\sigma,c}(x) = T_\theta\,(\mu_{c_1}(x_{11}), C_\gamma(S_\sigma(\mu_{c_2}(x_2), \mu_{c_3}(x_{10})), \mu_{c_4}(x_2))$$

Fuzzy Pattern Tree II

Here:

$$\mu_{c_i}(x) = \begin{cases} \frac{x}{c_i} & \text{if } 0 \leq x \leq c_i \\ \frac{1-x}{1-c_i} & \text{if } c_i \leq x \leq 1 \end{cases} \qquad \text{allow non-monotonicity}$$

$$T_\theta(u,v) = \frac{u\,v}{\max\{u,v,\theta\}} \qquad = \text{Dubois-Prade family}$$

$$S_\sigma(u,v) = 1 - T_\sigma(1-u, 1-v) \qquad = \text{corr. dual t-conorm}$$

$$C_\gamma(u,v) = \begin{cases} \gamma u + (1-\gamma)v & \text{if } u > v \\ (1-\gamma)u + \gamma v & \text{if } u \leq v \end{cases} \qquad = \text{ordered weighted operator}$$

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$\Rightarrow$ Piecewise smooth target function

$$(\theta^*, \gamma^*, \sigma^*, c^*) = \text{argmin}_{\theta,\gamma,\sigma,c} \sum_{i=1}^{N} (F_{\theta,\gamma,\sigma,c}(\mathbf{x}_i) - y_i)^2 \quad \text{with}$$

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Optimality Conditions

Generalized derivative concept required:

- directional derivative
- Clarke generalized gradient

$$\partial_C \varphi(x) := \operatorname{conv}\left\{ \lim_{i\to\infty} \nabla\varphi(x_i) : x_i \mapsto x, \nabla\varphi(x_i) \text{ exists} \right\} = \operatorname{conv}\{\partial^L \varphi(x)\}$$

F. Clarke: Optimization and Nonsmooth Analysis, SIAM, 1990

- Mordukhovich subgradient $\partial_M \varphi(x)$

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- Clarke stationarity: $0 \in \partial_C \varphi(x)$? $\partial_C(|x|) = \partial_C(-|x|)$!
- a little stronger: Mordukhovich stationarity: $0 \in \partial_M \varphi(x)$

Current (= Black Box) Approaches

- Use methods for smooth problems
  May fail, no convergence theory
- Subgradient method
  Very (!) slow convergence
- Bundle methods
  Lots of parameters, erratic convergence behaviour
  involves oracle
- Derivative-free methods
  No structure exploitation,
  difficult when number of optimization variables large

Hierarchy of Problems

locally Lipschitz continuous (LL)

$\cup$

piecewise smooth (PS)

$\cup$

piecewise linear (PL)

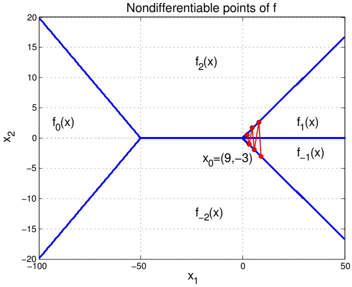
$\cup$

piecewise linear and convex (PL+C)

Observations

Solving min $\varphi(x)$ with φ PL+C not easy:

- Global minimization is NP-hard
- Steepest descent with exact line search may fail
- Zeno behaviour possible, i.e., solution trajectory with infinite number of direction changes in a finite amount of time



J.-B. Hiriart-Urruty, C. Lemaréchal: Convex Analysis and Minimization Algorithms I, Springer, 1993

New (= Gray Box) Approach

Goal: Locate stationary (?!) point of piecewise smooth function $\varphi(.)$ by

- successive approximation by piecewise linear (PL) models and
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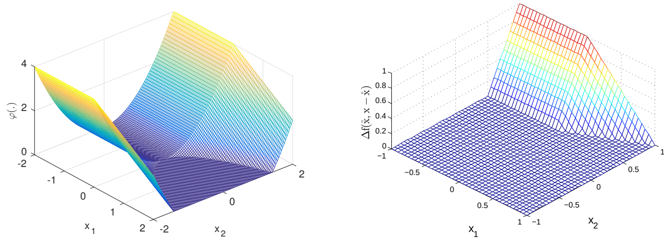
Example: Half-Pipe function

$$\varphi : \mathbb{R}^2 \mapsto \mathbb{R}, \quad \varphi(x) = \max\{x_2^2 - \max\{x_1, 0\}, 0\}$$



Nonlinear function $\varphi(.)$ and its piecewise linearization at $\mathring{x} = (1, 1)$

Abs-Linearisation I

Given: Target function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ piecewise smooth

Assumption: Non-smoothness caused by univariate piecewise linear elements like min, max or abs!

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For example:

$$\begin{aligned}\varphi(x) &= \min_{x\in\mathbb{R}^n} \max_{1\leq i\leq m} f_i(x)\\ &= \text{min max regret problem}\end{aligned}$$

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Then: φ can be written using switching variables

$$z_i, \quad i = 1, \ldots, s$$

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Hence:

Definition (Abs-normal form of PS function $\varphi : \mathbb{R}^n \to \mathbb{R}$)

$$F : \mathbb{R}^{n+s} \to \mathbb{R}^s, \quad z = F(x, |z|)$$
$$f : \mathbb{R}^{n+s} \to \mathbb{R}, \quad y = f(x, |z|) = \varphi(x)$$

with F and f at least twice differentiable.

Abs-Linearisation II

Defining

$$
\begin{array}{ll}
L = \frac{\partial}{\partial |z|} F(x, |z|) \in \mathbb{R}^{s \times s} & \text{strictly lower triangular} \\
Z = \frac{\partial}{\partial x} F(x, |z|) \in \mathbb{R}^{s \times n} & a = \frac{\partial}{\partial x} f(x, |z|) \in \mathbb{R}^{n}, \quad b = \frac{\partial}{\partial |z|} f(x, |z|) \in \mathbb{R}^{s}
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one obtains

Definition (Abs-linear form of abs-normal $\varphi : \mathbb{R}^n \to \mathbb{R}$ in x)

$$\begin{bmatrix} z \\ \Delta y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} Z & L \\ a & b \end{bmatrix} \begin{bmatrix} \Delta x \\ \Sigma \cdot z \end{bmatrix} \qquad \text{with}$$

$$c_1 \in \mathbb{R}^s, c_2 \in \mathbb{R}, \quad \sigma = \sigma(x) \equiv \mathbf{sign}(z(x)) \in \{-1,0,1\}^s, \Sigma \equiv \operatorname{diag}(\sigma)$$

as piecewise linearisation $\Delta\varphi$ of φ in x.

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Abs-normal form can be generated using appropriate variant of AD!

Example: Nesterov-Rosenbrock Function

Smooth variant:

$$\varphi_0(x) = \frac{1}{4}(x_1 - 1)^2 + \sum_{i=1}^{n-1}(x_{i+1} - 2x_i^2 + 1)^2$$

Example: Nesterov-Rosenbrock Function

PS variant:

$$\varphi_1(x) = \frac{1}{4}(x_1 - 1)^2 + \sum_{i=1}^{n-1} \left| x_{i+1} - 2x_i^2 + 1 \right|$$

M. Gürbüzbalaban, M.L. Overton: On Nesterov's nonsmooth Chebyshev–Rosenbrock functions, Nonlinear Anal: Theory, Methods Appl., 2012

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Abs-normal form:

$$z_i = F_i(x, |z|) = x_{i+1} - 2x_i^2 + 1, \qquad 1 \leq i \leq n-1,$$

$$y = f(x, |z|) = \frac{1}{4}(x_1 - 1)^2 + \sum_{i=1}^{n-1} |z_i| \qquad \Rightarrow$$

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$$Z = \begin{bmatrix} -4\,x_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -4\,x_2 & 1 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & -4\,x_s & 1 \end{bmatrix} \in \mathbb{R}^{(n-1)\times n}$$

$$L = 0 \in \mathbb{R}^{(n-1)\times(n-1)}, \quad a = \left(\frac{(x_1 - 1)}{2}, 0, \dots, 0 \right) \in \mathbb{R}^n, \quad b = \mathbb{1} \in \mathbb{R}^{n-1}$$

Open Questions I

Gap between class of abs-normal functions and PS functions?

Original Evaluation Procedure

For smooth functions, AD is based on

v_{i-n}	$=$	x_i	$i = 1 \dots n$
v_i	$=$	$\varphi_i(v_j)_{j \prec i}$	$i = 1 \dots l$
y	$=$	v_l	

Adapted Evaluation Procedure

For abs-normal functions, consider

$$
\begin{array}{|lcl|c|}
\hline
v_{i-n} & = & x_i & i = 1 \dots n \\
\hline
\left.\begin{array}{lcl} z_i & = & \psi_i(v_j)_{j \prec i} \\ \sigma_i & = & \mathrm{sign}(z_i) \\ v_i & = & \sigma_i z_i = \mathrm{abs}(z_i) \end{array}\right\} & & & i = 1 \dots s \\
\hline
y \equiv v_{s+1} & = & \psi_{s+1}(v_j)_{j \prec s+1} & \\
\hline
\end{array}
$$

- Declare z_i as independent variables
- adapt evaluation of abs() correspondingly

Abs-Linearisation via AD I

AD approach: tangent approximation for each elemental function

$$v_i(x + \Delta x) - v_i(x) \approx \Delta v_i \equiv \Delta v_i(\Delta x)$$

For smooth elementals:

$$\begin{aligned}
\Delta v_i &= \Delta v_j \pm \Delta v_k && \text{for } v_i = v_j \pm v_k, \\
\Delta v_i &= v_j * \Delta v_k + v_k * \Delta v_j && \text{for } v_i = v_j * v_k, \\
\Delta v_i &= \varphi'(v_j)_{j \prec i} * \Delta(v_j)_{j \prec i} && \text{for } v_i = \varphi_i(v_j)_{j \prec i} \neq \text{abs}(v_j), \\
\Delta v_i &= \text{abs}(v_j + \Delta v_j) - v_i && \text{for } v_i = \text{abs}(v_j).
\end{aligned}$$

$\Rightarrow$ If $y = F(x)$ involves no call of abs():

$$\Delta y = \Delta F(x; \Delta x) = F'(x)\Delta x, \qquad F'(x) \in \mathbb{R}^{m \times n} = \text{Jacobian}$$

standard AD!

Abs-Linearisation via AD II

For the absolute value function $v_i = \mathrm{abs}(v_j)$:

$$\begin{aligned} \Delta v_i &= \mathrm{abs}(v_j(\mathring{x}) + \Delta v_j) - v_j(\mathring{x}) \\ \Rightarrow \Delta y(\Delta x) &= \Delta F(\mathring{x}; \Delta x) : \mathbb{R}^n \mapsto \mathbb{R}^m \end{aligned}$$

is a piecewise linear continuous function for each fixed $x \in D$.

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Theorem

Suppose F is elementwise Lipschitz continuously differentiable on $D \subset K \subset \mathbb{R}^n$, D open, K closed and convex. Then there exists $\gamma > 0$ such that for all $x, \mathring{x} \in K$

$$\|F(x) - F(\mathring{x}) - \Delta F(\mathring{x}; x - \mathring{x})\| = \gamma \|x - \mathring{x}\|^2$$

A. Griewank. On stable piecewise linearization and generalized algorithmic differentiation, Optimization Methods and Software, 2013

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Derivatives a, b, c, Z, L required by abs-linear form provided by AD!

Open Questions II

Drivers/Interfaces of AD tools for abs-linearisation?

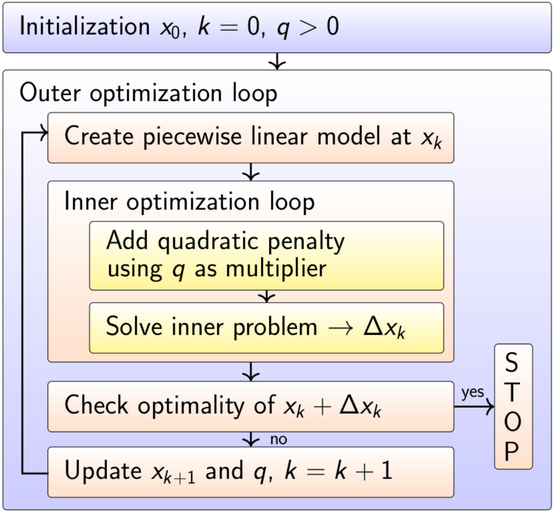
SALOP

Very brief description of the algorithm:

$$x_{k+1} = x_k + \underset{\Delta x}{\arg\min} \left\{ \Delta\varphi(x_k; \Delta x) + \frac{q}{2}\|\Delta x\|^2 \right\}$$

$=$ **S**uccessive **A**bs-**L**inear **OP**timization with a proximal term

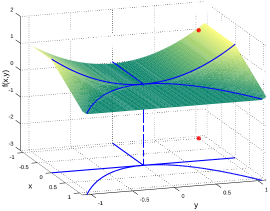
SALOP


Initialization x_0, $k = 0$, $q > 0$
Outer optimization loop
Create piecewise linear model at x_k
Inner optimization loop
Add quadratic penalty using q as multiplier
Solve inner problem $\to \Delta x_k$
Check optimality of $x_k + \Delta x_k$
yes
STOP
no
Update x_{k+1} and q, $k = k + 1$

Example

$$\varphi : \mathbb{R}^2 \to \mathbb{R}, \quad \varphi(x_1, x_2) = \max\{x_2^2 - \max\{x_1, 0\}, 0\}$$

$k = 0$



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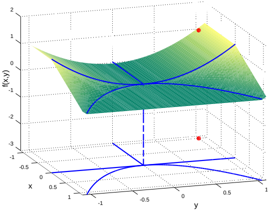
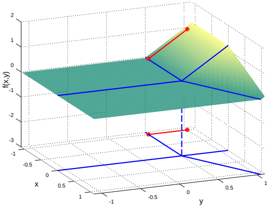


local QP in x_0 based on linearization

$\rightarrow$



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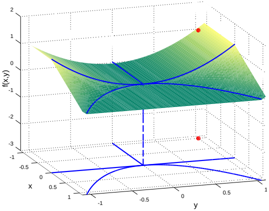

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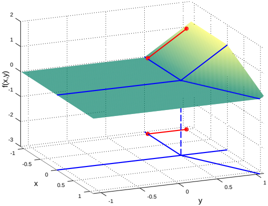
New iterate
$x_1 = x_0 + \Delta x_0$



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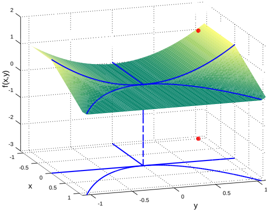

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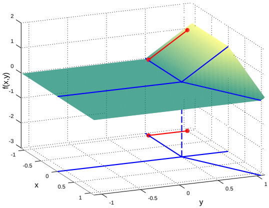
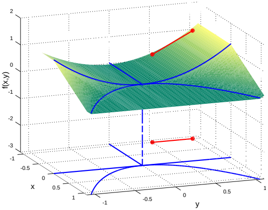


$k = 1$



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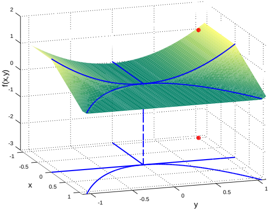
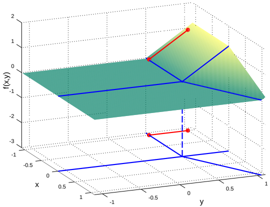


local QP in x_0 based on linearization

$\rightarrow$



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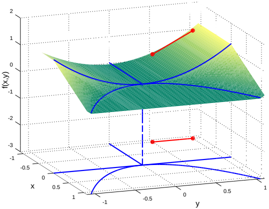

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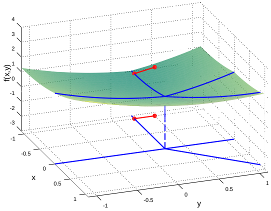

Convergence of SALOP

Finite convergence of inner loop:

- Argument space divided into finitely many polyhedra
- Function value decreased when switching polyhedra
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$\Rightarrow$ stationary point reached after finitely many steps

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Theorem

Assume that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a PS function as considered here with a bounded level set $\mathcal{N}_0 = \{x \in \mathbb{R}^n \mid f(x) \leq f(x_0)\}$. Let x_0 be the starting point of the generated sequence of iterates $\{x_k\}_{k\in\mathbb{N}}$ generated by SALOP. Then a cluster point x_* of the infinite sequence $\{x_k\}_{k\in\mathbb{N}}$ exists and all clusters points are Clarke stationary.

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S. Fiege, A. Walther, A. Griewank: An algorithm for nonsmooth optimization by successive piecewise linearization. Mathematical Programming, 2018

The Inner Optimisation Loop

Improved solver for inner loop:

- adaption of new optimality conditions for inner loop
- corresponding modification of QP solver

$\Rightarrow$ Active Signature Method (ASM)
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Iterations numbers:

n	1	2	3	4	5	6	7	8	9	10
ASM	2	4	8	16	32	64	128	256	512	1024
HANSO	3	61	494*	1341*	2521*	329*	357*	326*	307*	515*
MPBNGC	3	52	9859	9978*	3561*	4166*	2547*	1959*	9420*	9807*

A. Griewank, A. Walther: Finite convergence of an active signature method to local minima of piecewise linear functions. In revision + Matlab Implementierung von ASM

The LASSO Problem

In statistics and machine learning: Least Absolute Shrinkage and Selection Operator (LASSO)

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For given data $w \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$, the LASSO function is

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with the penalty factor $\rho > 0$.

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ASM with adapted quadratic term!!

LASSO: Iteration Numbers

Löser	$\rho = 100$ opt. value	$\rho = 100$ # iter.	$\rho = 17.353616$ opt. value	$\rho = 17.353616$ # iter.
Active Signature Method	13035.7	3	11452.1	3
LassoBlockCoordinate	13035.7	30	11452.1	29
LassoConstrained	13035.7	8	11452.1	6
LassoGaussSeidel	13035.7	12	11452.1	11
LassoGrafting	13087.2	10	11452.1	11
LassoIteratedRidge	13087.7	102	11452.1	102
LassoNonNegativeSquared	13035.7	64	11452.1	58
LassoPrimalDualLogBarrier	13035.7	9	11452.1	7
LassoProjection	13035.7	3	11452.1	5
LassoShooting	13035.7	54	11452.1	51
LassoSubGradient	13035.7	52	11452.1	23
LassoUnconstrainedApx v1	13035.7	50	11452.1	40
LassoUnconstrainedApx v2	13035.7	94	11452.1	27
LassoActiveSet	13288.9	14	11602.1	12
LassoLARS	13296.7	18	11602.1	14
LassoSignConstraints	13288.9	1	11602.1	4

Matlab interface for LASSO solvers: http://www.cs.ubc.ca/~schmidtm/Software/lasso.html

Quadratic Convergence

Proposition

If x_* is a sharp minimizer of φ then SALOP with $q \geq \gamma$ converges quadratically to x_* from all x_0 in some ball $B_\rho(x_*)$.

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Proof.

$$
\begin{aligned}
c\|x_{k+1} - x_*\| &\leq \varphi(x_{k+1}) - \varphi(x_*) = \varphi(x_{k+1}) - \varphi(x_k) - (\varphi(x_*) - \varphi(x_k)) \\
&\leq \Delta\varphi(x_k; x_{k+1} - x_k) - \Delta\varphi(x_k; x_* - x_k) \\
&\quad + \tfrac{\gamma}{2}(\|x_{k+1} - x_k\|^2 + \|x_* - x_k\|^2) \\
&\leq \tfrac{\gamma+q}{2}\|x_{k+1} - x_k\|^2 + \tfrac{\gamma-q}{2}\|x_k - x_*\|^2 \leq \gamma\|x_k - x_*\|^2 .
\end{aligned}
$$

□

Chained CB3 I

$$\varphi : \mathbb{R}^n \mapsto \mathbb{R}, \varphi(x) = \sum_{i=1}^{n-1} \max\{x_i^4 + x_{i+1}^2, (2 - x_i)^2 + (2 - x_{i+1})^2, 2e^{-x_i + x_{i+1}}\}$$

$s = 2(n-1)$, $x_* = (1 \ldots 1)^\top \in \mathbb{R}^n$ is sharp

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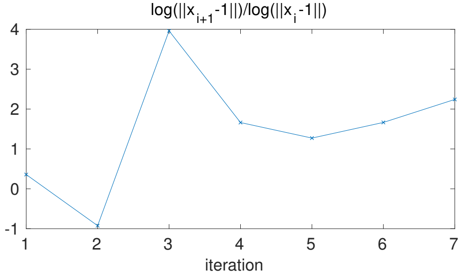
Implementation LiPsMin of SALOP yields for $n = 10$



Linear Convergence

Proposition

Suppose x_* *satisfies SSC with strict complementarity under LIKQ for* $\varphi(.)$. *Assume* $q > \max(\gamma, \|\check{U}_*^\top \check{H}_* \check{U}_*\|)$ *for the proximal parameter* q. *Then SALOP yields local and linear convergence with R-factor*

$$\|I - \tfrac{1}{q}\check{U}_*^\top \check{H}_* \check{U}_*\| \geq 1 - (\kappa(\check{U}_*^\top \check{H}_* \check{U}_*))^{-1} \ ,$$

where κ *denotes the condition number with respect to the spectral norm.*

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Proof.

- take care of nonlocalization
- formulation as fixed point problem, analysis of contraction rate

$\square$

A. Griewank and A. Walther: Relaxing kink qualifications and proving convergence rates in piecewise smooth optimization, in revision

Chained Crescent I

$$\varphi : \mathbb{R}^n \mapsto \mathbb{R}, \quad \varphi(x) = \max\{\varphi_1(x), \varphi_2(x)\}$$

$$\varphi_1(x) = \sum_{i=1}^{n-1}(x_i^2 + (x_{i+1}-1)^2 + x_{i+1} - 1),$$

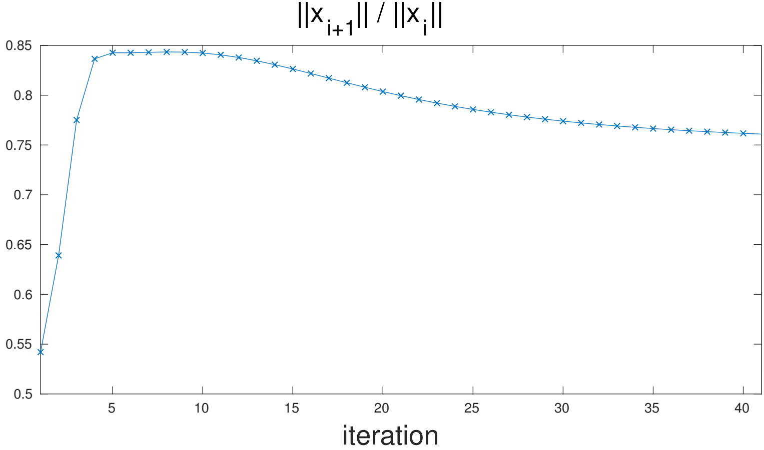
$$\varphi_2(x) = \sum_{i=1}^{n-1}(-x_i^2 - (x_{i+1}-1)^2 + x_{i+1} + 1)\ ,$$

$\Rightarrow$ PS, nonconvex function
isolated but not sharp minimizer $x_* = (1\ldots1)^\top \in \mathbb{R}^n$, $s = 1$,

$$Z = (0\ 4\ \ldots\ 4), \quad L = 0 \in \mathbb{R}, \quad a = (0\ 1\ \ldots\ 1), \quad b = 0.5\ ,$$

only switching variable is active at x_*, LIKQ holds

Chained Crescent I: Convergence



FPT Problem: Wine Quality

- data set contains 4 000 entries
- C implementation with old inner loop algo. could handle 200 entries
- Matlab implentation: up to 4 000 entries feasible!
- $n = 7$, $s = 18014$ for $m = 2\,000$ entries
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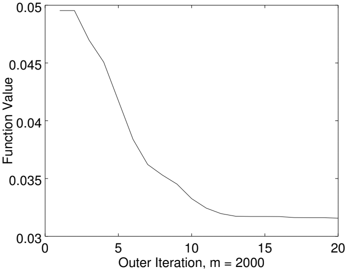

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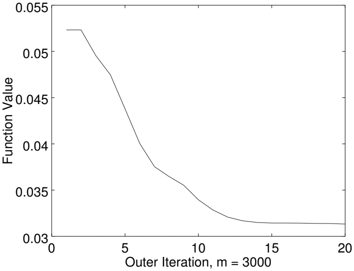

Open Questions III

- linear convergence with fewer assumptions?
- superlinear convergence?
- larger class of functions?

Signature Vectors

The signature vector

$$\sigma(x) = \mathrm{sign}(z(x))$$

and the corresponding diagonal matrix

$$\Sigma = \mathrm{diag}(\sigma)$$

define active switch set

$$\alpha = \alpha(x) \equiv \{1 \le i \le s \;|\sigma_i(x) = 0\,\} \qquad |\alpha(x)| = s - |\sigma(x)|.$$

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$$z = F(x, \Sigma z)$$

has unique solution z^σ with $\nabla z^\sigma = \frac{\partial}{\partial x} z^\sigma = (I - L\Sigma)^{-1} z.$

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Definition

We say that the linear independence kink qualification (LIKQ) is satisfied at a point $x \in \mathbb{R}^n$ if for $\sigma = \sigma(x)$ the active Jacobian

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Generalization of LICQ!

Generalized Gradients by AD

Definition

For a PS function $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ as considered here and a point $x \in \mathbb{R}^n$ the set of conical gradients is given by

$$\partial^K \varphi(x) = \left\{ g \in \mathbb{R}^n \,|\, g \in \partial^L \Delta\varphi(x; \Delta x)\big|_{\Delta x = 0} \right\} \ .$$

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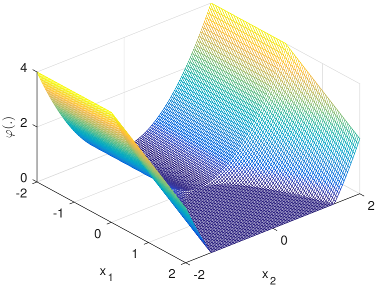
- Griewank (2013),
  considered also by Barton and Khan,
  see publications in 2013 and 2015
- Can be computed from the abs-normal form,
  i.e., they are available using AD
- A directional active gradient computed by AD is an element of the limiting gradients, i.e., $g \in \partial^L \varphi(x)$.

The Half-Pipe Example

$$\varphi : \mathbb{R}^2 \mapsto \mathbb{R}, \qquad \varphi(x_1, x_2) = \max(x_2^2 - \max(x_1, 0), 0)$$
$$= \begin{cases} x_2^2 & \text{if } x_1 \leq 0 \\ x_2^2 - x_1 & \text{if } 0 \leq x_1 \leq x_2^2 \\ 0 & \text{if } 0 \leq x_2^2 \leq x_1 \end{cases},$$



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Here, one has that

$$\widehat{\partial}^M \varphi(0) = \{(0,0)\} \subsetneq \partial^M \varphi(0) = \{(0,0), (-1,0)\} = \partial^L \varphi(0),$$
$$\Rightarrow \quad \partial^C \varphi(0) = \{(v,0) \,|\, v \in [-1,0]\},$$
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Gradient Cube Example ($n = 2$)

$$\varphi : \mathbb{R}^2 \mapsto \mathbb{R}, \quad \varphi(x_1, x_2) = |x_2 - |x_1|| + \varepsilon|x_1|, \qquad \varepsilon \in \mathbb{R}$$

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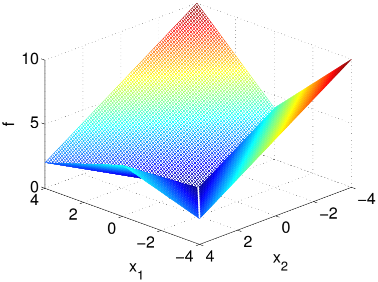

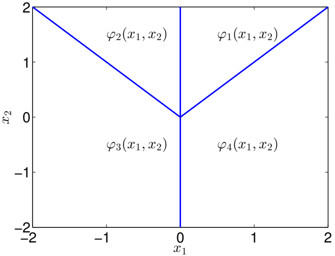

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Relations of different generalized gradients?

Mangasarin-Fromovitz-Kink Qualification

Definition

The Mangasarin-Fromovitz-Kink Qualification (MFKQ) holds at a point $\mathring{x}$ if

- for all $\sigma \succeq \mathring{\sigma}$ the vector inequality $J_\sigma v > 0$ is solvable for some $v \in \mathbb{R}^n$, where

$$\sigma \succeq \mathring{\sigma} \quad \text{in that} \quad \sigma_j \mathring{\sigma}_j \geq \mathring{\sigma}_j^2 \;\text{ for }\; j = 1, \ldots, n$$

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- strongly related to constraint qualification MFCQ
- much weaker than LIKQ

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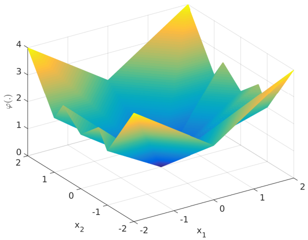

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What can we prove with these properties?

Proposition (Limiting, Mordukovich and Clark subdiff'tials)

For the abs-normal function $\varphi : \mathbb{R}^n \to \mathbb{R}$ *and* $x \in \mathbb{R}^n$*, the inclusions*

$$\partial^L \varphi(x) \subset \partial_M \varphi(x) \subset \partial_C \varphi(x)$$

hold. Furthermore, the function $\varphi(.)$ *is regular in* $\mathring{x} \in \mathbb{R}^n$ *if and only if*

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Proposition (Conical and limiting gradients)

For the abs-normal function $\varphi : \mathbb{R}^n \to \mathbb{R}$, one has

$$\partial^K \varphi(x) \subset \partial^L \varphi(x)$$

for all $x \in \mathbb{R}^n$. Furthermore, if MFKQ holds at $\mathring{x} \in \mathbb{R}^n$, then

$$\partial^K \varphi(\mathring{x}) = \partial^L \varphi(\mathring{x}) \; .$$

Definition (First order convexity (FOC))

The PS function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be *convex of first order* at a point $\mathring{x}$ if its piecewise linearization $\Delta\varphi(\mathring{x}; \cdot) : \mathbb{R}^n \to \mathbb{R}$ is convex on some ball about the argument $\Delta x = 0$.

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For the abs-normal function $\varphi : \mathbb{R}^n \to \mathbb{R}$, one has that $\varphi(.)$ is first order convex in some ball about x if $\varphi(.)$ is regular in x. Furthermore, if MFKQ holds at $x \in \mathbb{R}^n$, then $\varphi(.)$ is first-order convex in some ball about x if and only if $\varphi(.)$ is regular in x.

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A. Walther, A. Griewank. Characterizing and testing subdifferential regularity for piecewise smooth objective functions, in revision

Conclusion and Outlook

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