Resonances of the Laplacian on noncompact Riemannian symmetric spaces of low rank

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(joint work with Joachim Hilgert and Tomasz Przebinda)

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Statement of the problem

\( X = G/K \) is a Riemannian symmetric space of the noncompact type, where:

- \( G \) = connected noncompact real semisimple Lie group with finite center
- \( K \) = maximal compact subgroup of \( G \)

Examples:

- \( H^n(\mathbb{R}) = \text{SO}_0(1, n)/\text{SO}(n) \) real hyperbolic space
- \( \text{SU}(p, q)/\text{S}(\text{U}(p) \times \text{U}(q)) \), \( q \geq p \geq 1 \), Grassmannian of \( p \) subspaces of \( \mathbb{C}^{p+q} \) (complex hyperbolic space if \( p = 1 \))

\( \Delta \) = (positive) Laplacian on \( X \), with continuous spectrum \( \sigma(\Delta) = [\rho_X^2, +\infty[ \) with \( \rho_X^2 > 0 \).

The resolvent of \( \Delta \)

\[
R_\Delta(u) = (\Delta - u)^{-1}
\]

is a bdd operator on \( L^2(X) \) depending holomorphically on \( u \in \mathbb{C} \setminus \sigma(\Delta) \), i.e.

\[
\mathbb{C} \setminus \sigma(\Delta) \ni u \longrightarrow R_\Delta(u) = (\Delta - u)^{-1} \in \mathcal{B}(L^2(X))
\]

is a holomorphic operator-valued function.

As operator on \( L^2(X) \), the resolvent \( R_\Delta \) has no extension across \( \sigma(\Delta) \).

Letting \( R_\Delta \) act on a smaller dense subspace of \( L^2(X) \), e.g. \( C_c^\infty(X) \), a meromorphic continuation of \( R_\Delta \) across \( \sigma(\Delta) \) is possible.
Theorem (Strohmaier, Mazzeo-Vasy, 2005)

Let $X$ be an arbitrary Riemannian symmetric space of the noncompact type. There are $\Omega \subsetneq \mathbb{C}$ open with $\sigma(\Delta) \subset \Omega$ and $M$ Riemann surface above $\Omega$ such that

$$R_\Delta : \Omega \setminus \sigma(\Delta) \ni u \mapsto R_\Delta(u) \in \text{Hom}(C_\infty_c(X), C_\infty_c(X)')$$

admits holomorphic extension to $M$.

$\sim$ $\Omega$ is not large enough to find resonances.

Special cases showing that there might be resonances are classical:

Theorem (Guillopé-Zworski, 1995)

For $X = H^n(\mathbb{R})$ and $\Omega = \mathbb{C}$, the resolvent $R_\Delta$ has:

- holomorphic extension, if $n$ is odd
- meromorphic extension (with infinitely many poles) if $n$ even.

Problem 1: For general $X = G/K$, does $R_\Delta$ admit a meromorphic extension to a Riemann surface above $\Omega = \mathbb{C}$?

If so: what are the poles? What are the residues?

The poles of the meromorphically extended $R_\Delta$ are called the (quantum) resonances of the Laplacian.
(Quantum) resonances

In physics:

- Quantum mechanical systems which are bound can only assume certain discrete values of energy (=energy levels) which are constant in time.
- Quantum mechanical systems which are unbound might have states with energy that a certain starting time can assume certain discrete values, but are not constant in time, usually decreasing exponentially (=metastable states).
- Energy at a metastable state is described by a complex number $\zeta$ (a resonance): $\mathbb{Re} \zeta =$ energy at the starting time $\mathbb{Im} \zeta =$ rate of exponential time decreasing of the energy.
- The resonances are the poles of the meromorphic extension of the resolvent

$$\mathbb{C} \setminus \sigma(H) \ni u \longrightarrow R_H(u) = (H - u)^{-1}$$

of the Hamiltonian $H$, with continuous spectrum $\sigma(H)$, describing the unbound system.
In mathematics:

- **Classical situation:** Resonances for Schrödinger operators $H = \Delta_{\mathbb{R}^n} + V$
  - $\Delta_{\mathbb{R}^n} = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the positive Euclidean Laplacian
  - $V$ is a potential

  ($V$ chosen so that $H$ is s.a. and $\sigma(H) \subset [0, +\infty[$ is continuous; e.g. $V = 0$).

- **Geometric scattering:** Resonances for the Laplacian $\Delta$ of complete non-compact Riemannian manifolds (with bounded geometry).
  *Motivations:* scattering, dynamical systems, spectral analysis...

Very active field of research.

**Why studying resonances on symmetric spaces?**

- well understood geometry
- well developed Fourier analysis: HF (=Helgason-Fourier) transform
- radial part of $\Delta$ on a Cartan subspace is a Schrödinger operator
- tools from representation theory
Some usual renormalizations

$X = G/K$ Riemannian symmetric space of the noncompact type.

- Translate the spectrum $[\rho^2_X, +\infty)$ to $[0, +\infty)$
i.e. consider $\Delta - \rho^2_X$ instead of $\Delta$

- Change variables $u = z^2 \leadsto$ choice of square root: $\sqrt{-1} = i$
$u \in \mathbb{C} \setminus [0, +\infty[$ corresponds to $z \in \mathbb{C}^+ = \{w \in \mathbb{C} : \text{Im } w > 0\}$.

- Define

$$R(z) = R_{\Delta - \rho^2_X}(z^2) = (\Delta - \rho^2_X - z^2)^{-1}$$

So $R : \mathbb{C}^+ \rightarrow \mathcal{B}(L^2(X))$ is a holomorphic operator-valued function.

Goal:

Meromorphic continuation across $\mathbb{R}$ of $R : \mathbb{C}^+ \rightarrow \text{Hom}(C^\infty_c(X), C^\infty_c(X)')$

$\downarrow$

$C^\infty(X)$ instead of $C^\infty_c(X)'$
for $X = G/K$ symmetric
(Paley-Wiener theorem)
Residue operators

Suppose we have a meromorphic continuation of $R : \mathbb{C}^+ \to \text{Hom}(\mathcal{C}_c^\infty(X), \mathcal{C}^\infty(X))$ across $\mathbb{R}$, i.e.

- a Riemann surface $M$ with $\Omega \subset \mathbb{C}$ open, $\Omega \cap \mathbb{R} \neq \emptyset$

- $\tilde{R} : M \to \text{Hom}(\mathcal{C}_c^\infty(X), \mathcal{C}^\infty(X))$ meromorphic and extending a lift of $R$ to $M$:

\[
\begin{array}{ccc}
M & \xrightarrow{R} & \text{Hom}(\mathcal{C}_c^\infty(X), \mathcal{C}^\infty(X)) \\
\downarrow_{\pi} & & \\
\Omega & \xrightarrow{\tilde{R}} & \text{Hom}(\mathcal{C}_c^\infty(X), \mathcal{C}^\infty(X))
\end{array}
\]

- $z_0$ is a resonance (=pole of $\tilde{R}$).

The residue operator at $z_0$ is the linear operator

\[ \text{Res}_{z_0} \tilde{R} : \mathcal{C}_c^\infty(X) \to \mathcal{C}^\infty(X) \]

“defined” for $f \in \mathcal{C}_c^\infty(X)$ by

\[ \text{Res}_{z_0} \tilde{R}(f) : X \ni y \mapsto \text{Res}_{z=z_0}[\tilde{R}(z)(f)](y) \in \mathbb{C} \]

[“defined”: residues are computed wrt charts in $M$, so up to nonzero constant multiples]

**Well-defined:** the subspace $\text{Res}_{z_0} := \tilde{R}(\mathcal{C}_c^\infty(X))$ of $\mathcal{C}^\infty(X)$.

The rank of the residue operator at $z_0$ is $\dim(\text{Res}_{z_0})$. 
Problem 2: Find image and rank of the residue operator at \( z_0 \).

Additional properties appear as \( X \) is endowed with a \( G \)-invariant Riemannian metric.

The Laplacian \( \Delta \) of \( X \) is \( G \)-invariant
\( \leadsto \) \( R(z) \) and its mero extension \( \tilde{R}(z) \) are \( G \)-invariant
\( \leadsto \) the residue operator at a resonance \( z_0 \) is a \( G \)-invariant operator \( C^\infty_c(X) \to C^\infty(X) \)
\( \leadsto \) its image \( \text{Res}_{z_0} \subset C^\infty(X) \) is a \( G \)-module
(a \( K \)-spherical representation of \( G \) in our case)

Problem 3: Which (spherical) representations of \( G \) do we obtain?
Rank of residue operator \( \equiv \) dimension of the corresponding representation
Irreducible? Unitary?
Overview of results

General \( X \) of real rank one:

  meromorphic continuation of the resolvent (in the context of Damek-Ricci spaces).
  meromorphic continuation of the resolvent (using HF transform).

\( \diamond \) no resonances if \( X = H^n(\mathbb{R}) \) with \( n \) odd.
\( \diamond \) (infinitely many) resonances for \( X \neq H^n(\mathbb{R}) \) with \( n \) odd.
\( \diamond \) **Finite rank** residue operators, image: irreducible finite dim \( K \)-spherical reps of \( G \).

General \( X \) of real rank \( \geq 2 \):

(R. Mazzeo and A. Vasy (2005), A. Strohmaier (2005))

\( \diamond \) analytic continuation of the resolvent of \( \Delta \) from \( \mathbb{C}^+ \) across \( \mathbb{R} \)

\[ \begin{cases} 
\text{to an open domain in } \mathbb{C}, & \text{if the real rank of } X \text{ is odd} \\
\text{to a logarithmic cover of an open domain in } \mathbb{C}, & \text{if the real rank of } X \text{ is even}
\end{cases} \]

The open domain is **not large enough** to find resonances.

\( \diamond \) **If any**, resonances are along the negative imaginary axis.

\( \diamond \) **No resonances** in the even multiplicity case (\( = \) Lie algebra of \( G \) has one conjugacy class of Cartan subalgebras)

Specific \( X = G/K \) of real rank 2:

(J. Hilgert, A.P., T. Przebinda)

Complete answers to the three problems:

- for almost all rank 2 irreducible \( X \)
- for direct products \( X = X_1 \times X_2 \), with \( X_1, X_2 \) of rank one.
The resolvent of $\Delta$ on $X = G/K$

Explicit formula for the resolvent $R(z)$ of $\Delta$ on $C_c^\infty(X)$ via HF transform:

For $z \in \mathbb{C}^+$

$$R(z) = (\Delta - \rho_X^2 - z^2)^{-1} : C_c^\infty(X) \ni f \mapsto R(z)f \in C^\infty(X)$$

is given by

$$[R(z)f](y) \asymp \int_{\alpha^*} \frac{1}{\langle \lambda, \lambda \rangle - z^2} \left(f \times \varphi_{i\lambda}\right)(y) \frac{d\lambda}{c(i\lambda)c(-i\lambda)} \quad (y \in X),$$

where

- $\alpha^*$ = dual of a Cartan subspace $\alpha$ \(\mapsto\) real rank of $X := \dim \alpha^*$
- $\langle \cdot, \cdot \rangle$ = inner product on $\alpha^*$ induced by the Killing form of the Lie algebra of $G$
  \(\mapsto\) extend $\langle \cdot, \cdot \rangle$ to the complexification $\alpha^*_\mathbb{C}$ of $\alpha^*$ by $\mathbb{C}$-bilinearity

- $\varphi_\lambda$ = spherical function on $X$ of spectral parameter $\lambda \in \alpha^*_\mathbb{C}$
  \(\mapsto\) spherical functions = (normalized) $K$-invariant joint eigenfunctions of the commutative algebra of $G$-invariant diff ops on $X$

- $f \times \varphi_{i\lambda}$ = convolution on $X$ of $f$ and $\varphi_{i\lambda}$
  \(\mapsto\) by the Paley-Wiener thm for the HF-transform: entire and rapidly decreasing in $\lambda \in \alpha^*_\mathbb{C}$

- $c(\lambda) = \text{Harish-Chandra’s } c\text{-function}$

$$\frac{1}{c(i\lambda)c(-i\lambda)} = \text{Plancherel density for the HF-transform}$$
The Plancherel density \([c(i\lambda)c(-i\lambda)]^{-1}\)

\(\alpha (=\text{Cartan subspace}) \trianglelefteq \mathfrak{g} (=\text{Lie algebra of } G)\) by adjoint action \(\text{ad } H\) with \(H \in \alpha\)

\(\Sigma = \text{roots of } (\mathfrak{g}, \alpha)\)

\(\Sigma^+ = \text{choice of positive positive roots in } \Sigma\)

\(\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \text{ad } H(X) = \alpha(H)X \text{ for all } H \in \alpha\}\) = root space of \(\alpha \in \Sigma\)

\(m_\alpha = \dim_{\mathbb{R}} \mathfrak{g}_\alpha = \text{multiplicity of the root } \alpha\)

\(\rho = 1/2 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \in \mathfrak{a}^*\)

**Notation:** For \(\lambda \in \mathfrak{a}_C^*\) and \(\alpha \in \Sigma\) set \(\lambda_\alpha = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\)

**Harish-Chandra c-function:**

\(\Sigma^+_\ast = \{\beta \in \Sigma^+ : 2\beta \notin \Sigma\}\) (the unmultipliable positive roots)

\[c_\beta(\lambda) = \frac{2^{-2\lambda_\beta} \Gamma(2\lambda_\beta)}{\Gamma(\lambda_\beta + \frac{m_\beta}{4} + \frac{1}{2}) \Gamma(\lambda_\beta + \frac{m_\beta}{4} + \frac{m_\beta}{2})}\]

for \(\beta \in \Sigma^+_\ast\)

\[c(\lambda) = c_0 \prod_{\beta \in \Sigma^+_\ast} c_\beta(\lambda)\]

where \(c_0\) is a normalizing constant so that \(c(\rho) = 1\).

**Many rules:** e.g. if both \(\beta\) and \(\beta/2\) are roots, then \(m_{\beta/2}\) is even and \(m_\beta\) is odd.

**Many simplifications using classical formulas for }\Gamma\text{ : e.g. } \Gamma(ix)\Gamma(-ix) = \frac{i\pi}{x \sinh(\pi x)}\).

**Example:** If \(G/K\) of even multiplicities, then \([c(i\lambda)c(-i\lambda)]^{-1}\) is a polynomial.
\[ \tilde{\rho}_\beta = \frac{1}{2} \left( \frac{m_{\beta}/2}{2} + m_\beta \right) \]

**Lemma**

Set:
\[
\Pi(\lambda) = \prod_{\beta \in \Sigma^+_*} \lambda_\beta, \\
P(\lambda) = \prod_{\beta \in \Sigma^+_*} \left( \prod_{k=0}^{(m_{\beta}/2)/2-1} \left[ i\lambda_\beta - \left( \frac{m_{\beta}/2}{4} - \frac{1}{2} \right) + k \right] \prod_{k=0}^{2\tilde{\rho}_\beta-2} \left[ i\lambda_\beta - (\tilde{\rho}_\beta - 1) + k \right] \right), \\
Q(\lambda) = \prod_{\beta \in \Sigma^+_* \text{ odd}} \coth(\pi(\lambda_\beta - \tilde{\rho}_\beta)) .
\]

(Empty products are equal to 1)

Then:
\[
[c(\lambda)c(-\lambda)]^{-1} \asymp \Pi(\lambda)P(\lambda)Q(\lambda).
\]

Hence: \([c(i\lambda)c(-i\lambda)]^{-1}\) has at most first order singularities along the hyperplanes

\[ \mathcal{H}_{\beta,k,\pm} = \{ \lambda \in a^*_{\mathbb{C}} : \lambda_\beta = \pm i(\tilde{\rho}_\beta + k) \} \]

where \(\beta \in \Sigma^+_*\) has multiplicity \(m_\beta\) odd and \(k \in \mathbb{Z}_{\geq 0}\).

\[ \Sigma^+_{*,\text{odd}} = \{ \alpha \in \Sigma^+_* : m_\alpha \text{ is odd} \} \]
Extension of the resolvent of $\Delta$ on $X = G/K$

Suppose: real rank of $X = \dim a^* =: n \geq 2$.

Let $f \in C_c^\infty(X)$ and $y \in X$ be fixed.

Recall
\[
[R(z)f](y) \asymp \int_{a^*} \frac{1}{\langle \lambda, \lambda \rangle - z^2} (f \times \varphi_{i\lambda})(y) \frac{d\lambda}{c(i\lambda)c(-i\lambda)}
\]

singularities along $\mathbb{C}$-spheres radius $\pm z$ singularities along hyperplanes $\mathcal{H}_{\beta,k,\pm}$

**Polar coordinates** in $a^*$ give
\[
R(z) := [R(z)f](y) = \int_0^\infty \frac{1}{r^2 - z^2} F(r)r \, dr
\]

where
\[
F(r) = F_{f,y}(r) = r^{n-2} \int_{S^{n-1}} (f \times \varphi_{ir\sigma})(y) \frac{\omega(\sigma)}{c(ir\sigma)c(-ir\sigma)}
\]

and
\[
\omega(\sigma) = \text{pullback to } S^{n-1} \text{ of the } SO(n)\text{-invariant } (n-1)\text{-form}
\]

\[
\omega(z) = \sum_{j=1}^n (-1)^{j-1} z_j \, dz_1 \cdots \hat{dz}_j \cdots dz_n, \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^n \equiv a^*_\mathbb{C}
\]
Set \( a = \min\{\tilde{\nu}_\beta | \beta| : \beta \in \Sigma_{*, \text{odd}}^+ \} \) (and \( a = +\infty \) if \( m_\beta \) even for all \( \beta \in \Sigma^*_+ \))

**Lemma**

- For every fixed \( \sigma \in a^* \) with \( |\sigma| = 1 \), the function \( r \mapsto [c(ir\sigma)c(-ir\sigma)]^{-1} \) is holomorphic on \( \mathbb{C} \setminus i(-\infty, -a] \cup [a, +\infty) \).

The function

\[
\mathbb{C} \setminus i(-\infty, -a] \cup [a, +\infty) \ni w \to F(w) \in \mathbb{C}
\]

is holomorphic.

- Let \( U = \mathbb{C}^- \cup \{z \in \mathbb{C} : \Re z > 1, 0 \leq \Im z < 1\} \), where \( \mathbb{C}^- = \{z \in \mathbb{C} : \Im z < 0\} \).

Then \( \exists \) holo function \( H = H_{f,y} : U \to \mathbb{C} \) such that

\[
R(z) = H(z) + i\pi F(z) \quad \text{for} \ z \in U \cap \mathbb{C}^+
\]

**Corollary**

- The mero extension of \( R \) across the negative imaginary axis (where the resonances could be) is equivalent to that of \( F \).

- If any, the resonances are located on \( i(-\infty, -a] \).
The set $\Sigma_{*,\text{odd}}^+$

Let $\Sigma$ be an irreducible root system in $\alpha^*$ such that $\Sigma_{*,\text{odd}}^+ \neq \emptyset$.

- $\Sigma_*$ is a reduced and irreducible root system. So it has at most two root lengths.
- Roots of same length form a unique Weyl group orbit and have therefore same root multiplicity $m_\beta$.
- If there is a unique root length, then $m_\beta$ is constant and $\Sigma_{*,\text{odd}}^+ = \Sigma^+_*$.
  (This happens for $\Sigma = \Sigma_*$ of type A,D or E)
- If there are two root lengths (i.e. for $\Sigma_*$ of type B,C,F or G), then $\Sigma_{*,\text{odd}}^+ = \Phi_1 \sqcup \Phi_2$, where roots in $\Phi_j$ have same length, and $\Sigma_{*,\text{odd}}^+ \in \{\Sigma^+_*, \Phi_1, \Phi_2\}$.
  $\Sigma^+_* = \Phi_1 \sqcup \Phi_2$ is obtained from the following decompositions:

$$B_n = (A_1)^n \sqcup D_n \quad C_n = (A_1)^n \sqcup D_n \quad F_4^+ = D_4^+ \sqcup D_4^+ \quad G_2^+ = A_2^+ \sqcup A_2^+$$

Consequences: If $\Sigma_{*,\text{odd}}^+ \neq \emptyset$, then:

- The hyperplane arrangement $\mathcal{H} = \{\ker \beta : \beta \in \Sigma_{*,\text{odd}}^+\}$ is simplicial (= every connected component of $\alpha^* \setminus \cup \mathcal{H}$ is the intersection of $n = \dim \alpha^*$ open halfspaces, i.e. is the positive linear span of $n$ lin. indep. vectors).
- For some $\Sigma$ of types $B$, $C$ or $BC$, we have $\Sigma_{*,\text{odd}}^+ = (A_1)^n$. 

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Example: $G/K$ or rank 3 and root system $\Sigma$ of type $BC$, $B$ or $C$

$\Sigma^+ = \Sigma^+_s \sqcup \Sigma^+_m \sqcup \Sigma^+_l$, where:

$\Sigma^+_s = \{e_j; 1 \leq j \leq n\}$, multiplicity $m_s$,

$\Sigma^+_m = \{e_i \pm e_j; 1 \leq i \geq j \leq n\}$, multiplicity $m_m$,

$\Sigma^+_l = \{2e_j; 1 \leq j \leq n\}$, multiplicity $m_l$.

<table>
<thead>
<tr>
<th>$G/K$</th>
<th>$\Sigma$</th>
<th>$m_\alpha$</th>
<th>$\Sigma^+_{*,\text{odd}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SL(4, $\mathbb{R}$)/SO(3)</td>
<td>$A_3$</td>
<td>1</td>
<td>$\Sigma^+$</td>
</tr>
<tr>
<td>SU*(8)/Sp(8)</td>
<td>$A_3$</td>
<td>4</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>SU(3, $q$)/S(U(3) $\times$ U($q$)) ($q \geq 3$)</td>
<td>$C_3$ ($q = 3$) $BC_3$ ($q &gt; 3$)</td>
<td>$(2(q - 3), 2, 1)$</td>
<td>$\Sigma^+_l$</td>
</tr>
<tr>
<td>$\Sigma^+_m (q \text{ odd})$ $\Sigma^+_s \sqcup \Sigma^+_m (q \text{ even})$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SO$_0$(3, $q$)/SO(3) $\times$ SO($q$) ($q &gt; 3$)</td>
<td>$B_3$</td>
<td>$(q - 3, 1, 0)$</td>
<td>$\Sigma^+_l$</td>
</tr>
<tr>
<td>SO*(12)/U(6)</td>
<td>$BC_3$</td>
<td>$(4, 4, 1)$</td>
<td>$\Sigma^+_l$</td>
</tr>
<tr>
<td>Sp(6, $\mathbb{R}$)/U(3)</td>
<td>$C_3$</td>
<td>$(0, 1, 1)$</td>
<td>$\Sigma^+_m \sqcup \Sigma^+_l$</td>
</tr>
<tr>
<td>Sp(3, $q$)/Sp(3) $\times$ Sp($q$) ($q \geq 3$)</td>
<td>$BC_3$</td>
<td>$(4(q - 3), 4, 3)$</td>
<td>$\Sigma^+_l$</td>
</tr>
<tr>
<td>$\Sigma^+_m \sqcup \Sigma^+_l$</td>
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</tbody>
</table>

When $\Sigma^+_{*,\text{odd}} = \Sigma^+_l$, the mero extension of $F$ for $G/K$ can be deduced from that for a direct product of rank-one symmetric spaces.
Direct products of rank-one symmetric spaces

\( X = X_1 \times \cdots \times X_n \) where \( X_j \) = rank-one Riemannian symmetric noncompact type

(the index \( j \) indicates objects associated with \( X_j \))

\( a^* = a_1^* \oplus \cdots \oplus a_n^*, \quad \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 \oplus \cdots \oplus \langle \cdot, \cdot \rangle_n \)

\( \Sigma = \Sigma_1 \times \cdots \times \Sigma_n \) with \( \Sigma_j \in \{ A_1, BC_1 \} \)

\( \Delta = \sum_{j=1}^{n} (\text{id} \otimes \cdots \otimes \Delta_j \otimes \cdots \text{id}), \quad \sigma(\Delta) = [\rho_X^2, +\infty[, \quad \rho_X^2 = \rho_{X_1}^2 + \cdots + \rho_{X_n}^2 \)

\( c(\lambda) = c_1(\lambda_1) \cdots c_n(\lambda_n), \quad \lambda = \lambda_1 \cdots + \lambda_n \in a^*_C \) with \( \lambda_j \in a^*_C \)

- The Plancherel density of \( X_j \) is singular iff \( X_j \neq H^n(\mathbb{R}) \) with \( n \) odd.
- The Plancherel density of \( X \) is the product of the Plancherel densities of the \( X_j \)'s.

It has first order singularities along \( N \) mutually orthogonal families of hyperplanes parallel to the coordinate axes, where

\( N = \# \{ j \in \{ 1, \ldots, n \} : X_j \neq H^n(\mathbb{R}), \ n \text{ odd} \} \).
Example: product of two rank-one Riemannian symmetric spaces

J. Hilgert, A.P. and T. Przebinda (2017):

◇ meromorphic continuation of $R$ to suitable Riemann surfaces over $\mathbb{C}$
◇ No resonances if one of the two spaces is $H^n(\mathbb{R})$ with $n$ odd,
◇ infinitely many resonances in the other cases
◇ residue operators with finite rank
◇ range of the residue operators realized by finite direct sums of tensor products of finite dim irr $K$-spherical reps of $G_1$ and $G_2$

(where $X_1 = G_1/K_1$ and $X_2 = G_2/K_2$ are the symm spaces)
The integral defining $F$ for $X = X_1 \times \cdots \times X_n$

Suppose $X_j \neq H^n(\mathbb{R})$, $n$ odd, exactly for $j = 1, \ldots, N$ with $N \leq n$.

For $j = 1, \ldots, N$ define:

$p_j : \mathbb{C}^n \ni z = (z_1, \ldots, z_n) \rightarrow z_j \in \mathbb{C},$

$L_j = (a_j + b_j \mathbb{Z}_{\geq 0}) \cup (-a_j - b_j \mathbb{Z}_{\geq 0})$ with $a_j > 0$, $b_j > 0$

$L = \bigcup_{j=1}^N \ p_j^{-1}(L_j) = \bigcup_{j=1}^N \bigcup_{l_j \in L_j} \{ z \in \mathbb{C}^n : z_j = l_j \}$

$a = \min\{a_1, \ldots, a_N\}.$

$S^{n-1}(\mathbb{C}) = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 = 1 \} \quad \text{(the complex sphere)}$

$\omega(z) = \sum_{j=1}^n (-1)^{j-1} z_j \, dz_1 \cdots \, \hat{dz}_j \cdots \, dz_n,$ \quad $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be meromorphic on $\mathbb{C}^n$ and holomorphic on $\mathbb{C}^n \setminus iL$.

Since $f(z)\omega(z)$ is a closed form of top complex dimension on $S^{n-1}(\mathbb{C}) \setminus iL$ the function

$$C \setminus i((-\infty, -a] \cup [a, \infty)) \ni w \rightarrow F(w) = \int_{S^{n-1}} f(wz)\omega(z) \in \mathbb{C}$$

is well defined and holomorphic.

**Remark:** For the study of the resolvent on $X$, one chooses

$f(wz) = w^{n-2}(f \times \varphi_{iwz})(y) \left[ c(iwz)c(-iwz) \right]^{-1}$, having identified $a^*_C \ni \lambda \equiv z \in \mathbb{C}^n.$
Fix $v_0 \in ]-\infty, -a] \cup [a, \infty[$. Then $S^{n-1}(\mathbb{R}) \cap \frac{1}{v_0} L \neq \emptyset$ is possible and therefore the integral $\int_{S^{n-1}} f(wz)\omega(z)$, with $w = iv_0$, might diverge.

- Suppose $C_{iv_0} \subseteq S^{n-1}(\mathbb{C}) \setminus \frac{1}{v_0} L$ is a cycle homologous to $S^{n-1}$ in $S^{n-1}(\mathbb{C})$.

  $\rightsquigarrow C_{iv_0}$ is a “deformation” of $S^{n-1}$ within $S^{n-1}(\mathbb{C})$ which is disjoint with $\frac{1}{v_0} L$

Since $L$ is a locally finite family of hyperplanes, $\exists$ an open neighborhood $W \subseteq \mathbb{C}$ of $iv_0$ such that $C_{iv_0} \subseteq S^{n-1}(\mathbb{C}) \setminus \frac{i}{W} L$. So

$$W \ni w \rightarrow \int_{C_{iv_0}} f(wz)\omega(z) \in \mathbb{C}$$

is well defined and is holomorphic.

- Fix $w_0 \in W \cap \mathbb{C}_{Re>0}$. Suppose we have found finitely many cycles

$$C_k \subseteq S^{n-1}(\mathbb{C}) \setminus \frac{i}{w_0} L \quad (k = 1, 2, \ldots, M)$$

such that $[S^{n-1}] = [C_{iv_0}] + \sum_k [C_k]$ in $H_{n-1}(S^{n-1}(\mathbb{C}) \setminus \frac{i}{w_0} L)$.

Then, by Stokes Theorem, for $w \in \mathbb{C}_{Re>0}$ near $w_0$

$$\int_{S^{n-1}} f(wz)\omega(z) = \int_{C_{iv_0}} f(wz)\omega(z) + \sum_k \int_{C_k} f(wz)\omega(z).$$

The first integral on the RHS is holo on $W$. One hopes to choose the $C_k$’s so that residue computations in $z$ yield a mero function of $w \in W$. 

A. Pasquale (IECL, Lorraine)  Resonances of the Laplacian
- The homology of $S^{n-1}(\mathbb{C}) \setminus \{\text{hyperplane arrangement}\}$ is not known, unlike the case of $\mathbb{C}^n \setminus \{\text{hyperplane arrangement}\}$ (Goresky-MacPherson).

**Useful description:** $S^{n-1}(\mathbb{C})$ can be identified with the tangent bundle

$$TS^{n-1} = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : |u| = 1, u \cdot v = 0\}$$

to $S^{n-1}$ by means of the isomorphism

$$\tau : S^{n-1}(\mathbb{C}) \ni z = x + iy \to \left(\frac{x}{|x|}, y\right) \in TS^{n-1}$$

with inverse

$$\tau^{-1} : TS^{n-1} \ni (u, v) \to \sqrt{1 + |v|^2}u + iv \in S^{n-1}(\mathbb{C}).$$

- The general construction of the cycles is not yet achieved $C_{iv_0}$ and $C_k$, even in rank 3.

**Easiest possible case of rank 3:** $X = X_1 \times X_2 \times X_3$ with $X_1 \neq H^n(\mathbb{R})$, $n$ odd, and $X_2 = X = 3 = H^n(\mathbb{R})$, $n$ odd.

One family of parallel singular hyperplanes perpendicular to $x_1$-axis.

For $v_0 \in ]-\infty, -a] \cup [a, \infty[$: $S^2 \cap \frac{1}{v_0}L \neq \emptyset$ if and only if $\left|\frac{l}{v_0}\right| \leq 1$, and

$$\left|\frac{l}{v_0}\right| < 1 \Rightarrow \text{intersection is a circle perpendicular to } x_1 \text{ axis (generic case)}$$

$$\left|\frac{l}{v_0}\right| = 1 \Rightarrow \text{intersection is a single point } \in \{\pm 1, 0, 0\}.$$

**Theorem.** The resolvent $R$ extends holomorphically to $\mathbb{C}$ (no resonances).
Happy Birthday, Joachim!
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